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ON TWO CLASSES OF PARACOMPACT SPACES

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Using the Isbell's semiuniform product Tamano type characterizations of two classes of paracompact uniform spaces are given. Namely, the class of all spaces X such that each open cover of X is a uniform cover of the metric-fine coreflection $\mathbf{m}X$ of X, and the class of all X such that each finitely additive open cover of X is a uniform cover of $\mathbf{m}X$. A short survey of the theory of metric-fine spaces is given.

INTRODUCTION

By a space we mean a Hausdorff-uniform space. Following [F – H] a space is called paracompact if each open cover has a σ-dicrete (in the uniform sense) refinement (which may be taken to consist of cozero sets). This class was actually introduced in [Fe] under the name spaces "de caractère paracompact". These spaces are quite useful, see [F – H] and [Fe]. Note that every metric space is paracompact by a theorem of A. H. Stone.

A space X is called uniformly paracompact if the following equivalent conditions are satisfied:

(a) Each finitely additive open cover of X is uniform.

(b) Each open cover has a uniformly locally finite refinement.

(c) For some, and then any, compactification K of X, and for each compact $C \subset K \setminus X$ there exists a uniform cover $\mathcal{U}$ of X such that $\overline{U}^K \cap C = \emptyset$ for each $U \in \mathcal{U}$.

The equivalence of (a) and (b) is proved in [R_2] and (c) for $K = \beta X$ is proved in [H]. Of course, by a compactification of X we mean any compact space K containing the induced topological space of X as a dense subspace. The concept of uniform paracompactness is basic because many concepts of paracompact type can be defined as uniform paracompactness of some modification of X.

Recall that the result of Tamano [T] says that each of the following two conditions is necessary and sufficient for a completely regular topological space X to be paracompact in the usual topological sense:

(i) $X \times \beta X$ is normal

(ii) For each compact $C \subset \beta X \setminus X$ there exists a continuous function on $X \times \beta X$ which is 0 on $X \times C$, and 1 on the diagonal $\Delta_X = \{\langle x, x\rangle \mid x \in X\}$.

In this kind of results it does not matter which compactification is taken. In [F_4] it is shown that if the product is interpreted as the usual (= categorial) product in uniform spaces, and if we take for separation of closed sets uniformly continuous functions or **coz**-functions or h^1 **coz**-functions or h **coz**-functions, then (i) and (ii) are equivalent, and characterize, respectively, compact spaces, Lindelöf spaces, paracompact spaces, and the spaces such that the locally fine coreflection λX of X is paracompact.

Here we use the semi-uniform product $*$ of Isbell [I], and prove various Tamano type characterizations of spaces X such that the metric-fine coreflection $\mathbf{m}X$ of X is uniformly paracompact, and spaces such that $\mathbf{m}X$ is the fine uniformity and the topology is paracompact.

In § 1 a short survey of metric-fine spaces is given. In § 2 the properties of the Isbell's product are recalled and the action of **m** on this product is investigated. The main results are formulated in § 3, and proved in § 4.

The results of this paper were included into the author's lecture at the International topological conference in Leningrad in 1982.

§ 1. METRIC-FINE COREFLECTION

For the convenience of the reader we recall the basic facts about metric-fine spaces. The main reason for that is that the main results are published in Seminar Uniform Spaces 1972–3 and 1973–4, and these seminar notes are not available. We add the recent description (e) below from [F_3] which was done just to prove the results of this paper.

In general we use the notation from [Č]. The set of all uniformly continuous maps from X into Y is denoted by $\mathbf{U}(X, Y)$, if Y is the space $\mathbb{R}$ of the reas then Y is usually omitted.

1.1. For each uniform space X we denote by $\mathbf{t}_f X$ the finest uniform space topologically equivalent to X; note that Isbell [I] uses α for $\mathbf{t}_f$. The space $\mathbf{t}_f X$ is called the topologically fine coreflection of X. Clearly $\mathbf{U}(\mathbf{t}_f X, Y)$ is just the set of all continuous maps from X into Y, which is denoted by $\mathbf{C}(X, Y)$.

