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TIME-PERIODIC SOLUTIONS OF TELEGRAPH EQUATIONS IN n SPATIAL VARIABLES

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1. INTRODUCTION

The aim of this paper is to extend to the case of n spatial variables a result of P. H. Rabinowitz [3] on the existence of classical solutions of the equation

$$u_{tt} - u_{xx} + \alpha u_t + \varepsilon g(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0\,, \quad x \in (0, \pi) \tag{1.1}$$

with the boundary conditions

$$u(t, 0) = u(t, \pi) = 0 \tag{1.2}$$

and the periodicity conditions

$$u(t + \omega, x) = u(t, x)\,. \tag{1.3}$$

Under the assumption that g is a sufficiently smooth function, periodic in t with period ω, $\alpha > 0$ and ε is close to zero, P. H. Rabinowitz proved that there is a classical solution to (1.1)–(1.3).

The classical Newton method cannot be used for proving this result since the "loss of derivatives" appears as a consequence of the presence of the second order derivatives in the composition operator $g(t, x, u, \ldots, u_{xx})$. To overcome this difficulty P. H. Rabinowitz applied a Moser theorem [2] in which the existence of an approximate solution to the linearized equation is required. This means that a viscosity term is added to the linearized equation and the exact solution of the equation so obtained stands for the approximate solution of the linearized equation.

This approach is also applied in [5] to the equation

$$u_{tt} + u_{xxxx} + \alpha u_t = \varepsilon\, G(u)$$

and in a more general form in [6] to the equations of the type

$$u_{tt} + (-1)^p\, u_{x^{2p}} + \alpha u_t - u = \varepsilon\, G(u)$$

where $G(u) = g(t, x, u, u_t, u_x, \ldots, u_{x^p}, u_{tx}, u_{tt})$ or $G(u) = g(t, x, u, u_t, u_x, \ldots, u_{x^p}, u_{tx}, u_{x^{2p}})$.

We are going to extend the result of [3] to the case of n spatial variables. The elliptic operator occurring in the equation is changed accordingly. We shall apply the theorem from [1] which requires to solve the linearized equation only. As the "loss of derivatives" occurs in the t-variable we shall use a sequence of spaces whose elements improve their differentiability properties in the variable t. Since all the functions are periodic in t it is easy to introduce the truncation operator from [1] by means of truncated Fourier series.

We shall suppose that $\Omega \subset R^n$ is a bounded open domain with a smooth boundary. We set

$$Q = [0, 2\pi] \times \Omega ,$$

and from now on we shall suppose that all functions occurring in the problem are real-valued and periodic in t with the period 2π. Other periods than 2π can be treated similarly.

The following notation will be used. For $\alpha = (\alpha_1, \ldots, \alpha_n)$ we shall denote $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$. Similarly, $D_t^j = \partial^j/\partial t^j$. For $\gamma = (\gamma_0, \gamma')$, $\gamma' = (\gamma_1, \ldots, \gamma_n)$, we put $|\gamma| = \gamma_0 + |\gamma'|$ and $D_{t,x}^\gamma = D_t^{\gamma_0} D_x^{\gamma'}$.

By $H^p(Q)$, p a positive integer, we denote the space of all functions v periodic in t with the period 2π with $D_{t,x}^\gamma v \in L^2(Q)$ for $|\gamma| \leqq p$. We set

$$\|v\|_p = \max \{\|D_{t,x}^\gamma v\|;\ |\gamma| \leqq p\}$$

where

$$\|v\| = \left(\int_Q v^2(t, x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} .$$

By

$$G(u) = g(t, x, u, u_t, u_{tt}, \nabla u, \nabla u_t, \nabla\nabla u)$$

we denote the composition operator containing all derivatives $D_{t,x}^\gamma u$, $|\gamma| \leqq 2$. The function g is supposed to be smooth and periodic in t with the period 2π on the set $R \times \Omega \times \mathcal{O}$, where $\mathcal{O}$ is a neighbourhood of zero in $R^\varkappa$, $\varkappa = 3(n + 1) + n(n - 1)/2$. Further let us suppose that we are given an operator A by

$$(1.4) \qquad (Au)(x) = \sum_{|\alpha| \leqq 1, |\beta| \leqq 1} (-1)^\beta D_x^\beta (A_{\alpha\beta}(x) D_x^\alpha u(x)) ,$$

where the functions $A_{\alpha\beta}$ are smooth functions on $\bar{\Omega}$ satisfying

$$A_{\alpha\beta} = A_{\beta\alpha} \quad \text{for} \quad |\alpha| = |\beta| = 1 .$$

Finally, let $d(x)$ be a smooth function on $\bar{\Omega}$ satisfying

$$(1.5) \qquad d(x) \geqq d_0 > 0 \quad \text{for} \quad x \in \Omega .$$

We shall deal with the problem given by the equation

$$(1.6) \qquad u_{tt} + d(x)\, u_t + Au + \varepsilon\, G(u) = 0$$

and the boundary and periodicity conditions

$$u(t, x) = 0 \quad \text{for} \quad t \in R\,, \quad x \in \partial\Omega\,, \tag{1.7}$$

$$u(t + 2\pi, x) = u(t, x) \quad \text{for} \quad t \in R\,, \quad x \in \Omega\,. \tag{1.8}$$

To specify the assumptions under which this problem is to be examined we denote by A_0 the bilinear form associated with the operator A, namely,

$$A_0(v, \varphi) = \sum_{|\alpha| \leq 1, |\beta| \leq 1} \langle A_{\alpha\beta}(x)\, D_x^\alpha v,\, D_x^\beta \varphi \rangle\,, \tag{1.9}$$

where

$$\langle v, \varphi \rangle = \int_Q v(t, x)\, \varphi(t, x)\, \mathrm{d}t\, \mathrm{d}x\,,$$

i.e., $\|v\| = \langle v, v \rangle^{1/2}$. Further, let B_0 be the bilinear form associated with the linear part of the equation (1.6),

$$B_0(v, \varphi) = \langle v_{tt} + d(x)\, v_t, \varphi \rangle + A_0(v, \varphi)\,. \tag{1.10}$$

Denoting

$$\Lambda v = 2v_t + d_0 v\,, \tag{1.11}$$

we shall suppose that

$$B_0(v, \Lambda v) \geqq d_1 \|v\|_1^2\,, \quad d_1 > 0\,, \tag{1.12}$$

holds for all v such that $v, v_t \in H^1(Q)$ and $v(t, \cdot) = 0$ on $\partial\Omega$.

