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INTEGRAL OF MULTIVALUED MAPPINGS AND ITS CONNECTION WITH DIFFERENTIAL RELATIONS

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1. INTRODUCTION

The notion of integral of a multivalued mapping has been introduced by a number of authors. Let us mention R. J. Aumann [1] (1965) and Z. Artstein, J. A. Burns [2] (1975), whose approaches are essentially different. While the former defined the integral in terms of measurable selections, the latter used a Riemann-type definition following the idea due to J. Kurzweil [3, 4] (1957), [5] (1980), R. Henstock [6] (1963) and E. J. MacShane [7] (1969). Let us recall these definitions.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $T = [a, b]$ a compact interval, $F : T \to \mathscr{S}^n$ a multivalued mapping ($\mathscr{S}^n$ denotes the family of all subsets of $\mathbb{R}^n$).

Definition 1 (Aumann [1]). Let

$$\mathscr{F} = \{f : T \to \mathbb{R}^n \mid f \text{ Lebesgue integrable, } f(t) \in F(t) \text{ for a.e. } t \in T\} .$$

The set

$$I_{\mathscr{A}} = \left\{\int_T f(t)\, \mathrm{d}t \mid f \in \mathscr{F}\right\} \subset \mathbb{R}^n$$

is called *the $\mathscr{A}$-integral of F over T* and we write

$$I_{\mathscr{A}} = (\mathscr{A}) \int_T F(t)\, \mathrm{d}t .$$

Before giving the other definition, let us recall the requisite notions.

A partition of T is a collection

(0) $$\Delta = \{(t_j, [a_{j-1}, a_j]),\ j = 1, \dots, k;\ a = a_0 \leqq a_1 \dots \leqq a_k = b\} ,$$

a gauge on T is a positive real-valued function $\delta : T \to \mathbb{R}^+ = (0, +\infty)$. We say that Δ is subordinate to δ (briefly: Δ sub δ) if

$$[a_{j-1}, a_j] \subset (t_j - \delta(t_j),\ t_j + \delta(t_j)) ,\quad j = 1, \dots, k .$$

The Riemann sum (for F) corresponding to a partition Δ is constructed as

$$S(F, \Delta) = \{\sum_{j=1}^{k} \varphi_j(a_j - a_{j-1}) \mid \varphi_j \in F(t_j),\ j = 1, \dots, k\}$$

(we write briefly $S(F, \Delta) = \sum_{j=1}^{k} F(t_j)(a_j - a_{j-1})$, using the usual definition of the sum and the multiple of sets).

Definition 2 (Artstein, Burns [2]). *Let $F(t)$ be compact for $t \in T$. A compact set $I_{\mathscr{B}} \subset \mathbb{R}^n$ is the $\mathscr{B}$-integral of F over T* if for every $\varepsilon > 0$ there is a gauge δ such that

$$h(S(F, \Delta), I_{\mathscr{B}}) < \varepsilon$$

for every partition Δ of T subordinate to δ (h is the Hausdorff distance of sets). We write

$$I_{\mathscr{B}} = (\mathscr{B}) \int_T F(t)\, \mathrm{d}t\,.$$

Let us briefly mention some properties of the integrals introduced in Definitions 1, 2. (For detailed accounts, the reader is refered to [1], [2].)

The $\mathscr{A}$-integral always exists, nonetheless, it can be empty. On the other hand, the $\mathscr{B}$-integral need not exist but if it does, it is a nonempty set. Both the $\mathscr{A}$- and $\mathscr{B}$-integrals assume convex values; moreover, the identity $\int_T F(t)\, \mathrm{d}t = \int_T \operatorname{conv} F(t)\, \mathrm{d}t$ holds for the $\mathscr{B}$-integral, but not generally for the $\mathscr{A}$-integral (conv stands for the closed convex hull of a set). Finally, if the $\mathscr{B}$-integral of F over T exists (which happens if and only if F is integrably bounded (see Definition 4) and conv F is measurable), then it coincides with the $\mathscr{A}$-integral of F over T.

Apparently, the Riemann-type definition has some advantages. On the other hand, if for example $F(t) = [0, 1]$ for $t \in T$, then of course $(\mathscr{B}) \int_T F(t)\, \mathrm{d}t$ exists, while the ($\mathscr{B}$)-integral of G, $G(t) = F(t) \cup \{-\chi_M(t)\}$, where χ_M is the characteristic function of a nonmeasurable set $M \subset T$, does not. This situation seems rather unnatural. Therefore, the aim of the present paper is to give a modification of the Riemann-type definition, which would avoid such phenomena.

Our definition will be equivalent to that of the $\mathscr{A}$-integral provided F is integrably bounded, measurable and assumes compact values (measurability can be replaced by convexity of the values $F(t)$). It follows from the above mentioned conditions of existence of the $\mathscr{B}$-integral (cf. Theorems C and D in [2]) that our integral also coincides with the $\mathscr{B}$-integral provided the latter exists.

Further results of the present paper can be described as follows: for a multifunction F, denote by $\Theta(F)$ the multifunction with the following properties:

(i) $\Theta(F)(t)$ is a closed subset of $\mathbb{R}^n$ for all t;
(ii) $\Theta(F)$ is measurable;

(iii) $\Theta(F)(t) \subset \mathrm{cl}\, F(t)$ for all t (cl denotes the closure);

(iv) $\Theta(F)$ is maximal in the following sense: if U is a multifunction satisfying (i), (ii), (iii) (with $\Theta(F)$ replaced by U), then $U(t) \subset \Theta(F)(t)$ for a.e. t.

It is evident that $\Theta(F)$ is uniquely defined (up to a set of measure zero), provided it exists. Moreover, $\Theta(F)$ exists for every multifunction F (cf. Theorem 4). In Theorem 5 we prove – our definition of the integral being used on both sides – that

$$\int_T \Theta(\mathrm{conv}\, F)(t)\, \mathrm{d}t = \int_T F(t)\, \mathrm{d}t$$

provided the multifunction F is integrably bounded (here of course $(\mathrm{conv}\, F)(t) = \mathrm{conv}\, F(t)$ for $t \in T$). Further, if F is measurable and integrably bounded, then

$$(\mathscr{A}) \int_T \mathrm{cl}\, F(t)\, \mathrm{d}t = \int_T F(t)\, \mathrm{d}t\,.$$

As a consequence, we obtain

$$(\mathscr{A}) \int_T \Theta(\mathrm{conv}\, F)(t)\, \mathrm{d}t = \int_T F(t)\, \mathrm{d}t\,.$$

On the other hand,

$$(\mathscr{A}) \int_T \mathrm{cl}\, F(t)\, \mathrm{d}t = (\mathscr{A}) \int_T \Theta(F)(t)\, \mathrm{d}t$$

(F need be neither measurable nor integrably bounded). Obviously $\Theta(F)(t) \subset \Theta(\mathrm{conv}\, F)(t)$ for a.e. $t \in T$ and the above formulas provide some insight into why it can occur that

$$(\mathscr{A}) \int_T F(t)\, \mathrm{d}t \subsetneqq \int_T F(t)\, \mathrm{d}t$$

(even if $F(t)$ is compact for $t \in T$).

Finally, we prove a theorem which is an analogue of that on the equivalence of the differential and the corresponding integral equation.

The case of multifunctions that are not integrably bounded exhibits certain specific features. We intend to pursue its study and present more complete results later.

2. PRELIMINARY

In this section we will introduce an operator Φ by a definition strongly resembling that of the Artstein-Burns integral [2] but not involving the Hasudorff distance. Theorem 3 will show that if we restricted ourselves to multifunctions that are integrably bounded and measurable, we could use it as an equivalent definition for both the integrals mentioned in Introduction. However, Examples 1–3 will demonstrate that in general the definition requires some modification.