1.2. For any function f we denote by $\mathbf{coz}(f)$, the cozero set of f, the set $\{x \mid fx \neq 0\}$. By a cozero-set in a space X we mean the cozero set of some $f \in \mathbf{U}(X)$, and

$\mathbf{coz}(X)$ stands for all cozero sets in X. We denote by $\mathbf{Coz}(X, Y)$ the set of all **coz**-mappings of X into Y; recall that $f \in \mathbf{Coz}(X, Y)$ if $f^{-1}[\mathbf{coz}(Y)] \subset \mathbf{coz}(X)$. While $\mathbf{U}(X)$ need not be inversion-closed (i.e., if $\mathbf{coz}(f) = X, f \in \mathbf{U}(X)$, then $1/f$ is not necessarily in $\mathbf{U}(X)$), $\mathbf{Coz}(X)$ is obviously inversion closed. For properties we refer to [F_2] and papers by A. Hager. If X is metrizable then $\mathbf{coz}(X)$ coincides with open sets, and hence $\mathbf{Coz}(X) = \mathbf{C}(X)$ for metrizable spaces.

1.3. Following [Ha], see also [Fe], a space is called metric-fine if for each $f \in \in \mathbf{U}(X, M)$, M metric, the map $f : X \to \mathbf{t}_f M$ is also uniformly continuous. A. Hager observed that the class of all metric-fine spaces is coreflective (obviously it is closed under inductive generation) and showed that for a separable X (i.e. countable uniform covers form a basis for all uniform covers) the coreflection $\mathbf{m}X$ of X has all countable coz-covers of X for a basis. It is proved in [F_1] and [R_1] that:

(a) for any X, $\mathbf{m}X$ has for a basis all covers of the form $f^{-1}[\mathscr{U}]$, where $f \in \in \mathbf{U}(X, M)$, $\mathbf{M}$ is metric, and $\mathscr{U}$ is an open cover of M.

(b) all completely coz(X)-additive σ-discrete (in X) covers of X form a basis of uniform covers of $\mathbf{m}X$ ([F_1]).

Recall that a family $\{X_a\}$ is completely $\mathscr{M}$-additive if the union of each sub-family of $\{X_a\}$ belongs to $\mathscr{M}$. Of course, discrete is understood in the uniform sense; a family $\{X_a\}$ is discrete iff it is metrically discrete for some uniformly continuous pseudometric.

For the next description [F_2] note that the uniform vicinities of the diagonal of a uniform space X are elements of the filter on $X \times X$ generated by $\bigcup\{U \times U \mid U \in \in \mathscr{U}\}$ where $\mathscr{U}$ runs over all uniform covers of X.

(c) The cozero sets in $X \times X$ containing the diagonal form a basis of the vicinities of $\mathbf{m}X$.

Also the following description [F_2] suggests that $\mathbf{m}$ may be important for studying cozero sets.

(d) $\mathbf{m}$ is the finest functor F with the property that

$$\mathbf{coz}(X) = \mathbf{coz}(FX)$$

for each X.

Following two descriptions show the relationship to partitions of unity. It is proved in [F_3]:

(e) The covers of the form $\{\mathbf{coz}(f_a)\}$, where $\{f_a\}$ runs over equi-uniform (locally finite) partitions of unity on X, form a basis for uniform covers of $\mathbf{m}X$.

By a partition of unity on X it is understood a family $\{f_a\}$ of non-negative functions on X such that $\Sigma\{f_a x\} = 1$ for each x.

(f) A space X is metric-fine iff every equi-uniform partition of unity is l_1-uniformly continuous.

Recall that a partition $\{f_a \mid a \in A\}$ is l_1-uniformly continuous, or simply an l_1-partition, if the mapping

$$f = \{f_a\} : X \to l_1(A)$$

is uniformly continuous, where $fx = \{f_a x \mid a \in A\}$, or equivalently, if the finite partial sums form an equi-uniform family. This result is proved in [F_2], and it is an easy corollary to (e).

From the various descriptions given above it is clear that the class of metric-fine spaces may be useful in many considerations. We shall need the following properties which are obvious from the above descriptions.

(g) For any space X:

$$\mathbf{coz}(X) = \mathbf{coz}(\mathbf{m}X)$$

$$\mathbf{Coz}(X) = \mathbf{Coz}(\mathbf{m}X) = \mathbf{U}(\mathbf{m}X).$$

It follows from (f) that each metric-fine space is inversion-closed (and hence it has the Daniel property: if $f_n \downarrow 0$ and f_n ranges in $\mathbf{U}(X)$ then $\{f_n\}$ is equi-uniform) by a result of Zahradník, or directly from the definition. Of course the converse is not true, and the two notions coincide for measure-fine spaces.