We shall prove the following result.

Theorem 1.1. *Let d, $A_{\alpha\beta}$ and g be sufficiently smooth functions satisfying the assumptions listed above. Then for every ε close to zero there exists a classical solution to* (1.6)–(1.8).

In fact, we will show that the solution is much smoother. The assumptions of regularity of the functions d, $A_{\alpha\beta}$, g allow to satisfy high regularity demands of the theorem from [1]. The assumption that $|\varepsilon|$ is small means that $u = 0$ is "close" to a solution of (1.6)–(1.8). The hypothesis (1.12) suggests that the problem is "nonresonant" and can easily be shown to be satisfied for some equations. For example, for the equation of the form

$$u_{tt} + \alpha u_t - \Delta u + \varepsilon g(t, x, u, \ldots, \nabla\nabla u) = 0$$

with (1.7) and (1.8), which is a direct generalization of (1.1)–(1.3), we have

$$B_0(v, \varphi) = \langle v_{tt} + \alpha v_t, \varphi \rangle + \sum_{j=1}^{n} \langle v_{x_j}, \varphi_{x_j} \rangle\,.$$

Then

$$B_0(v, 2v_t) = 2\, \alpha \|v_t\|^2$$

as a consequence of the 2π-periodicity in t and, similarly,

$$B_0(v, \alpha v) = -\|v_t\|^2 + \alpha \sum_{j=1}^{n} \|v_{x_j}\|^2 .$$

Hence

$$B_0(v, 2v_t + \alpha v) = \alpha(\|v_t\|^2 + \sum_{j=1}^{n} \|v_{x_j}\|^2) ,$$

which shows that (1.12) is satisfied with $\Lambda v = 2v_t + \alpha v$.

In Section 2 we give Moser's theorem. Spaces are introduced in Section 3. The Moser theorem is applied in Section 4. The linearized equation is solved in Section 5. Section 6 contains the proofs of three auxiliary assertions used in Section 5.

2. MOSER'S THEOREM

In this section a slightly changed version of the theorem from [1] is given. Let us suppose that we are given two sequences of Banach spaces U_N, F_N, and a sequence of operators T_N, $N = 1, 2, \ldots$, such that

(2.1) $$U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots , \quad F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots ,$$

(2.2) $$\|T_N u\|_{U_{r+s}} \leqq a N^{s+\delta} \|u\|_{U_r} , \quad r, s \geqq 0 ,$$

(2.3) $$\|(\mathrm{id} - T_N)\, u\|_{U_r} \leqq a N^{-s+\delta} \|u\|_{U_{r+s}} , \quad r \geqq 0 , \quad s \geqq \delta$$

with constants $a > 0$ and $\delta \geqq 0$. Further, let r, ϱ and σ be nonnegative integers, $r \geqq \sigma$. Let us suppose that for a nonnegative integer k depending on r, ϱ, σ and δ, the mapping f satisfies the following hypotheses with a constant $\alpha > 0$:

(2.4) For any $u \in U_r$, $\|u\|_{U_r} \leqq \alpha^{-1}$,

(i) $f(u) \in F_\varrho$,

(ii) $f'(u)$ is linear and bounded from U_r into F_ϱ.

(2.5) For any $u, v \in U_r$, $\|u\|_{U_r} \leqq \alpha^{-1}$, $\|u + v\|_{U_r} \leqq \alpha^{-1}$,

we have

$$\|f(u + v) - f(u) - f'(u)\, v\|_{F_\varrho} \leqq \alpha \|v\|^2_{U_r} .$$

(2.6) If $u \in U_{r+k}$ and $N \geqq 1$ satisfy

$\|u\|_{U_{r+\lambda}} \leqq \alpha^{-1} N^\lambda$ for $\lambda = 0, 1, \ldots, k$,

then

$\|f(u)\|_{F_{\varrho+\lambda}} \leqq \alpha N^\lambda$ for $\lambda = 0, 1, \ldots, k$.

(2.7) If $u \in U_{r+k}$, $h \in F_{\varrho+k}$ and $N \geqq 1$ satisfy

$\|u\|_{U_{r+\lambda}} \leqq \alpha^{-1}N^{\lambda}$, $\|h\|_{F_{\varrho+\lambda}} \leqq \alpha N^{\lambda}$ for $\lambda = 0, 1, \ldots, k$,

then the equation

$f'(u)\, v = h$

has a solution $v \in U_{r-\sigma+k}$ satisfying

(i) $\|v\|_{U_{r-\sigma}} \leqq \alpha^2 \|h\|_{F_\varrho}$,

(ii) $\|v\|_{U_{r-\sigma+\lambda}} \leqq \alpha^2 N^{\lambda}$ for $\lambda = 0, 1, \ldots, k$.

Theorem 2.1. *Under the above hypotheses there is $\eta > 0$ such that the equation $f(u) = 0$ has a solution in $\{u;\ \|u\|_{U_r} \leqq \alpha^{-1}\}$ provided $\|f(0)\|_{F_\varrho} < \eta$.*

3. SPACES

We set $n_0 = [\frac{1}{2}(n + 1)] + 1$, $n_1 = 2n_0 - 1$ and $m = n_1 + 2$. All function spaces mentioned below are spaces of real-valued functions periodic in t with the period 2π. To include the boundary condition (1.7) into the spaces we set

$$B = \{u \in H^1(Q);\ u(t, \cdot) = 0 \text{ on } \partial\Omega\}.$$

By the same symbol we denote also the space

$$B = \{u \in H^1(\Omega);\ u = 0 \text{ on } \partial\Omega\}.$$

Further, we denote

$$U_p = \{u \in B;\ u_{t^q} \in H^m(Q),\ q = 0, \ldots, p\},$$
$$F_p = \{u;\ u_{t^q} \in H^{n_1}(Q),\ q = 0, \ldots, p\}$$

with the norms

$$\|u\|_{U_p} = \max\{\|u_{t^q}\|_m;\ q = 0, \ldots, p\},$$
$$\|u\|_{F_p} = \max\{\|u_{t^q}\|_{n_1};\ q = 0, \ldots, p\}.$$

Here $H^1(Q)$ has its usual meaning in which it has been used in Section 1.