In addition to the notation introduced in Section 1, we shall use the following symbols: d will denote the Euclidean distance of points or sets, e.g. $d(x, y)$, $d(x, A)$, $d(A, B) = \inf \{d(x, y) \mid x \in A, y \in B\}$; $B(x, \delta)$ and $\bar{B}(x, \delta)$ will be the open and closed ball, respectively, with center x and radius δ (in $\mathbb{R}^n$); similarly $\Omega(A, \delta)$, $\bar{\Omega}(A, \delta)$ will denote the open δ-neighborhood of a set A and its closure, respectively; m stands for the Lebesgue measure, m_i for the inner measure (the dimension will be clear from the context). $\mathscr{K}^n$ is the family of all nonempty convex compact subsets of $\mathbb{R}^n$, $\mathscr{K}^n_0 = \mathscr{K}^n \cup \{\phi\}$. As mentioned in Introduction, conv and cl stand for the closed convex hull and the closure, respectively.

Definition 3. Let $T = [a, b]$, $F : T \to \mathscr{S}^n$. Then we denote by $\Phi(F, T)$ the set of all $z \in \mathbb{R}^n$ such that for every $\varepsilon > 0$ there is a gauge $\delta : T \to \mathbb{R}^+$ such that for every partition Δ of T subordinate to δ we have $d(z, S(F, \Delta)) < \varepsilon$. That is,

$$\Phi(F, T) = \{z \in \mathbb{R}^n \mid \forall \varepsilon > 0 \; \exists \text{ gauge } \delta : \Delta \text{ sub } \delta \Rightarrow d(z, S(F, \Delta)) < \varepsilon\} . \tag{1}$$

Remarks. 1. A real-valued function $f : T \to \mathbb{R}^n$ may be viewed as a multivalued mapping from T into $\mathscr{S}^n$. In that case, the functional from Definition 3 equals the Lebesgue integral iff it is nonempty (hence obviously one-point); it is empty, iff the Lebesgue integral does not exist. (Cf. [5], [6].)

2. Evidently we can write

$$\Phi(F, T) = \bigcap_{\varepsilon > 0} \bigcup_{\delta} \bigcap_{\Delta \text{sub} \delta} \Omega(S(F, \Delta), \varepsilon) , \tag{2}$$

where Δ sub δ indicates that the intersection is taken over all partitions Δ subordinate to δ.

Definition 4. We shall say that $F : T \to \mathscr{S}^n$ is *integrably bounded* if there is an integrable function $\varrho : T \to [0, +\infty)$ such that $F(t) \subset \bar{B}(0, \varrho(t))$ for a.e. $t \in T$.

Theorem 1. *Let $F : T \to \mathscr{S}^n$ be such a map that $F(t)$ is bounded for all $t \in T$. Then the set $\Phi(F, T)$ is closed and convex; it is compact provided F is integrably bounded.*

Proof. The closedness immediately follows from the relation (1); the same relation implies that $\Phi(F, T)$ is bounded provided F is integrably bounded. The convexity easily follows from

Proposition. *If $F(t)$ is bounded for $t \in T$, then*

$$\Phi(F, T) = \bigcap_{\varepsilon > 0} \bigcup_{\delta} \bigcap_{\Delta \text{sub} \delta} \operatorname{conv} \Omega(S(F, \Delta), \varepsilon) . \tag{3}$$

Indeed, let us assume that Proposition is true. The set

$$P_{\delta, \varepsilon} = \bigcap_{\Delta \text{sub} \delta} \operatorname{conv} \Omega(S(F, \Delta), \varepsilon) \tag{4}$$

is obviously convex. If $x, y \in \bigcup_{\delta} P_{\delta,\varepsilon}$, then there are gauges δ_i, $i = 1, 2$, such that $x \in P_{\delta_1,\varepsilon}$, $y \in P_{\delta_2,\varepsilon}$. Put $\delta_3(t) = \min(\delta_1(t), \delta_2(t))$ for $t \in T$. Then $x, y \in P_{\delta_3,\varepsilon}$, which is convex, hence any point $\lambda x + (1 - \lambda) y$, $0 \leqq \lambda \leqq 1$, belongs to $P_{\delta_3,\varepsilon} \subset \bigcup_{\delta} P_{\delta,\varepsilon}$. This proves that $\bigcup_{\delta} P_{\delta,\varepsilon}$ is convex and since the intersection of convex sets is again a convex set, we conclude that $\Phi(F, T)$ is convex.

Proof of Proposition: Remark 2 makes the inclusion $\subset$ in (3) obvious. To prove the converse inclusion, we use a lemma quoted in [2], which is due to Shapley, Folkman (cf. [8], Theorem 9, p. 396):

Lemma. *Let* $L > 0$, $A_j \subset R^n$, $A_j \subset B(0, L)$, $j = 1, \ldots, k$. *Then*

$$\operatorname{conv} \sum_{j=1}^{k} A_j \subset \Omega\Big(\sum_{j=1}^{k} A_j,\ L\sqrt{n}\Big).$$

Let

$$z \in \bigcap_{\varepsilon > 0} \bigcup_{\delta > 0} \bigcap_{\Delta \operatorname{sub} \delta} \operatorname{conv} \Omega(S(F, \Delta), \varepsilon) \tag{5}$$

and $\eta > 0$. Then there is a gauge δ such that $z \in P_{\delta,\eta/2}$ (cf. (4)). Put $\delta_0(t) = \min(\delta(t), \eta[4\sqrt{(n)}(1 + c(t))]^{-1})$, where $c : T \to R^+$ is such that $F(t) \subset \bar{B}(0, c(t))$ for $t \in T$; then obviously

$$z \in P_{\delta_0,\eta/2}\,. \tag{6}$$

Let us denote $A_j = F(t_j)(a_j - a_{j-1})$ where t_j, a_j are from Δ (cf. (0)); then $S(F, \Delta) = \sum_{j=1}^{k} A_j$. If Δ is subordinate to δ_0, then $A_j \subset B(0, 2c(t_j)\,\delta_0(t_j))$, where $2c(t)\,\delta_0(t) < \frac{1}{2}\eta\, c(t)\,[\sqrt{(n)}(1 + c(t))]^{-1}$. Using the above lemma with $L = \frac{1}{2}\eta/\sqrt{n}$, we obtain

$$\operatorname{conv} S(F, \Delta) \subset \Omega(S(F, \Delta), L\sqrt{n}) = \Omega(S(F, \Delta), \tfrac{1}{2}\eta)\,. \tag{7}$$

The elementary inclusion

$$\operatorname{conv} \Omega(M, \varepsilon) \subset \Omega(\operatorname{conv} M, \varepsilon)$$

$(M \subset R^n, \varepsilon > 0)$ yields

$$\operatorname{conv} \Omega(S(F, \Delta), \tfrac{1}{2}\eta) \subset \Omega(\operatorname{conv} S(F, \Delta), \tfrac{1}{2}\eta) \subset \Omega(S(F, \Delta), \eta)$$

in virtue of (7). Hence

$$P_{\delta_0,\eta/2} \subset \bigcap_{\Delta \operatorname{sub} \delta_0} \Omega(S(F, \Delta), \eta)\,,$$

which immediately implies $z \in \Phi(F, T)$ for z satisfying (5). The proof of Theorem 1 is complete.

Theorem 2. *Let $F : T \to \mathscr{S}^n$, let $F(t)$ be bounded for all $t \in T$. Then*

$$\Phi(F, T) = \Phi(\operatorname{conv} F, T).$$

Proof. Notice that for $\lambda_j \in \mathbb{R}$, $A_j \subset \mathbb{R}^n$, $j = 1, \ldots, k$, the identity $\sum_{j=1}^{k} \lambda_j \operatorname{conv} A_j = \operatorname{conv}\left(\sum_{j=1}^{k} \lambda_j A_j\right)$ holds. Consequently, if $z \in \Phi(\operatorname{conv} F, T)$, then for every $\varepsilon > 0$ there is a gauge δ such that for every partition Δ subordinate to δ,

$$z \in \Omega(S(\operatorname{conv} F, \Delta), \varepsilon) = \Omega(\operatorname{conv} S(F, \Delta), \varepsilon).$$

Thus the proof of the inclusion $z \in \Phi(F, T)$ is based on the inclusion (7) (with $\eta = \varepsilon$), which again is established via Lemma as in the proof of Proposition.