In conclusion we state a simple result which will be used in the proof of Theorem 2:

(h) if $\{U_a\}$ is a σ-discrete completely $\mathbf{coz}(X)$-additive cover of a metric-fine space X, then there exists an l_1-partition of unity $\{f_a\}$ on X such that $\mathbf{coz}(f_a) = U_a$ for each a.

Proof. Observe that one can assume that $X = \mathbf{m}M = \mathbf{t}_f M$ where M is a metric space. The rest is routine.

§ 2. COZERO SETS ON THE ISBELL PRODUCT

Recall that the semi-uniform product $X * Y$ of two spaces X and Y is the set $X \times Y$ endowed with the uniformity projectively generated (= initial) by all mappings $f : X \times Y \to M$, M metric, such that for each $x \in X$ the mapping $\{y \to f\langle x, y\rangle\} : Y \to M$ is uniformly continuous, and the family of mappings

$$\{\{x \to f\langle x, y\rangle\} : X \to M \mid y \in Y\}$$

is equi-uniformly continuous on X.

This product was introduced by Isbell as a suitable tensor-product on the category of uniform spaces. Note that the product $*$ is not commutative. We need the following three facts from Isbell [I].

Fact 1. Each uniform cover of $X * Y$ is refined by a cover of the form $\{U_a \times V_{ab}\}$ where $\{U_a\}$ is a uniform cover of X, and for each a $\{V_{ab}\}$ is a uniform cover of Y. In fact, a cover $\mathscr{W}$ is uniform iff there exists a sequence $\{\mathscr{W}_n\}$ of covers of this form such that $\mathscr{W}_0$ refines $\mathscr{W}$, and each $\mathscr{W}_{n+1}$ star-refines $\mathscr{W}_n$. If X has a basis consisting of point-finite covers then the covers described above form a basis for uniform covers of $X * Y$ (the proof is like [I], VII.5., see also [F–F]).

Fact 2. $X \times Y$ and $X * Y$ are topologically equivalent.

Fact 3. If Y is compact then $(\mathbf{t}_f(X \times Y) =)\, \mathbf{t}_f(X * Y) = \mathbf{t}_f X * Y$.

In dealing with covering properties it is inconvenient that the description of uniform covers on $X * Y$ is quite indirect.

If E is a coreflection of the category of uniform spaces then obviously the identity mappings $F(X \times Y) \to FX \times FY$ are uniformly continuous. However, the identity mappings

$$(*) \qquad F(X * Y) \to FX * FY$$

need not be uniformly continuous (an example has been found recently by J. Vilímovský). It would be useful to find general sufficient conditions for uniform continuity of these mappings.

Proposition 1. *For any two spaces X and Y the identity mapping $\mathbf{m}(X * Y) \to \mathbf{m}X * \mathbf{m}Y$ is uniformly continuous.*

Proposition 2. *If either X is discrete or Y is compact then*

$$\mathbf{m}(X * Y) = \mathbf{m}X * \mathbf{m}Y.$$

Corollary to Proposition 1. $\mathbf{coz}(X * Y) = \mathbf{coz}(\mathbf{m}X * \mathbf{m}Y)$.

Proof. $\mathbf{coz}(\mathbf{m}Z) = \mathbf{coz}(Z)$ for each Z. See 1,4(g).

Proof of Proposition 2 using Proposition 1. If X is discrete then the statement is trivial. Assume that Y is compact. By Proposition 1 it is enough to show that $\mathbf{m}X * Y$ is finer than $\mathbf{m}(X * Y)$. Let $f : X * Y \to M$ be uniformly continuous with M metric. By definition of $*$ there exists a uniformly continuous pseudometric d on X such that

$$f\colon \langle X, d\rangle * Y \to M \in \mathbf{U}$$

and hence

$$f\colon \mathbf{t}_f(\langle X, d\rangle * Y) \to \mathbf{t}_f M \in \mathbf{U}\,.$$

By Fact 3,

$$\mathbf{t}_f(\langle X, d\rangle * Y) = \mathbf{t}_f\langle X, d\rangle * Y\,.$$

Since the identity $\mathbf{m}X \to \mathbf{t}_f\langle X, d\rangle$ is uniformly continuous, we get that $f\colon \mathbf{m}X * Y \to \mathbf{t}_f M \in \mathbf{U}$, and hence $\mathbf{m}X * Y$ is finer than $\mathbf{m}(X * Y)$.