For a positive integer j we denote $e_j(t) = (1/\pi) \sin jt$ and $e_{-j}(t) = (1/\pi) \cos jt$. For $j = 0$ we set $e_0(t) = 1/2\pi$. Then every function $u \in U_p$ can be written in the form

$$u = \sum_{j \in Z} u_j(x)\, e_j(t)$$

with

$$\|u\|^2_{U_p} = \max\{\sum_{j \in Z} |j|^{2s}\, \|D^\alpha_x u_j\|^2_{L^2(\Omega)};\ |s| + |\alpha| \leqq m + p,\ |\alpha| \leqq m\}.$$

Hence, setting, for a positive N,

$$(T_N u)(t, x) = \sum_{|j| \leqq N} u_j(x)\, e_j(t), \tag{3.1}$$

we have $T_N : U_r \to U_{r+s}$ and

(3.2) $$\|T_N u\|_{U_{r+s}} \leqq N^s \|u\|_{U_r},$$

(3.3) $$\|(\mathrm{id} - T_N)\, u\|_{U_r} \leqq N^{-s} \|u\|_{U_{r+s}}.$$

Further, for a positive integer M, we denote

$$Z_M = \{u;\ u = \sum_{|j| \leqq M} u_j(x)\, e_j(t)\}.$$

For nonnegative integers p and s we put

$$X^{s,p} = \{u;\ D_t^l D_x^\alpha u \in L^2(Q),\ l + |\alpha| \leqq s,\ |\alpha| \leqq p\}$$

with

$$\|u\|_{X^{s,p}} = \max\{\|D_t^l D_x^\alpha u\|;\ l + |\alpha| \leqq s,\ |\alpha| \leqq p\}$$

and

$$H^{0,p} = \{u;\ D_x^\alpha u \in L^2(Q),\ |\alpha| \leqq p\}$$

with

$$\|u\|_{H^{0,p}} = \max\{\|D_x^\alpha u\|;\ |\alpha| \leqq p\}.$$

The norm in $H^p(\Omega)$ will be given by

$$\|v\|_{H^p(\Omega)} = \max\{\|D_x^\gamma v\|_{L^2(\Omega)};\ |\alpha| \leqq p\}.$$

Thus if $v = \sum v_j(x)\, e_j(t) \in H^{0,p}(Q)$, then

$$\|v\|_{H^{0,p}(Q)} = (\sum_j \|v_j\|^2_{H^p(\Omega)})^{1/2}.$$

Finally, we denote by $C^m(Q)$ the space of functions on $\bar{Q}$ having continuous derivatives on $\bar{Q}$ up to the order m. For $u \in H^{m+n_0}(Q)$ we have, by Sobolev's embedding theorem,

$$\|u\|_{C^m(Q)} \leqq c_s \|u\|_{H^{m+n_0}(Q)}.$$

4. APPLICATION OF MOSER'S THEOREM

The spaces U_N, F_N and the operators T_N defined in Section 3 satisfy (2.1)–(2.3) with $\delta = 0$ and $a = 1$. If $u \in U_0$ is such that $\|u\|_{U_0}$ is sufficiently small, then the composition operator $g(t, x, u, u_t, \ldots, \nabla\nabla u)$ is well-defined and we can set

$$f_\varepsilon(u) = u_{tt} + d(x)\, u_t + Au + \varepsilon g(t, x, u, u_t, u_{tt}, \nabla u, \nabla u_t, \nabla\nabla u).$$

We take $r = \varrho = \sigma = 2$ and we shall apply Theorem 2.1 to the mapping f_ε. Obviously, for $|\gamma| \leqq 2$, $D^\gamma_{t,x}$ is a linear and bounded mapping of $U_{2+\lambda}$ into $F_{2+\lambda}$. Reasoning as in Lemma 4.2 below it is easy to show that f_ε satisfies the hypotheses (2.4)–(2.6) of Theorem 2.1.

Lemma 4.1. *If $u, v \in H^{n_1}(Q)$, then $uv \in H^{n_1}(Q)$ and $\|uv\|_{n_1} \leqq c\|u\|_{n_1} \|v\|_{n_1}$.*

Proof. This holds even for functions from $H^{n_0}(Q)$, [2]. In the case of $H^{n_1}(Q)$ the proof is particularly simple. We have to estimate $\|(D^{\gamma_1}_{t,x}u)\, D^{\gamma_2}_{t,x}v\|$ for $|\gamma_1| + |\gamma_2| \leqq n_1$. At least one of the indices γ_i, say γ_1, has to satisfy $|\gamma_1| \leqq n_0 - 1$. Then $\|(D^{\gamma_1}_{t,x}u)\, D^{\gamma_2}_{t,x}v\| \leqq \|D^{\gamma_1}_{t,x}u\|_{C(Q)}\, \|D^{\gamma_2}_{t,x}v\| \leqq c_s\|u\|_{\gamma_1+n_0}\, \|v\|_{n_1} \leqq c_s\|u\|_{n_1}\, \|v\|_{n_1}$. This completes the proof.

Lemma 4.2. *Let $g(t, x, \zeta_1, \ldots, \zeta_\varkappa)$ be a smooth function on $R \times \Omega \times \mathcal{O}$, where $\mathcal{O}$ is a neighbourhood of zero in $R^\varkappa$. Let β be such that*

$$G(w)\,(t, x) = g(t, x, w_1(t, x), \ldots, w_\varkappa(t, x))$$

is well defined on $\mathcal{M} = \{w = (w_1, \ldots, w_\varkappa);\ \|w_j\|_{F_2} < \beta,\ j = 1, \ldots, \varkappa\}$. Then

(i) *G is a continuous mapping of $\mathcal{M}$ into F_2;*

(ii) *if for a fixed positive integer k, $\|w_j\|_{F_{2+\lambda}} \leqq \beta N^\lambda$ for $j = 1, \ldots, \varkappa$ and $\lambda = 0, 1, \ldots, k$, then $\|G(w)\|_{F_{2+\lambda}} \leqq bN^\lambda$ for $\lambda = 0, 1, \ldots, k$.*