Theorem 3. *Let $F : T \to \mathscr{S}^n$ be measurable, integrably bounded and let $F(t)$ be compact and nonempty for $t \in T$. Then*

$$(\mathscr{A}) \int_T F(t)\, \mathrm{d}t = \Phi(F, T). \tag{8}$$

Proof. The inclusion $\subset$ is obvious. Indeed, if $z \in (\mathscr{A}) \int_T F(t)\, \mathrm{d}t$, then $z = \int_T f(t)\, \mathrm{d}t$, where f is a measurable selection, i.e. $f(t) \in F(t)$ for a.e. $t \in T$. Since $F(t) \neq \emptyset$ we can assume that f is defined for all $t \in T$ and hence by Remark 1 (and by [5]) we conclude that $z \in \Phi(f, T) \subset \Phi(F, T)$.

Thus we only have to prove the inclusion $\supset$.

Let $P = \{p_1, p_2, \ldots\}$ be a countable dense subset of $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ and let $(.,.)$ denote the scalar product. Then

$$\operatorname{conv} F(t) = \bigcap_{i=1}^{\infty} \{x \in \mathbb{R}^n \mid (p_i, x) \leqq \lambda_i(t)\},$$

where

$$\lambda_i(t) = \sup \{(p_i, x) \mid x \in F(t)\}.$$

Denote $\Lambda_i = \int_T \lambda_i(t)\, \mathrm{d}t$; the integral exists since F (and hence also λ_i) is measurable and integrably bounded. We have

$$\Phi(F, T) \subset \bigcap_{i=1}^{\infty} \{x \in \mathbb{R}^n \mid (p_i, x) \leqq \Lambda_i\}. \tag{9}$$

Indeed, let $y \in \Phi(F, T)$ and suppose on the contrary that there is a positive integer r,

$$(p_r, y) - \Lambda_r = \eta > 0. \tag{10}$$

Since $y \in \Phi(F, T)$, we find a gauge δ_0 such that for every partition Δ subordinate to δ_0 the inequality

$$d(y, S(F, \Delta)) < \tfrac{1}{3}\eta$$

holds. Hence there is $z \in S(F, \Delta)$ such that

$$|(p_r, y) - (p_r, z)| < \tfrac{1}{3}\eta$$

(recall that $\|p_r\|_* = 1$) and hence

$$(p_r, y) \leqq S(\lambda_r, \Delta) + \tfrac{1}{3}\eta \,. \tag{11}$$

On the other hand, using the Riemann-type definition of the Lebesgue integral (cf. Remark 1), for the gauge δ_0 we find a partition Δ_0 subordinate to δ_0 such that

$$S(\lambda_r, \Delta_0) \leqq \Lambda_r + \tfrac{1}{3}\eta \,. \tag{12}$$

Combining (11) (with $\Delta = \Delta_0$), (12) and (10) we conclude that

$$(p_r, y) \leqq \Lambda_r + \tfrac{2}{3}\eta = (p_r, y) - \tfrac{1}{3}\eta \,,$$

a contradiction since $\eta > 0$. Hence (9) holds.

On the other hand, let us denote

$$\Psi_i = \sup \left\{(p_i, x) \mid x \in (\mathscr{A}) \int_T F(t) \, \mathrm{d}t\right\} .$$

Then (since $F(t)$ is compact and hence the $\mathscr{A}$-integral is both compact and convex)

$$(\mathscr{A}) \int_T F(t) \, \mathrm{d}t = \bigcap_{i=1}^{\infty} \{x \in \mathbb{R}^n \mid (p_i, x) \leqq \Psi_i\} \,.$$

Assume that there exists a positive integer r such that

$$\Psi_r < \Lambda_r \,. \tag{13}$$

Denote

$$F_r(t) = F(t) \cap \{x \in \mathbb{R}^n \mid (p_r, x) = \lambda_r(t)\} \,.$$

Then $F_r(t)$ is nonempty (since $F(t)$ is nonempty and compact) and measurable and hence there exists a measurable selection $f_r : T \to \mathbb{R}^n$, $f_r(t) \in F_r(t)$, which obviously satisfies $\int_T f_r(t) \, \mathrm{d}t \in (\mathscr{A}) \int_T F(t) \, \mathrm{d}t$. However, this inclusion implies

$$\Psi_r \geqq \left(p_r, \int_T f_r(t) \, \mathrm{d}t\right) = \int_T (p_r, f_r(t)) \, \mathrm{d}t = \int_T \lambda_r(t) \, \mathrm{d}t = \Lambda_r \,,$$

which contradicts the inequality (13). Hence $\Psi_i \geqq \Lambda_i$ for $i = 1, 2, \ldots$. Taking into account the inclusion (9), we conclude that

$$(\mathscr{A}) \int_T F(t) \, \mathrm{d}t \supset \Phi(F, T) \,,$$

which completes the proof of the identity (8).

Theorem 3 together with Remark 1 may tempt us to conjecture $(\mathscr{A}) \int_T F(t) \, \mathrm{d}t = \Phi(F, T)$ for any F with compact values; this however is disproved by a simple example given by Aumann [1]. Modifying it a little we even demonstrate that the Aumann integral can be empty while $\Phi(F, T)$ is nonempty.

Example 1. Let $T = [0, 1] = M_1 \cup M_2$, M_i non-measurable disjoint sets both with inner measure zero and outer measure one. Set

$$F(t) = \begin{cases} \{0, 2\} & \text{for } t \in M_1 \\ \{-1, 1\} & \text{for } t \in M_2 . \end{cases}$$

Then evidently $(\mathscr{A}) \int_T F(t) \, dt = \emptyset$ but $\Phi(F, T) = \Phi(\operatorname{conv} F, T) \supset$ $\supset (\mathscr{A}) \int_T \operatorname{conv} F(t) \, dt = [0, 1]$. (Actually, $\Phi(F, T) = [0, 1]$ as can be seen from Definition 3 or Theorem 5.)

Even if we put up with the fact that generally $(\mathscr{A}) \int_T F(t) \, dt \neq \Phi(F, T)$ there are other properties of Φ that would avert us from trying to introduce a new notion of integral by setting $\Phi(F, T) = \int_T F(t) \, dt$.

First of all, we should expect continuity of Φ; however, examples show that if $\beta \to b_-$, $T = [a, b]$, then $\Phi(F, [a, b]) = \lim_{\beta \to b_-} \Phi(F, [a, \beta])$ need not necessarily hold.

Example 2. Let M_1, M_2 be the same as in Example 1. Set

$$F(t) = \begin{cases} \{\ \ 1\} & \text{for } t \in M_1 , \\ \{-1\} & \text{for } t \in M_2 , \\ \{-1, 1\} & \text{for } t \in (1, 2] . \end{cases}$$

Then $\Phi(F, [0, \beta]) = \emptyset$ for $0 < \beta < 2$ but $\Phi(F, [0, 2]) = \{0\}$. Indeed, the latter identity follows from the fact that, provided the gauge δ is sufficiently fine then, whatever the contribution of the terms involving subintervals from $[0, 1]$ may be, it can always be "balanced" (up to an arbitrarily prescribed $\varepsilon > 0$) by choosing properly the elements 1 or -1 from $F(t)$ in the subintervals from $[1, 2]$, so that $0 \in \Phi(F, [0, 2])$.

On the other hand, this is not possible if instead of $[1, 2]$ there is only a shorter interval $[1, \beta]$ available.

The above example also shows that additivity cannot be expected either; we have

$$\Phi(F, [0, \beta]) + \Phi(F, [\beta, 2]) = \emptyset \neq \Phi(F, [0, 2])$$

provided the sum of sets in $\mathbb{R}^n$ is defined as usual.

Let us present one more example that demonstrates that similar phenomena as in Example 2 can occur even if the Aumann integral is nonempty.

Example 3. Let M_1, M_2 be the same as above, and define $F : [0, 2] \to \mathbb{R}^2$ as follows:

$$F(t) = \begin{cases} \{(\ \ 1, 0);\ (0, 1)\} & \text{for } t \in M_1 , \\ \{(-1, 0);\ (0, 1)\} & \text{for } t \in M_2 , \\ \{(-1, 0);\ (1, 0)\} & \text{for } t \in [1, 2] . \end{cases}$$

Then

$$\Phi(F, [0, \beta]) = \{(0, \beta)\} \quad \text{for} \quad 0 \leqq \beta \leqq 1 ,$$

$$\Phi(F, [1, \beta]) = \{(x, 0) \mid x \in [-\beta + 1, \beta - 1]\} \quad \text{for} \quad 1 < \beta \leqq 2 ,$$

$$\Phi(F, [0, \beta]) = \{(x, y) \mid x \in [-y - \beta + 2, y + \beta - 2], y \in [2 - \beta, 1]\}$$
$$\text{for } 1 < \beta \leqq 2 .$$

In particular,

$$\Phi(F, [0, 1]) + \Phi(F, [1, 2]) = \{(x, 1) \mid x \in [-1, 1]\}$$

but

$$\Phi(F, [0, 2]) = \{(x, y) \mid x \in [-y, y], y \in [0, 1]\} .$$

The examples show that the crucial fact is that the "bad" behavior of the multifunction on part of the interval can be compensated by its "good" behavior on the rest of it (which is not possible if we consider the two parts separately). This suggests the way out, which will be followed in the next section.

3. THE INTEGRAL OF A MULTIFUNCTION: ALTERNATIVE DEFINITION

Definition 5. Let $T = [a, b]$, $F : T \to \mathscr{S}^n$. We define

$$\int_T F(t) \, \mathrm{d}t = \bigcap_D \sum_{j=1}^m \Phi(F, [\sigma_{j-1}, \sigma_j]) ;$$

the intersection is taken over all finite decompositions D of the interval T:

$$D = \{\sigma_0, \sigma_1, \ldots, \sigma_m\} ,$$

m positive integer, $a = \sigma_0 < \sigma_1 < \ldots < \sigma_m = b$ and the sum of sets in $\mathbb{R}^n$ is defined in the usual way.

Further, let $A \subset \mathbb{R}$ be bounded. Then for $F : A \to \mathscr{S}^n$ we define

$$\int_T F(t) \, \mathrm{d}t = \int_{T(A)} F_A(t) \, \mathrm{d}t ,$$

where $T(A)$ is a compact interval, $A \subset T(A)$, and $F_A : T(A) \to \mathscr{S}^n$ is defined by $F_A(t) = F(t)$ for $t \in A$, $F_A(t) = \{0\}$ otherwise.

Remarks. 3. In what follows, we shall use the letter $\mathscr{A}$ to specify that the integral involved is the Aumann integral; the integral sign without any additional symbol will stand for the integral according to the above definition. Notice that

$$(\mathscr{A}) \int_T F(t) \, \mathrm{d}t \subset \int_T F(t) \, \mathrm{d}t$$

without any assumptions on F.

4. Theorems 1, 2 imply that $\int_T F(t)\,\mathrm{d}t$ is convex and closed (compact if F is integrably bounded) and, moreover, $\int_T F(t)\,\mathrm{d}t = \int_T \operatorname{conv} F(t)\,\mathrm{d}t$. Since $F(t) \subset \operatorname{cl} F(t) \subset \operatorname{conv} F(t)$, we also have $\int_T F(t)\,\mathrm{d}t = \int_T \operatorname{cl} F(t)\,\mathrm{d}t$.

Theorem 4. *Let $F : A \to \mathscr{S}^n$ (A bounded and measurable). Then $\Theta(F)$ exists.*

Theorem 4 is due to Rzeżuchowski. It appears (in a slightly modified form) as Theorem 2 in [9].

Sketch of proof of Theorem 4. (For simplicity, let us assume $A = T$, T a compact interval.) Denote

$$Z = \{z \in \mathbb{R}^n \mid z \text{ has rational coordinates}\} = \{z_1, z_2, \ldots\}.$$

Further, let

$$\xi_i(t) = \inf\{r \in \mathbb{R} \mid F(t) \cap \bar{B}(z_i, r) \neq \emptyset\} \quad \text{if} \quad t \in T, \quad z_i \notin \operatorname{cl} F(t),$$

$$\xi_i(t) = -\infty \quad \text{if} \quad t \in T, \quad z_i \in \operatorname{cl} F(t).$$

Then there exists such a measurable function $\psi_i : T \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ that

(i) $\psi_i(t) \geqq \xi_i(t)$ for a.e. $t \in T$;
(ii) if $\vartheta : T \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is measurable and $\vartheta(t) \geqq \xi_i(t)$ for a.e. $t \in T$, then $\vartheta(t) \geqq \psi_i(t)$ for a.e. $t \in T$.

Denote

$$\begin{aligned} V_i(t) &= \mathbb{R}^n \setminus B(z_i, \psi_i(t)) && \text{if} \quad \psi_i(t) \in \mathbb{R}, \\ V_i(t) &= \mathbb{R}^n && \text{if} \quad \psi_i(t) = -\infty, \\ V_i(t) &= \emptyset && \text{if} \quad \psi_i(t) = \infty, \end{aligned}$$

$$V(t) = \bigcap_{i=1}^{\infty} V_i(t).$$

Then evidently $V(t) \subset \operatorname{cl} F(t)$ for a.e. $t \in T$. Let $Q : T \to \mathbb{R}^n$ be measurable, $Q(t) \subset \operatorname{cl} F(t)$ for a.e. $t \in T$. Then $Q(t) \cap B(z_i, \psi_i(t)) = \emptyset$ if $\psi_i(t) \in \mathbb{R}$ and the measurability of Q implies that $Q(t) \subset V_i(t)$ for a.e. $t \in T$, so that $Q(t) \subset V(t)$ for a.e. $t \in T$. We may put $\Theta(F) = V$.

Remarks. 5. Under the conditions of Theorem 4 we have

$$(\mathscr{A}) \int_A \operatorname{cl} F(t)\,\mathrm{d}t = (\mathscr{A}) \int_A \Theta(F)(t)\,\mathrm{d}t.$$

6. Let $F : A \to \mathscr{S}^n$ (A bounded and measurable) be measurable (i.e. $\{t \mid F(t) \cap E \neq \emptyset\}$ is measurable for every compact subset $E \subset \mathbb{R}^n$), $F(t)$ being bounded for $t \in A$. Then the functions ξ_i from the proof of Theorem 4 are measurable, so that $\psi_i = \xi_i$ for $i = 1, 2, \ldots$ and $\Theta(F) = \operatorname{cl} F$. Hence $\operatorname{cl} F$ is measurable.

7. Assume that F is measurable and integrably bounded. Then

$$(\mathscr{A}) \int_T \operatorname{cl} F(t) \, \mathrm{d}t = \Phi(F, T)$$

(cf. Remarks 6, 4 and Theorem 3). Consequently, if $T = [a, b]$, $a < c < b$, then by the properties of the $\mathscr{A}$-integral we have $\Phi(F, [a, c]) + \Phi(F, [c, b]) = \Phi(F, [a, b])$, which yields

$$\int_T F(t) \, \mathrm{d}t = \Phi(F, T).$$

The main aim of this section is to prove the following result:

Theorem 5. *Let $A \subset R$ be measurable and bounded. Let $F : A \to \mathscr{S}^n$ be integrably bounded. Put $M = \Theta(\operatorname{conv} F)$. Then*

$$\int_A M(t) \, \mathrm{d}t = \int_A F(t) \, \mathrm{d}t .$$

(Since M is measurable and integrably bounded, we have

$$(\mathscr{A}) \int_A M(t) \, \mathrm{d}t = \int_A M(t) \, \mathrm{d}t = \Phi(M_A, T(A)).)$$

Before proceeding to the proof we shall present several auxiliary results.