Proof. We shall deal only with the case (ii) since (i) is similar. For $l \leqq 2 + \lambda$ we shall estimate $D^l_t\, G(w)$ in $H^{n_1}(Q)$. This means to estimate in $H^{n_1}(Q)$

$$(\mathcal{D}g)\,(w) \prod_{p=1}^{\varkappa} \prod_{s=1}^{l} (D^s_t w_p)^{\alpha_{sp}}$$

with nonnegative integers α_{sp} satisfying

$$\sum_{s=1}^{l} \sum_{p=1}^{\varkappa} s\alpha_{sp} \leqq l\,. \tag{4.1}$$

Here $\mathcal{D}g$ denotes a certain derivative of g whose order and form need not be specified and which satisfies $\|\mathcal{D}\, g(w)\|_{n_1} \leqq c$. By assumption, $\|D^s_t w_p\|_{n_1} \leqq cN^{(s-2)^+}$, where $j^+ = \max(0, j)$. Hence the estimate of $D^l_t\, G(w)$ in $H^{n_1}(Q)$ is cN^χ, where $\chi \leqq \leqq \sum_{s=1}^{l} \sum_{p=1}^{\varkappa} \alpha_{sp}(s - 2)^+$. In virtue of (4.1), $\chi \leqq (l - 2)^+ \leqq \lambda$. This completes the proof.

To show that f_ε satisfies the assumption (2.7) we shall deal with the operator $f'_\varepsilon(u)$ which has the form

$$f'_\varepsilon(u)\, v = v_{tt} + d(x)\, v_t + Av +$$
$$+ \varepsilon \sum_{|\alpha|,|\beta| \leqq 1} (-1)^{|\beta|}\, a_{\alpha\beta}(t, x)\, D^\beta_x D^\alpha_x v + \varepsilon \sum_{|\alpha| \leqq 1} a_\alpha(t, x)\, D^\alpha_x v_t + \varepsilon \bar{a}(t, x)\, v_{tt}\,,$$

where

$$a_{\alpha\beta}(t, x) = -\frac{\partial g}{\partial(D^{\alpha+\beta}_x u)}\,(t, x, u, \ldots, \nabla\nabla u) \quad \text{for} \quad \alpha = \beta\,, \quad |\alpha| = |\beta| = 1\,,$$

$$a_{\alpha\beta}(t, x) = -\frac{1}{2}\,\frac{\partial g}{\partial(D^{\alpha+\beta}_x u)}\,(t, x, u, \ldots, \nabla\nabla u) \quad \text{for} \quad \alpha \neq \beta\,, \quad |\alpha| = |\beta| = 1$$

(this means $a_{\alpha\beta} = a_{\beta\alpha}$ for $|\alpha| = |\beta| = 1$),

$$a_{\alpha\beta}(t, x) = \frac{\partial g}{\partial(D_x^\alpha u)}(t, \ldots, \nabla\nabla u) \quad \text{for} \quad |\alpha| \leqq 1\,, \quad \beta = 0\,,$$

$$a_{\alpha\beta}(t, x) = 0 \quad \text{for} \quad \alpha = 0\,, \quad |\beta| = 1\,,$$

$$a_\alpha(t, x) = \frac{\partial g}{\partial(D_x^\alpha u_t)}(t, \ldots, \nabla\nabla u)\,, \quad \bar{a}(t, x) = \frac{\partial g}{\partial(u_{tt})}(t, \ldots, \nabla\nabla u)\,.$$

By Lemma 4.2, if $u \in U_{2+k}$ and $\|u\|_{U_{2+\lambda}} \leqq \alpha N^\lambda$ for $\lambda = 0, 1, \ldots. k$, then $a \in F_{2+k}$ and

$$\|a\|_{F_{2+\lambda}} \leqq bN^\lambda \quad \text{for} \quad \lambda = 0, 1, \ldots, k\,, \tag{4.2}$$

where b is independent of N. Here, we have shorten the notation writting a instead of $a_{\alpha\beta}$, a_α or $\bar{a}$. This convention will be used throughout the paper. Further, let $h \in F_{2+k}$ satisfy

$$\|h\|_{F_{2+\lambda}} \leqq \alpha N^\lambda\,, \quad \lambda = 0, 1, \ldots, k\,. \tag{4.3}$$

In the next section we shall give the proof of the following lemma.

Lemma 4.3. *There are positive constants ε_0 and c (independent of N) such that the following implication holds:*

If a and h satisfy (4.2) *and* (4.3), *respectively, then for every ε, $|\varepsilon| \leqq \varepsilon_0$, there is a unique $v \in U_k$ satisfying $f_\varepsilon'(u)\, v = h$. Moreover,*

$$\|v\|_{U_0} \leqq c\|h\|_{F_2}\,, \tag{4.4}$$

$$\|v\|_{U_\lambda} \leqq cN^\lambda \quad \textit{for} \quad \lambda = 0, 1, \ldots, k\,. \tag{4.5}$$

This lemma shows that $f_\varepsilon'(u)$ satisfies the hypothesis (2.7) with $\sigma = 2$. Theorem 1.1 now follows from Theorem 2.1 applied to the mapping f_ε.

5. SOLUTION OF THE LINEARIZED EQUATION

The bilinear form B_ε associated with $f_\varepsilon'(u)\, v$ is given by

$$B_\varepsilon(v, \varphi) = B_0(v, \varphi) + \varepsilon \sum_{|\alpha| = |\beta| = 1} \langle a_{\alpha\beta} D_x^\alpha v, D_x^\beta \varphi \rangle +$$
$$+ \varepsilon \sum_{|\alpha| = |\beta| = 1} \langle (D_x^\beta a_{\alpha\beta})\, D_x^\alpha v, \varphi \rangle + \varepsilon \sum_{|\alpha| \leqq 1, \beta = 0} \langle a_{\alpha\beta} D_x^\alpha v, \varphi \rangle +$$
$$+ \varepsilon \sum_{|\alpha| \leqq 1} \langle a_\alpha D_x^\alpha v_t, \varphi \rangle + \varepsilon \langle \bar{a} v_{tt}, \varphi \rangle\,,$$

where B_0 is defined by (1.10) and the functions $a_{\alpha\beta}$, a_α and $\bar{a}$ satisfy (4.2). Further, we put

$$A_\varepsilon(v, \varphi) = A_0(v, \varphi) + \varepsilon \sum_{|\alpha|,|\beta| \leq 1} \langle a_{\alpha\beta} D_x^\alpha v, D_x^\beta \varphi \rangle + \\ + \varepsilon \sum_{|\alpha|=|\beta|=1} \langle (D_x^\beta a_{\alpha\beta}) D_x^\alpha v, \varphi \rangle .$$

We begin by showing that

$$B_\varepsilon(v, \Lambda v) \geqq d_1 \|v\|_1^2 \tag{5.1}$$

for every $v \in H^1(Q) \cap B$ with $v_t \in H^1(Q)$ $(\cap B)$. The estimate of $B_0(v, \Lambda v)$ is given by (1.12). All the remaining terms can be estimated by $\varepsilon\, c\|v\|_1^2$. For instance, for $\alpha \neq \beta$, $|\alpha| = |\beta| = 1$, we have $a_{\alpha\beta} = a_{\beta\alpha}$ and therefore

$$|\langle a_{\alpha\beta} D_x^\alpha v, D_x^\beta v_t \rangle + \langle a_{\beta\alpha} D_x^\beta v, D_x^\alpha v_t \rangle| = \\ = |\langle (D_t^1 a_{\alpha\beta}) D_x^\alpha v, D_x^\beta v \rangle| \leqq c\|a\|_{C^1(Q)} \|v\|_1^2 .$$

Similarly,

$$|\langle a_\alpha D_x^\alpha v_t, 2v_t \rangle| = |\langle (D_x^\alpha a_\alpha) v_t, v_t \rangle| \leqq \|a\|_{C^1(Q)} \|v\|_1^2 .$$

Hence we have (5.1) for $|\varepsilon| \leqq \varepsilon_0$, ε_0 sufficiently small. In what follows the value ε_0 will be further reduced sometimes without any particular reference.

If $u \in H^1(\Omega) \cap B$, i.e., if u does not depend on t, then $A_0(u, u) = B_0(u, \Lambda u)/d_0$. Hence, by (1.12), we have

$$A_0(u, u) \geqq d_2 \|u\|_{H^1(\Omega)}^2 \quad \text{for} \quad u \in H^1(\Omega) \cap B \tag{5.2}$$

with a positive d_2. Since any $v \in H^{0,1}(Q) \cap B$ can be written as $v = \sum_j v_j(x)\, e_j(t)$ with $v_j \in H^1(\Omega) \cap B$, we get

$$A_0(v, v) = \sum_j A_0(v_j, v_j) \geqq d_2 \sum_j \|v_j\|_{H^1(\Omega)}^2 = d_2 \|v\|_{H^{0,1}(Q)}^2 .$$

This implies

$$A_\varepsilon(v, v) \geqq d_3 \|v\|_{H^{0,1}(Q)}^2 \quad \text{for all} \quad v \in H^{0,1}(Q) \cap B \tag{5.3}$$

for sufficiently small ε and $d_3 > 0$.

We shall suppose that the functions a and h satisfy (4.2) and (4.3). Let us fix a positive integer M. The mapping Λ is a linear homeomorphism of $H^1(Q) \cap B \cap Z_M$ onto itself. Using (5.1) we find a (unique) $v \in H^1(Q) \cap B \cap Z_M$ satisfying

$$B_\varepsilon(v, \varphi) = \langle h, \varphi \rangle \quad \text{for} \quad \varphi \in H^1(Q) \cap B \cap Z_M . \tag{5.4}$$

Putting $\varphi = \Lambda v$ and using (5.1) we have

$$\|v\|_{H^1(Q)} \leqq c\|h\| . \tag{5.5}$$

In what follows we shall show that v satisfies

$$\|v\|_{U_0} \leqq c\|h\|_{F_1} , \tag{5.6}$$

$$\|v\|_{U_\lambda} \leqq cN^\lambda \quad \text{for} \quad \lambda = 0, 1, \ldots, k , \tag{5.7}$$

with a constant c independent of h and N. Thus for every positive integer M we can find $v = v_M$ and then let $M \to \infty$. As a limit we shall obtain a (unique) v satisfying (5.4) for all $\varphi \in H^1(Q) \cap B$, (5.6) and (5.7). Hence $f'_\varepsilon(u)\, v = h$ and Lemma 4.3 will be proved.

Now we prove the estimates (5.6) and (5.7) for v with a particular fixed M. By definition of B_ε, we easily find that for $1 \leqq l \leqq 2n_0 + k$,

$$
\begin{aligned}
(5.8) \qquad & B_\varepsilon(v_{t^l}, \Lambda v_{t^l}) = B_\varepsilon(v, (-1)^l \Lambda v_{t^{2l}}) + \\
&+ \varepsilon \sum_{p=1}^{l} \binom{l}{p} \Big\{ \sum_{|\alpha|=|\beta|=1} \langle (D_t^p a_{\alpha\beta})\, D_x^\alpha D_t^{l+1-p} v, D_x^\beta \Lambda v_{t^{l-1}} \rangle - \\
&- \sum_{|\alpha|=1} \langle (D_t^p a_\alpha) D_x^\alpha D_t^{l+1-p} v, \Lambda v_{t^l} \rangle - \\
&- \langle (D_t^p \bar{a})\, D_t^{l+2-p} v, \Lambda v_{t^l} \rangle + \sum_{|\alpha|=|\beta|=1} \langle (D_t^{p+1} a_{\alpha\beta})\, D_x^\alpha D_t^{l-p} v, D_x^\beta \Lambda v_{t^{l-1}} \rangle - \\
&- \sum_{|\alpha|=|\beta|=1} \langle (D_t^p D_x^\beta a_{\alpha\beta})\, D_x^\alpha D_t^{l-p} v, \Lambda v_{t^l} \rangle - \sum_{|\alpha| \leqq 1, |\beta|=0} \langle (D_t^p a_{\alpha\beta})\, D_x^\alpha D_t^{l-p} v, \Lambda v_{t^l} \rangle \Big\} .
\end{aligned}
$$