Let $F : A \to \mathscr{S}^n$, A measurable and bounded, let $F(t)$ be bounded for $t \in A$. In what follows we shall make use of the set P introduced in the proof of Theorem 3, that is, $P = \{p_1, p_2, \ldots\}$ is a countable dense subset of $\{x \in R^n \mid \|x\| = 1\}$. Then defining

$$\omega_i(t) = \sup \{(p_i, x) \mid x \in F(t)\}$$

we may write

$$\operatorname{conv} F(t) = \bigcap_{i=1}^{\infty} \{x \in R^n \mid (p_i, x) \leqq \omega_i(t)\} .$$

Further, for every i let λ_i be such an integrable function defined on A that

$$\lambda_i(t) \leqq \omega_i(t) ,$$

and if μ is measurable on A, $\mu(t) \leqq \omega_i(t)$ for a.e. $t \in A$, then $\mu(t) \leqq \lambda_i(t)$ for a.e. $t \in A$. (Observe that $\int_A \lambda_i(t) \, \mathrm{d}t = \int_A \omega_i(t) \, \mathrm{d}t$ – see Appendix.)

We define

(14) $$G(t) = \bigcap_{i=1}^{\infty} \{x \in R^n \mid (p_i, x) \leqq \lambda_i(t)\} .$$

Lemma 1. $G = \Theta(\operatorname{conv} F)$ *(i.e. $G(t) = \Theta(\operatorname{conv} F)(t)$ for a.e. $t \in A$).*

Proof. G is measurable, $G(t) \subset \operatorname{conv} F(t)$ for $t \in A$ and therefore $G(t) \subset \subset \Theta(\operatorname{conv} F)(t)$ for a.e. $t \in A$. On the other hand, put

$$v_i(t) = \sup\{(p_i, x) \mid x \in \Theta(\operatorname{conv} F)(t)\} ;$$

then v_i is measurable by the measurability of $\Theta(F)$ and $v_i(t) \leqq \omega_i(t)$ as $\Theta(\operatorname{conv} F)(t) \subset \subset \operatorname{conv} F(t)$ for $t \in A$. Hence $v_i(t) \leqq \lambda_i(t)$ for a.e. $t \in A$ and

$$\Theta(F)(t) \subset \bigcap_{i=1}^{\infty} \{x \in \mathbb{R}^n \mid (p_i, x) \leqq v_i(t)\} \subset G(t)$$

for a.e. $t \in A$.

From Lemma 1 and (14) we obtain

$$\Theta(\operatorname{conv} F)(t) = \bigcap_{i=1}^{\infty} \{x \in \mathbb{R}^n \mid (p_i, x) \leqq \lambda_i(t)\} \tag{15}$$

for a.e. $t \in A$.

Lemma 2. *Let $A \subset \mathbb{R}$ be measurable and bounded, $F : A \to \mathscr{S}^n$ integrably bounded with an integrable majorant ϱ, that is, $\varrho : A \to \mathbb{R}^+$, $\int_A \varrho(t)\,\mathrm{d}t < \infty$, $F(t) \subset \bar{B}(0, \varrho(t))$ for $t \in A$.*

Let there exist measurable disjoint sets A_1, A_2, a number $\varepsilon > 0$ and a convex compact set $Q \subset \mathbb{R}^n$ such that

(i) $A = A_1 \cup A_2$,

(ii) $t \in A_1 \Rightarrow h(M(t), Q) \leqq \varepsilon$

($M \equiv \Theta$ (conv F) is given by (15)).

Then

$$\Phi(F, A) \subset \bar{\Omega}\left(Q\, m(A_1); \varepsilon\, m(A_1) + \int_{A_2} \varrho(t)\,\mathrm{d}t\right),$$

where $\Phi(F, A) = \Phi(F_A, T(A))$ with F_A, $T(A)$ having the same meaning as in Definition 5.

Proor. Let us fix a positive integer k, a positive real $\eta > 0$ and $y \in \Phi(F, A)$. Then it follows from the identity $\int_A \omega_k(t)\,\mathrm{d}t = \int_A \lambda_k(t)\,\mathrm{d}t$ that the set

$$C = C_k = \{t \in A \mid \omega_\kappa(t) \geqq \lambda_k(t) + \eta\} \tag{16}$$

has inner measure zero: $m_i(C) = 0$.

In what follows, χ_S stands for the characteristic function of a set $S \subset \mathbb{R}$. For brevity, let us write $T(A) = T$.

Find a gauge δ such that for any partition

$$\Delta = \{(\tau_j, [t_{j-1}, t_j]),\ j = 1, \ldots, m;\ a = t_0 < t_1 < \ldots < t_m = b\}$$

of the interval T subordinate to δ the following inequalities hold:

(17) $$\left|\sum_{j=1}^{m} \chi_{A_i}(\tau_j)(t_j - t_{j-1}) - m(A_i)\right| < \eta, \quad i = 1, 2;$$

(18) $$\left|\sum_{j=1}^{m} \varrho(\tau_j)\chi_{A_i}(\tau_j)(t_j - t_{j-1}) - \int_{A_i} \varrho(t)\,\mathrm{d}t\right| < \eta, \quad i = 1, 2;$$

(19) $$d(y, S(F_A, \Delta)) < \eta.$$

(Existence of such a gauge δ is a consequence of the Riemann-type definition of integral.)

Now there is a partition Δ_0 subordinate to δ and such that

(20) $$\sum_{j,\tau_j \in C} \varrho(\tau_j)(t_j - t_{j-1}) < \eta;$$

indeed, since $m_i(C) = 0$ we have $\int_C \varrho(t)\,\mathrm{d}t = 0$ and in virtue of Lemma 4 (see Appendix) the sum on the left-hand side of (20) can be made as close to zero as required by choosing a suitable partition Δ_0.

The inequality (19) implies that there is $z \in S(F_A, \Delta_0)$ such that

$$|(p_k, y) - (p_k, z)| < \eta$$

(recall that $\|p_k\| = 1$) and hence

(21) $$(p_k, y) < (p_k, z) + \eta.$$

By definition there are $f_j \in F_A(\tau_j)$ such that

$$(p_k, z) = \sum_{j=1}^{m} (p_k, f_j)(t_j - t_{j-1}).$$

We divide the sum on the right-hand side into three parts according to whether $\tau_j \in A_1 \setminus C$, $\tau_j \in A_1 \cap C$ or $\tau_j \in A_2$. (In fact, there is still one more possibility, namely $\tau_j \in T \setminus A$; however, in that case $F_A(\tau_j) = \{0\}$ and the corresponding sum vanishes.) We estimate

$$\sum_{\tau_j \in A_1 \cap C} (p_k, f_j)(t_j - t_{j-1}) \leqq \sum_{\tau_j \in A_1 \cap C} \varrho(\tau_j)(t_j - t_{j-1}) < \eta$$

by (20),

$$\sum_{\tau_j \in A_2} (p_k, f_j)(t_j - t_{j-1}) \leqq \sum_{\tau_j \in A_2} \varrho(\tau_j)(t_j - t_{j-1}) < \eta + \int_{A_2} \varrho(t)\,\mathrm{d}t$$

by (18).