It is easy to check that every term in $\{\ \}$ is estimated by

$$
(5.9) \qquad \|(D_t^p D_{t,x}^\alpha a)\,(D_t^{l-p} D_{t,x}^\beta v)\|\ \|v_{t^l}\|_1
$$

with $p = 0, 1, \ldots, l$, $|\alpha| \leqq 1$, $|\beta| \leqq 1$. For $p = 0$ these terms are estimated by

$$
\|D_{t,x}^\alpha a\|_{C(Q)}\ \|v_{t^l}\|_1^2 \leqq bc_s \|v_{t^l}\|_1^2 .
$$

Putting $\varphi = (-1)^l \Lambda v_{t^{2l}}$ in (5.4) we get

$$
B_\varepsilon(v, (-1)^l \Lambda v_{t^{2l}}) = \langle h_{t^l}, \Lambda v_{t^l} \rangle \leqq c\|h_{t^l}\|\ \|v_{t^l}\|_1 .
$$

Using this in (5.8) with all terms of the type (5.9) with $p = 0$ shifted to the left-hand side, we get

$$
(5.10) \qquad (d_1 - \varepsilon c_1)\, \|v_{t^l}\|_1 \leqq \|h_{t^l}\| + \varepsilon c\, Z(l) ,
$$

where

$$
Z(l) = \sum_{|\alpha| \leqq 1, |\beta| \leqq 1} \sum_{s=1}^{l} \|(D_t^s D_{t,x}^\alpha a)\, D_t^{l-s} D_{t,x}^\beta v\| .
$$

The following lemma provides an estimate of $Z(l)$.

Lemma 5.1.

(i) *Let* $1 \leqq l \leqq m - 1$ *and* $v \in H^l(Q)$. *Then* $Z(l) \leqq c\|a\|_{F_2}\ \|v\|_l$.

(ii) *Let* $\lambda = 0, 1, \ldots, k - 1$ *and* $v \in U_\lambda$. *Then* $Z(m + \lambda) \leqq c\{\|a\|_{F_{3+\lambda}}\ \|v\|_{U_0} + \sum_{\nu=1}^{\lambda} \|a\|_{F_{2+\nu}} \|v\|_{U_{\lambda-\nu}}\}$.

We postpone the proof to the next section.

Using part (i) of this lemma in (5.10) we get

(5.11) $$\|v_t\|_1 \leqq c\|h\|_{F_1} .$$

Hence (5.4) has the form

(5.12) $$A_\varepsilon(v, \varphi) = \langle b, \varphi\rangle \quad \text{for} \quad \varphi \in H^1(Q) \cap B \cap Z_M ,$$

where

$$b = h - v_{tt} - d(x)\, v_t + \varepsilon\big(\sum_{|\alpha| \leqq 1} a_\alpha D_x^\alpha v_t + \bar{a} v_{tt}\big) .$$

By (5.11), $b \in L^2(Q)$. We now use the following lemma whose proof will be given in the next section.

Lemma 5.2. *There is ε_0 such that if $g \in H^{0,s}(Q)$ for some $s \leqq n_1$ and $|\varepsilon| \leqq \varepsilon_0$, then, for every positive integer M, there exists a unique $w \in H^{0,s+2}(Q) \cap B \cap Z_M$ satisfying*

$$A_\varepsilon(w, \varphi) = \langle g, \varphi\rangle \quad \textit{for} \quad \varphi \in H^1(Q) \cap B \cap Z_M .$$

Moreover

$$\|w\|_{H^{0,s+2}(Q)} \leqq c\|g\|_{H^{0,s}(Q)} .$$

By this lemma and (5.12), $v \in H^{0,2}(Q)$. Thus (5.12) takes on the form

$$\sum_{|\alpha| \leqq 1, |\beta| \leqq 1} (-1)^{|\beta|} \langle D_x^\beta A_{\alpha\beta}(x)\, D^\alpha v + \varepsilon a_{\alpha\beta} D_x^{\beta+\alpha} v, \varphi\rangle = \langle b, \varphi\rangle .$$

Substituting $(-1)^{l-p}\varphi_{t^{l-p}}$ for φ and integrating by parts, we get

$$A_\varepsilon(v_{t^{l-p}}, \varphi) = \langle g_{l,p}, \varphi\rangle \quad \text{for} \quad \varphi \in H^1(Q) \cap B \cap Z_M .$$

Here

$$g_{l,p} = b_{t^{l-p}} - A_p^l v ,$$

$$A_p^l = \sum_{s,s',\beta} (-1)^{|\beta|} \binom{l-p}{s} (D_t^s a_{\alpha\beta})\, D_x^\beta D_t^{s'} v ,$$

where the sum is taken over $|\beta| \leqq 2$ and nonnegative integers s, s', $s + s' = l - p$, for which $s \geqq 1$ in the case $|\beta| = 2$.

The following lemma provides estimates of $g_{l,p}$.

Lemma 5.3.

(i) *Let $l = 1, \ldots, m-1$ and $p = 1, \ldots, l$ be fixed. Let $v \in H^l(Q) \cap X^{l+1,p}$. Then $\|g_{l,p}\|_{H^{0,p-1}} \leqq c\{\|h_{t^{l-p}}\|_{H^{0,p-1}(Q)} + \|a\|_{F_2}(\|v\|_l + \|v\|_{X^{l+1,p}})\}$.*

(ii) *Let $\lambda = 0, 1, \ldots, k-1$ and $p = 1, 2, \ldots, m-1$ be fixed. Let $v \in U_\lambda \cap X^{m+\lambda+1,p}$. Then $\|g_{m+\lambda,p}\|_{H^{0,p-1}} \leqq c\{\|h_{t^{m+\lambda-p}}\|_{H^{0,p-1}(Q)} + \|a\|_{F_2}\|v\|_{X^{m+\lambda+1,p}} + \sum_{\nu=0}^{\lambda} \|a\|_{F_{2+\nu}} \|v\|_{U_{\lambda-\nu}}\}$.*

Now using Lemma 5.1 – 5.3 we prove the following two implications; $l = 1, \ldots \ldots, m - 1$, $p = 1, \ldots, l$:

(5.13) If $\|v\|_l \leqq c\|h\|_{F_1}$, then $\|v_{t^l}\|_1 = \|v\|_{X^{l+1,1}} \leqq c\|h\|_{F_1}$.

(5.14) If $\|v\|_l + \|v\|_{X^{l+1,p}} \leqq c\|h\|_{F_1}$, then $\|v\|_{X^{l+1,p+1}} \leqq c\|h\|_{F_1}$.

(5.13) is a consequence of (5.10) and part (i) of Lemma 5.1. If $\|v\|_l + \|v\|_{X^{l+1,p}} \leqq \leqq c\|h\|_{F_1}$, we use part (i) of Lemma 5.3 to get a similar estimate for $g_{l,p}$ and then Lemma 5.2 with $s = p - 1$ to get $\|v_{t^{l-p}}\|_{H^{0,p+1}(Q)} \leqq c\|h\|_{F_1}$. This proves (5.14).

For $l = m - 1$ and $p = m - 1$, (5.14) yields $\|v\|_m \leqq c\|h\|_{F_1}$. As $\|\cdot\|_m = \|\cdot\|_{U_0}$, the estimate (5.6) is proved. Using parts (ii) of Lemmas 5.1 and 5.3, we similarly get (5.7). This completes the proof of Lemma 4.3.

6. PROOFS OF AUXILIARY LEMMAS

In this section we give the proofs of Lemmas 5.1 – 5.3.

Proof of Lemma 5.1. For $1 \leqq l \leqq m$ and $0 \leqq \lambda \leqq k - 1$ we have

$$Z(l + \lambda) = \sum_{|\alpha| \leqq 1, |\beta| \leqq 1} \sum_{s=1}^{l+\lambda} \|(D_t^s D_{t,x}^\alpha a)\, D_t^{l+\lambda-s} D_{t,x}^\beta v\| .$$

For j a nonpositive integer, we set $D_t^j v = v$. On estimating $Z(l + \lambda)$ we shall distinguish two cases.

(1) If $1 \leqq s \leqq n_0 + \lambda$, then

$$\|(D_t^s D_{t,x}^\alpha a)\, D_t^{l+\lambda-s} D_{t,x}^\beta v\| \leqq c_s \|D_t^s a\|_{n_0+1} \|D_t^{\lambda-s+1} v\|_l \leqq$$
$$\leqq c_s \|D_t^{s-n_0+2} a\|_{n_1} \|D_t^{\lambda-s+1} v\|_l \leqq c_s \|a\|_{F_{2+\nu}} (\|D_t^{\lambda-\nu} v\|_l + \|v\|_l),$$

where $\nu = \max(0, s - n_0)$.

(2) If $n_0 + \lambda < s \leqq l + \lambda$, then

$$\|(D_t^s D_{t,x}^\alpha a)\, D_t^{l+\lambda-s} D_{t,x}^\beta v\| \leqq c_s \|D_t^{s-2n_0+2} a\|_{n_1} \|D_t^{l+\lambda-s} v\|_{n_0+1} \leqq$$
$$\leqq c_s \|a\|_{F_{2+\mu}} \|v\|_l ,$$

where $\mu = \max(0, l + \lambda - n_1 - 1)$. This follows from the following computation:

$$s - 2n_0 + 2 \leqq l + \lambda - 2n_0 + 2 = (l + \lambda - n_1 - 1) + 2 .$$

To finish the proof we take $\lambda = 0$ and $l \leqq m - 1$. Then $\nu = 0$, $\mu = 0$ and part (i) follows from the derived estimates. For $l = m$, ν ranges over the set $0, 1, \ldots, \lambda$, $\mu = \lambda + 1$ and part (ii) follows immediately.

Proof of Lemma 5.2. We begin by showing that for every $b \in H^{0,s}(Q)$ we can find a unique $\Psi b \in H^{0,s+2}(Q) \cap B \cap Z_M$ satisfying

(6.1) $$A_0(\Psi b, \varphi) = \langle b, \varphi \rangle \quad \forall \varphi \in H^{0,1}(Q) \cap B \cap Z_M .$$

Writing $b = \sum b_j(x)\, e_j(t)$, we have $b_j \in H^s(\Omega) \cap B$ and $\|b\|_{H^{0,s}(Q)} = (\sum_j \|b_j\|^2_{H^s(\Omega)})^{1/2}$.

Since A_0 satisfies (5.2), we apply Theorem 9.8 from [4]. For every $j \in Z$ we obtain a unique $v_j \in H^{s+2}(\Omega) \cap B$ such that $A_0 v_j = b_j$ and $\|v_j\|_{H^{s+2}(\Omega)} \leqq c_3 \|b_j\|_{H^s(\Omega)}$. We put $\Psi b = \sum_{|j| \leqq M} v_j(x)\, e_j(t)$. It is easy to verify that Ψb satisfies (6.1) and that

$$\|\Psi b\|_{H^{0,s+2}(Q)} \leqq c\|b\|_{H^{0,s}(Q)} .$$

Further, we show that, for a satisfying (4.2) and $|\gamma| \leqq 2$,

$$\|a D_x^\gamma v\|_{H^{0,s}(Q)} \leqq c\|v\|_{H^{0,s+2}(Q)} . \tag{6.2}$$

This means, for $|\alpha| + |\alpha'| \leqq s$, to estimate $\|(D_x^\alpha a)\, D_x^{\alpha'+\gamma} v\|$. We shall distinguish two cases. (1) For $|\alpha| \leqq n_0 - 1$, we have

$$\|(D_x^\alpha a)\, D_x^{\alpha'+\gamma} v\| \leqq c_s \|a\|_{2n_0 - 1} \|v\|_{H^{0,s+2}(Q)} \leqq c_s b \|v\|_{H^{0,s+2}(Q)} .$$

(2) For $|\alpha| \geqq n_0$, we have $|\alpha'| \leqq s - n_0$ and thus

$$\|(D_x^\alpha a)\, D_x^{\alpha'+\gamma} v\|^2 = \int_0^{2\pi} \int_\Omega |D_x^\alpha a|^2\, |D_x^{\alpha'+\gamma} v|^2 \,\mathrm{d}x\,\mathrm{d}t \leqq$$

$$\leqq c_s \int_0^{2\pi} \|a(t, \cdot)\|^2_{H^{n_1}(\Omega)} \|v(t, \cdot)\|^2_{C^{|\alpha'|+2}(\Omega)} \,\mathrm{d}x\,\mathrm{d}t \leqq$$

$$\leqq c_s \max\{\|a(t, \cdot)\|^2_{H^{n_1}(\Omega)};\ t \in [0, 2\pi]\} \int_0^{2\pi} \|v(t, \cdot)\|_{H^{|\alpha'|+n_0+2}(\Omega)} \,\mathrm{d}t \leqq$$

$$\leqq c\{\|a\|_{n_1} + \|a_t\|_{n_1}\}^2 \|v\|^2_{H^{0,s+2}(Q)} ,$$

since $|\alpha'| + n_0 + 2 \leqq s + 2$. (6.2) is proved and this implies that the operator $\mathscr{A} v = \sum_{|\alpha|, |\beta| \leqq 1} a_{\alpha\beta} D_x^{\alpha+\beta} v$ is a bounded mapping of $H^{0,s+2}(Q)$ into $H^{0,s}(Q)$.Therefore, for ε sufficiently small, the mapping $\Psi(g - \varepsilon \mathscr{A} v)$ is a contraction on $H^{0,s+2}(Q) \cap B$ and its fixed point is the desired function w. This completes the proof of the lemma.