It remains to estimate the first sum corresponding to $\tau_j \in A_1 \setminus C$. Denote

$$q_k = \sup\{(p_k, x) \mid x \in Q\};$$

since $M(\tau_j) \subset \Omega(Q, \varepsilon)$ for $\tau_j \in A_1$, we have $\lambda_k(\tau_j) \leqq q_k + \varepsilon$. Then, by definition of A_1, C (cf. (16)) we have

$$\sum_{\tau_j \in A_1 \setminus C} (p_k, f_j)(t_j - t_{j-1}) \leqq \sum_{\tau_j \in A_1 \setminus C} \lambda_k(\tau_j)(t_j - t_{j-1}) + \eta\, m(A) \leqq$$

$$\leqq \sum_j \chi_{A_1}(\tau_j)\, \lambda_k(\tau_j)(t_j - t_{j-1}) + \sum_j \chi_{A_1 \cap C}(\tau_j)\, |\lambda_k(\tau_j)|\, (t_j - t_{j-1}) + \eta\, m(A) \leqq$$

$$\leqq (q_k + \varepsilon) \sum_j \chi_{A_1}(\tau_j)(t_j - t_{j-1}) + \sum_j \chi_C(\tau_j)\, \varrho(\tau_j)(t_j - t_{j-1}) + \eta\, m(A) \leqq$$

$$\leqq (q_k + \varepsilon)\, m(A_1) + |q_k + \varepsilon|\, \eta + \eta + \eta\, m(A) =$$

$$= (q_k + \varepsilon)\, m(A_1) + \eta(|q_k + \varepsilon| + 1 + m(A))$$

by (17), (20).

Combining the three estimates and (21) we obtain

$$(p_k, y) \leqq (q_k + \varepsilon)\, m(A_1) + \eta(|q_k + \varepsilon| + 4 + m(A)) + \int_{A_2} \varrho(t)\, dt .$$

Since $\eta > 0$ was arbitrary, this yields

$$(p_k, y) \leqq q_k\, m(A_1) + \varepsilon\, m(A_1) + \int_{A_2} \varrho(t)\, dt .$$

Using this inequality we complete the proof of Lemma 2 by way of contradiction.

Assume $d = d(y, m(A_1)\, Q) = \varepsilon\, m(A_1) + \int_{A_2} \varrho(t)\, dt + \zeta$, $\zeta > 0$ (recall that $y \in \Phi(F, A)$). Since Q is compact and convex, there is $x \in Q$ such that $\|x - y\| = d$ and at the same time

$$(x, p) = \sup \{(z, p) \mid z \in Q\} ,$$

$$p = \frac{y - x}{\|y - x\|} .$$

Given $\xi > 0$, there is a positive integer r such that $\|p - p_r\| < \xi$. Consequently,

$$d = \|y - x\| = \left(y - x, \frac{y - x}{\|y - x\|}\right) = (y - x, p) =$$

$$= (y, p_r) - (x, p) + (y, p - p_r) \leqq$$

$$\leqq \varepsilon\, m(A_1) + \int_{A_2} \varrho(t)\, dt + |q_r - q|\, m(A_1) + (y, p - p_r) ,$$

where $q = \sup \{(p, x) \mid x \in Q\}$. As Q is convex compact we can find ξ such that the last two summands contribute less than ζ. This contradicts the assumption on d. The proof of Lemma 2 is complete.

Lemma 3. *Let $A \subset \mathbb{R}$ be a measurable set, $M : A \to \mathcal{S}^n$ a measurable mapping with convex compact values. Let $\varepsilon > 0$. Then there are measurable pairwise disjoint sets $A_i \subset \mathbb{R}^n$, $i = 1, 2, \dots$ with $\bigcup_i A_i = A$, and convex compact polyhedrons Q_i such that the mapping L: $A \to \mathcal{S}^n$ defined by*

$$L(t) = Q_i \quad \text{for} \quad t \in A_i\,,$$

satisfies

$$h(M(t), L(t)) < \varepsilon \quad \text{for} \quad t \in A\,. \tag{22}$$

Proof. Let $\{r_k\}$ be the set of all points in $\mathbb{R}^n$ with rational coordinates. Then the sets

$$\operatorname{conv}\{r_{i_1}, \dots, r_{i_k}\}$$

form a countable set of convex compact polyhedrons $\{Q_i \mid i = 1, 2, \dots\}$. Denote

$$B_i = \{t \in A \mid h(M(t), Q_i) < \varepsilon\}\,,$$

$$A_i = B_i \setminus \bigcup_{k=1}^{i-1} B_k\,.$$

It is easy to verify that A_i, Q_i satisfy the assertion of Lemma 3.

Let us now proceed to the proof of Theorem 4. Let us assume that F is integrably bounded with a majorant ϱ. As $M(t) \subset \operatorname{conv} F(t)$ for a.e. $t \in A$, it follows from Theorem 2 that

$$\int_A M(t)\,\mathrm{d}t \subset \int_A F(t)\,\mathrm{d}t\,. \tag{23}$$

Thus, the proof reduces to that of the converse inclusion.

Let the symbols $\omega_i(t)$, $\lambda_i(t)$, $M(t)$ have the meaning introduced above (cf. (14), (15)); let ϱ be an integrable majorant of F (see Lemma 2), let A_i, Q_i, L be from Lemma 3 and let $\varepsilon > 0$.

Choose $\eta > 0$ so that ($T = T(A)$ has the meaning from Definition 5)

$$m(\Gamma) < \eta\, m(T) \Rightarrow \int_\Gamma \varrho(t)\,\mathrm{d}t < \varepsilon\,; \tag{24}$$

for this η let us find a positive integer ν such that

$$\sum_{j=\nu+1}^{\infty} m(A_j) < \eta\, m(T)\,. \tag{25}$$

We introduce the notation

$$H_i^+ = \{t \mid t \in A_i,\ t \text{ is not a point of metrical density of } A_i\}\,,$$
$$H_i^- = \{t \mid t \in T \setminus A_i,\ t \text{ is not a point of metrical density of } T \setminus A_i\}\,,$$

$$H = \bigcup_{i=1}^{\nu} (H_i^+ \cup H_i^-)\,.$$

Evidently

$$\mathscr{m}(H) = 0 .$$

We find a gauge δ such that for every $i \in \{1, \ldots, \nu\}$

(26) $\quad t \in A_i \setminus H , \quad t - \delta(t) \leqq s < t \Rightarrow \mathscr{m}(A_i \cap [s, t]) \geqq (1 - \eta)(t - s) ,$

(27) $\quad t \in A_i \setminus H , \quad t < s \leqq t + \delta(t) \Rightarrow \mathscr{m}(A_i \cap [t, s]) \geqq (1 - \eta)(s - t) .$

Moreover, δ is chosen so that for every partition Δ of T subordinate to δ

(28) $$\sum_j^{**} \int_{t_{j-1}}^{t_j} \varrho(t) \, \mathrm{d}t < \varepsilon ,$$

where the two stars indicate that the sum is taken over all j such that $\tau_j \in \bigcup_{i=\nu+1}^{\infty} A_i \cup H$. (This is possible due to the choice of ν and η – cf. (24), (25).)

Assume $y \in \int_A F(t) \, \mathrm{d}t$; taking the points

$$a = t_0 < t_1 < \ldots < t_m = b$$

belonging to a partition Δ subordinate to the gauge δ chosen above, we have by our definition of integral

$$y = z_1 + \ldots + z_m ,$$

$$z_j \in \Phi(F_A, [t_{j-1}, t_j]) .$$

Fix $j \in \{1, \ldots, m\}$. Then either there exists $i \in \{1, \ldots, \nu\}$ such that $\tau_j \in A_i \setminus H$ or $\tau_j \in \bigcup_{i=\nu+1}^{\infty} A_i \cup H$.

In the first case, we have by Lemma 2

$$\Phi(F_A, [t_{j-1}, t_j]) \subset \bar{\Omega}\left(Q_i \, \mathscr{m}(A_i \cap [t_{j-1}, t_j]) ; \quad \varepsilon(t_j - t_{j-1}) + \int_{T_{ji}} \varrho(t) \, \mathrm{d}t\right)$$

where $T_{ji} = [t_{j-1}, t_j] \setminus A_i$. By (22) we have

$$Q_i \, \mathscr{m}(A_i \cap [t_{j-1}, t_j]) \subset \Omega\left(\int_{A_i \cap [t_{j-1}, t_j]} M(t) \, \mathrm{d}t ; \; \varepsilon(t_j - t_{j-1})\right) ,$$

$$\int_{A_i \cap [t_{j-1}, t_j]} M(t) \, \mathrm{d}t \subset \bar{\Omega}\left(\int_{t_{j-1}}^{t_j} M(t) \, \mathrm{d}t ; \int_{T_{ji}} \varrho(t) \, \mathrm{d}t\right) .$$

This implies

$$Q_i \, \mathscr{m}(A_i \cap [t_{j-1}, t_j]) \subset \bar{\Omega}\left(\int_{t_{j-1}}^{t_j} M(t) \, \mathrm{d}t ; \; \varepsilon(t_j - t_{j-1}) + \int_{T_{ji}} \varrho(t) \, \mathrm{d}t\right) .$$

Hence we conclude

$$\Phi(F_A, [t_{j-1}, t_j]) \subset \bar{\Omega}\left(\int_{t_{j-1}}^{t_j} M(t) \, \mathrm{d}t ; \; 2\varepsilon(t_j - t_{j-1}) + 2 \int_{T_{ji}} \varrho(t) \, \mathrm{d}t\right) .$$

In the latter case (if $\tau_j \in \bigcup_{i=\nu+1}^{\infty} A_i \cup H$) we have

$$\Phi(F_A, [t_{j-1}, t_j]) \subset \bar{B}\left(0; \int_{t_{j-1}}^{t_j} \varrho(t)\, \mathrm{d}t\right).$$

Combining the last two inclusions, we conclude by (26)–(28)

$$y = z_1 + \ldots + z_m \in$$

$$\in \bar{\Omega}\left(\int_T M(t)\, \mathrm{d}t;\ 2\varepsilon\, m(T) + 2 \sum_{j,i}^{*} \int_{T_{ji}} \varrho(t)\, \mathrm{d}t + \sum^{**} \int_{t_{j-1}}^{t_j} \varrho(t)\, \mathrm{d}t\right),$$

where $\sum_{j,i}^{*}$ extends over such couples (j, i), $i = 1, \ldots, \nu$, $j = 1, \ldots, m$, that $\tau_j \in \in \bigcup_{i=1}^{\nu} A_i \setminus H$.

By (26), (27) we have

$$m(\bigcup_{j,i}^{*} T_{ji}) < \eta\, m(T)$$

and by (24)

$$\sum_{j,i}^{*} \int_{T_{ji}} \varrho(t)\, \mathrm{d}t < \varepsilon\,.$$

Making use of (28) we obtain that

$$y \in \bar{\Omega}\left(\int_T M(t)\, \mathrm{d}t;\ 2\varepsilon\, m(T) + 3\varepsilon\right).$$

Since $\varepsilon > 0$ was arbitrary, we conclude $y \in \int_T M(t)\, \mathrm{d}t$. However, $M(t) \subset \operatorname{conv} F_A(t) = \{0\}$ for $t \in T \setminus A$ so that $\int_T M(t)\, \mathrm{d}t = \int_A M(t)\, \mathrm{d}t$, which completes the proof of Theorem 5.

Remark 8. Using Theorem 5 we can give the following supplement to Remark 7: If $F : T \to \mathscr{K}_0^n$ is integrably bounded, then $(\mathscr{A}) \int_T F(t)\, \mathrm{d}t = \int_T F(t)\, \mathrm{d}t$. Indeed, for a given F we find M from Theorem 4. Since $F(t)$ is convex, we have $M(t) \subset F(t)$ and hence $\int_T F(t)\, \mathrm{d}t = \int_T M(t)\, \mathrm{d}t = (\mathscr{A}) \int_T M(t)\, \mathrm{d}t \subset (\mathscr{A}) \int_T F(t)\, \mathrm{d}t$. Since the converse inclusion is evident, our assertion is proved.

4. RELATION BETWEEN A DIFFERENTIAL RELATION AND THE CORRESPONDING "INTEGRAL RELATION"

Our definition of integral of a multivalued mapping together with Theorem 5 enables us to generalize the classical result on equivalence of a differential and the corresponding integral equation to differential relations.

If $F : T \times \mathbb{R}^n \to \mathscr{K}^n$ is a multivalued mapping (with nonempty convex compact values), then a function $x : I \to \mathbb{R}^n$, I a subinterval of T, is a solution of

$$\dot{x} \in F(t, x) \tag{29}$$

if it is absolutely continuous and

$$\dot{x}(t) \in F(t, x(t)) \quad \text{for a.e.} \quad t \in I\,.$$

Let us denote the set of all solutions of (29) by Sol F. On the other hand, let us denote by Int F the set of all functions $x : I \to \mathbb{R}^n$, $I \subset T$ an interval, such that for any $t \in I$, $t + h \in I$,

$$x(t + h) - x(t) \in \int_t^{t+h} F(\tau, x(\tau))\,\mathrm{d}\tau \tag{30}$$

holds.

Theorem 6. *If $F : T \times \mathbb{R}^n \to \mathscr{K}^n$ is integrably bounded (i.e. if there exists such an integrable function $\varrho: T \to [0, +\infty)$ that $F(t, x) \subset \bar{B}(0, \varrho(t))$ for $t \in T$, $x \in \mathbb{R}^n$), then*

$$\operatorname{Sol} F = \operatorname{Int} F\,. \tag{31}$$

Proof. (i) Sol $F \subset$ Int F. Let $x : I \subset T \to \mathbb{R}^n$, $x \in$ Sol F. Then $\dot{x}$ exists a.e. in I and it is a measurable selection in $F(.\,, x(.))$. Hence

$$x(t + h) - x(t) = \int_t^{t+h} \dot{x}(\tau)\,\mathrm{d}\tau$$

for t, $t + h \in I$ and, using the Riemann-type definition of the Lebesgue integral of a real function (cf. Remark 1) we easily obtain (30).

(ii) Sol $F \supset$ Int F. Let $x : I \subset T \to \mathbb{R}^n$, $x \in$ Int F. Denote by $F_x : I \to \mathscr{K}^n$ the mapping defined by $F_x(t) = F(t, x(t))$ for $t \in I$. Let $Q = \Theta(F_x)$.

Obviously, we have $x \in$ Int Q. Let us assume $x \notin$ Sol Q. Since F and hence also Q is integrably bounded, x is absolutely continuous, $\dot{x}(t)$ exists for a.e. $t \in I$ and $|\dot{x}(t)| \leqq$ $\leqq \varrho(t)$, where ϱ is integrable. Denote again

$$\lambda_i(t) = \sup \{(p_i, x) \mid x \in Q(t)\}\,,$$

$$\Lambda_i(a, b) = \int_a^b \lambda_i(t)\,\mathrm{d}t$$

and notice that

$$\Lambda_i(a, b) = \sup \left\{(p_i, z) \mid z \in \int_a^b Q(t)\,\mathrm{d}t\right\}.$$

(Recall that by Theorem 3 the integral on the right-hand side equals the Aumann integral, which implies the identity.) The assumption $x \notin$ Sol Q implies that there is a set $M \subset I$, $m(M) > 0$, such that for every $t \in M$ there is a positive integer $j = j(t)$ such that

$$(p_j, \dot{x}(t)) > \lambda_j(t).$$

Consequently, there is a positive integer r such that

$$(p_r, \dot{x}(t)) > \lambda_r(t)$$

on a set of positive measure. Consequently, there exist $t_0 \in I$ and $h > 0$ such that

$$\int_{t_0}^{t_0+h} (p_r, \dot{x}(t))\,\mathrm{d}t > \int_{t_0}^{t_0+h} \lambda_r(t)\,\mathrm{d}t = \Lambda_r(t_0, t_0 + h),$$

which contradicts the assumption $x \in \mathrm{Int}\, Q$. Hence $x \in \mathrm{Sol}\, Q$ and consequently $x \in \mathrm{Sol}\, F$, which completes the proof.