Proof of Lemma 5.3. For $l = 1, \ldots, m$, $p = 1, \ldots, \min(l, m-1)$ and $\lambda = 0, 1, \ldots, k-1$ we shall estimate $\|g_{l+\lambda,p}\|_{H^{0,p-1}(Q)}$. If $l = 1, \ldots, m-1$, $p = 1, \ldots, l$ and $\lambda = 0$, we get the estimate in (i); if $l = m$, $p = 1, \ldots, m-1$ and $\lambda = 0, 1, \ldots, k-1$, we get (ii). To obtain an estimate for $\|g_{l+\lambda,p}\|_{H^{0,p-1}(Q)}$ we shall deal with

$$\|(D_t^s D_x^\alpha a)\, D_t^{\sigma+s'} D_x^{\alpha'+\beta} v\| , \tag{6.3}$$

where $s, s', \sigma, \alpha, \alpha'$ and β satisfy

$$s + s' \leqq l + \lambda - p , \quad |\alpha| + |\alpha'| \leqq p - 1 , \quad \sigma + |\beta| \leqq 2$$

and the following implication holds:

$$\text{if} \quad |\beta| = 2 , \quad \text{then} \quad s \geqq 1 . \tag{6.4}$$

We shall distinguish three cases.

(1) Let $s + |\alpha| \leqq n_0 + \lambda + 1$ and $|\alpha| \leqq n_0 - 1$. Then (6.3) does not exceed

$$c_s \| D_t^s D_x^\alpha a \|_{n_0} \| D_t^{\sigma+s'} D_x^{\alpha'+\beta} v \| .$$

Setting

$$\nu = \max(0, s + |\alpha| - n_0 - 1)$$

we have $\| D_t^s D_x^\alpha a \|_{n_0} \leqq \| a \|_{F_{2+\nu}}$, since $s + |\alpha| + n_0 \leqq n_1 + 2 + \nu$. If $\lambda = 0$, then $\nu = 0$. For $\lambda > 0$, ν is an element of the set $0, \ldots, \lambda$.

Further, we shall estimate $\| D_t^{\sigma+s'} D_x^{\alpha'+\beta} v \|$. Obviously,

$$s' + |\alpha'| + (\sigma + |\beta|) \leqq l + \lambda + 1 - s - |\alpha| . \tag{6.5}$$

If $s + |\alpha| = 0$, then $\nu = 0$ and, by (6.4), $|\beta| \leqq 1$. Hence $|\alpha'| + |\beta| \leqq p$ and this yields

$$\| D_t^{\sigma+s'} D_x^{\alpha'+\beta} v \| \leqq \| v \|_{X^{l+\lambda+1,p}} .$$

Thus (6.3) does not exceed $c_s \| a \|_{F_2} \| v \|_{X^{l+\lambda+1,p}}$. If $s + |\alpha| > 0$, then $\nu \leqq s + |\alpha| - 1$. Further $|\alpha'| + |\beta| \leqq p + 1 \leqq m$ and, by (6.5),

$$\| D_t^{\sigma+s'} D_x^{\alpha'+\beta} v \| \leqq \| v \|_{X^{l+\lambda-\nu,m}} .$$

For $\lambda = 0$ we have $\| v \|_{X^{l+\lambda-\nu,m}} \leqq \| v \|_l$ and therefore (6.3) does not exceed $c_s \| a \|_{F_2} \| v \|_l$. For $\lambda > 0$ and $l = m$, $\| v \|_{X^{l+\lambda-\nu,m}} \leqq \| v \|_{U_{\lambda-\nu}}$ and (6.3) does not exceed $\| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}}$.

(2) Let $s + |\alpha| \leqq n_0 + \lambda + 1$ and $|\alpha| \geqq n_0$. From $s + |\alpha| \leqq \nu + n_0 + 1$ we have $\| D_t^s D_x^\alpha a \| \leqq \| a \|_{F_{2+\nu}}$ and (6.3) is estimated by $c_s \| a \|_{F_{2+\nu}} \| D_t^{\sigma+s'} D_x^{\alpha'+\beta} v \| n_0$. Now $|\alpha'| + |\beta| < p$ and $s' + |\alpha'| + (\sigma + |\beta|) + n_0 \leqq l + \lambda + 1 + n_0 - |\alpha| - s$. Thus, if $\nu = 0$, then $\| D_t^{\sigma+s'} D_x^{\sigma'+\beta} v \|_{n_0} \leqq \| v \|_{X^{l+\lambda+1,p}}$. If $\nu > 0$, i.e. $\nu = s + |\alpha| - n_0 - 1$, then $\| D_t^{\sigma+s'} D_x^{\alpha'+\beta} v \|_{n_0} \leqq \| v \|_{X^{l+\lambda-\nu,p}}$. Thus in this case (6.3) is estimated by $c_s(\| a \|_{F_2} \| v \|_{X^{l+\lambda+1}} + \| a \|_{F_{2+\nu}} \| v \|_{X^{l+\lambda-\nu,p}})$.

(3) Let $s + |\alpha| \geqq n_0 + \lambda + 2$. Then, obviously, $\| D_t^s D_x^\alpha a \| \leqq \| a \|_{F_{2+\lambda}}$ and $\| D_t^{\sigma+s'} D_x^{\alpha'+\beta} v \|_{n_0} \leqq \| v \|_l$ since $s' + |\alpha'| + (\sigma + |\beta|) + n_0 \leqq l + \lambda + 1 + n_0 - s - |\alpha| \leqq l - 1$.

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