APPENDIX: RIEMANN-TYPE DEFINITION OF LOWER INTEGRAL

Let us first recall some well-known facts.

Definition 6. Let $\omega : T \to \mathbb{R}$. The numbers

$$\overline{\int_T} \omega(t)\,\mathrm{d}t = \inf \left\{ \int_T \lambda(t)\,\mathrm{d}t \mid \lambda \text{ integrable, } \lambda(t) \geqq \omega(t) \text{ for } t \in T \right\},$$

$$\underline{\int_T} \omega(t)\,\mathrm{d}t = \sup \left\{ \int_T \lambda(t)\,\mathrm{d}t \mid \lambda \text{ integrable, } \lambda(t) \leqq \omega(t) \text{ for } t \in T \right\}$$

are called *the upper and the lower integral of ω over T*, respectively.

Remark 9. It is easily seen that for every $\omega : T \to \mathbb{R}$ integrably bounded there exist integrable functions $\lambda_0, \lambda^0 : T \to \mathbb{R}$, $\lambda_0(t) \leqq \omega(t) \leqq \lambda^0(t)$ for $t \in T$, such that

$$\overline{\int_I} \omega(t)\,\mathrm{d}t = \int_I \lambda^0(t)\,\mathrm{d}t, \quad \underline{\int_I} \omega(t)\,\mathrm{d}t = \int_I \lambda_0(t)\,\mathrm{d}t$$

for any interval $I \subset T$; the functions λ_0, λ^0 are uniquely determined (up to sets of zero measure).

The following theorem provides an equivalent definition of the lower integral, based on Riemann sums. We use the symbol $S(f, \Delta)$ for real-valued functions in the sense of Remark 1, viewing f as a set-valued map with one-point values.

Lemma 4. *Let $\omega : T \to \mathbb{R}$ be integrably bounded. Put*

$$S = \{ s \in \mathbb{R} \mid \forall \text{ gauge } \delta\ \exists \text{ partition } \Delta \text{ sub } \delta : S(\omega, \Delta) \leqq s \}.$$

Then

$$\int_T \omega(t)\,\mathrm{d}t = \inf S\,.$$

Proof. Denote $I(\omega) = \int_T \omega(t)\,\mathrm{d}t$.

(i) $I(\omega) \leqq \inf S$. Let $\varepsilon > 0$ and $s \in S$. Let λ_0 be the function from Remark 9. Then there is a gauge δ such that for every partition Δ subordinate to δ the inequality

$$\left|S(\lambda_0, \Delta) - \int_T \lambda_0(t)\,\mathrm{d}t\right| < \varepsilon$$

holds (cf. Remark 1). Since $s \in S$, we find (for δ just found) a partition Δ_0 sub δ such that

$$S(\omega, \Delta_0) \leqq s\,.$$

Recalling the properties of λ_0, we conclude that

$$s \geqq S(\omega, \Delta_0) \geqq S(\lambda_0, \Delta_0) \geqq \int_T \lambda_0(t)\,\mathrm{d}t - \varepsilon = I(\omega) - \varepsilon\,.$$

Since both s and ε were arbitrary, we complete the proof of the desired inequality by a standard argument.

(ii) $I(\omega) \geqq \inf S$. Let again λ_0 be the function from Remark 9 and set

$$N = \{t \in T \mid \omega(t) > \lambda_0(t)\}\,.$$

Then evidently $m_i(N) = 0$, where m_i is the inner measure of a set.

Let $\varepsilon > 0$, let δ be a gauge on T. Our aim is now to find a partition Δ_1 subordinate to δ and such that the inequality $S(\omega, \Delta_1) \leqq I(\omega) + \varepsilon$ holds, which implies $I(\omega) + {} + \varepsilon \in S$. Since $\int_T \lambda_0(t)\,\mathrm{d}t = I(\omega)$, we find a gauge δ_0 such that

$$S(\lambda_0, \Delta) \leqq I(\omega) + \tfrac{1}{2}\varepsilon \tag{32}$$

for every partition Δ subordinate to δ_0. Put $\delta_1(t) = \min(\delta(t), \delta_0(t))$ for $t \in T$. Then (32) holds for all partitions Δ subordinate to δ_1.

Denote $Q = T \setminus N$; then $m_e(Q) = m(T)$, where m_e is the outer measure of a set. Consider the family of intervals

$$\{[t - \xi,\ t + \xi] \mid t \in Q,\ 0 < \xi < \delta_1(t)\}\,. \tag{33}$$

This family is a covering of Q in the sense of Vitali, hence there exists a disjoint countable subfamily

$$\{[\sigma_j - \xi_j,\ \sigma_j + \xi_j],\ j = 1, 2, \ldots\}$$

such that

$$m(Q \setminus \bigcup_{j=1}^{\infty} [\sigma_j - \xi_j,\ \sigma_j + \xi_j]) = 0\,.$$

Let $\varepsilon_1 > 0$. Then there is a finite number of (disjoint) intervals $J_j = [\sigma_{k_j} - \xi_{k_j}, \sigma_{k_j} + \xi_{k_j}]$, $j = 1, \ldots, r$, such that

$$m(\bigcup_{j=1}^{r} J_j) \geqq m_e(Q) - \varepsilon_1 = m(T) - \varepsilon_1\,. \tag{34}$$

Now we construct a partition Δ_1 such that

(a) all pairs (σ_{k_j}, J_j), $j = 1, \ldots, r$, belong to Δ_1;

(b) Δ_1 is subordinate to δ_1.

Since $0 < \xi_j \lessdot \delta_1(\sigma_j)$ (cf. (33), such a partition exists according to [5], Lemma 3.20. (Roughly speaking, this lemma asserts that any "sub-partition compatible with a gauge δ" can be completed to obtain a partition subordinate to δ.)

Let us estimate the Riemann sum corresponding to the partition Δ_1 and the function ω. We have

$$(35) \qquad S(\omega, \Delta_1) = S(\lambda_0, \Delta_1) + S(\omega - \lambda_0, \Delta_1)\,.$$

The first right-hand side term is estimated by (32). As concerns the second term, we may omit all summands corresponding to the pairs (σ_j, J_j) since $\sigma_j \in Q$ and hence $\omega(\sigma_j) = \lambda_0(\sigma_j)$. Thus

$$(36) \quad S(\omega - \lambda_0, \Delta_1) \leqq m(T \setminus \bigcup_{j=1}^{r} J_j) \sup_{t\in T} (\omega(t) - \lambda_0(t)) \leqq \varepsilon_1 \sup_{t\in T} (\omega(t) - \lambda_0(t))\,.$$

If $\sup_{t\in T} (\omega(t) - \lambda_0(t)) = d < +\infty$, we complete the proof by a standard argument putting $\varepsilon_1 = \frac{1}{2}\varepsilon d^{-1}$ and combining (35) with (32), (36), which eventually yields $I(\omega) + \varepsilon \in S$ with $\varepsilon > 0$ arbitrary.

If $\sup_{t\in T} (\omega(t) - \lambda_0(t)) = +\infty$, we have to use the fact that ω is integrably bounded. If the integrable bound is ϱ, we find a set $M \subset T$ such that $\int_M \varrho(t)\,dt$ is sufficiently small while $\varrho(t) \leqq$ const for $t \in T \setminus M$. Then the second right-hand side term in (35) splits into two summands; the one corresponding to the set $T \setminus M$ is dealt with as above, while the other is small due to the choice of the set M.

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