Remark. If it is known that for each metric space M and each compact space Y we have $\mathbf{m}(M * Y) = \mathbf{t}_f(M * Y)$ then Proposition 1 is not needed because then

$$f\colon \mathbf{m}(X * Y) \to \mathbf{t}_{\mathrm{f}}\langle X, d\rangle * Y$$

is uniformly continuous, and these mappings then projectively generate $\mathbf{m}(X * Y)$.

For the proof of Proposition 1 we shall use the following

Lemma 1. *If $\{U_a \mid a \in A\}$ is a σ-discrete completely* **coz**-*additive family in X, and if $\{V_a \mid a \in A\}$ ranges in* $\mathbf{coz}(Y)$ *then*

$$L = \bigcup\{U_a \times V_a \mid a \in A\} \in \mathbf{coz}(X * Y)\,.$$

Proof. We shall use two-times the following easy characterization of cozero sets ([F_2]).

$E \in \mathbf{coz}(Z)$ iff there exists a sequence $\{\mathscr{W}_n\}$ of uniform covers (which may be assumed finite-dimensional and hence point-finite) such that

$$E = \bigcup\{\{x \mid \operatorname{st}(x, \mathscr{W}_n) \subset E\} \mid n \in \omega\}\,.$$

We shall construct $\{\mathscr{W}_n\}$ for $Z = X * Y$ and $E = L$ as follows. For convenience we may and shall assume that $\{U_a\}$ is discrete. Choose a sequence $\{\mathscr{G}_n\}$ of finite-dimensional covers of X such that

$$\bigcup\{U_a\} = \bigcup\{x \mid \operatorname{st}(x, \mathscr{G}_n) \subset \bigcup\{U_a\}\} \mid n \in \omega\}$$

and since $\{U_a\}$ is discrete, we may choose $\mathscr{G}_n$'s so that no $G \in \mathscr{G}_n$ meets two distinct U_a. For each a in A choose a sequence $\{\mathscr{H}^a_n\}$ of uniform covers of Y such that

$$V_a = \bigcup\{\{y \mid \operatorname{st}(y, \mathscr{H}^a_n) \subset V_a\} \mid n \in \omega\}\,.$$

Finally, let $\{\mathscr{W}_{nk}\}$ be the collection of all $G \times Y$ with $G \in \mathscr{G}_n$, and $G \subset U_a$ for no $a \in A$, and all $G \times H$ with $G \subset U_a$ for some $a \in A$, and $H \in \mathscr{H}^a_k$. Since all $\mathscr{G}_n$ are point-finite, and admit a star-refining sequence of point-finite covers, by Fact 1 all $\mathscr{W}_{nk}$ are uniform covers of $X * Y$. Arange $\{\mathscr{W}_{nk}\}$ in a sequence $\{\mathscr{W}_m\} \mid m \in \omega\}$ to obtain the required sequence.

Remark. We have proved that L in Lemma 1 is a cozero set in $X' * Y$ where X' has all finite-dimensional uniform covers of X for a basis.

For the proof of Proposition 1 we also need the following obvious

Lemma 2. *If $\{X_a\}$ is discrete in X, and if for each a $\{Y_{ab} \mid b \in B_a\}$ is discrete in Y then $\{X_a \times Y_{ab}\}$ is discrete in $X * Y$.*

Proof of Proposition 1. Recall 1.3(b) that $\mathbf{m}Z$ has for a basis all σ-discrete in Z completely $\mathbf{coz}(Z)$-additive covers of Z. It is easy to see that one can take just point-finite covers of this form to get a basis. The space $\mathbf{m}X * \mathbf{m}Y$ has all covers of the following form for a basis:

$$\{U_a \times V_{ab} \mid \langle a, b\rangle \in \Sigma\{B_a \mid a \in A\}\}$$

with $\{U_a\}$ point-finite σ-discrete completely $\mathbf{coz}(X)$-additive, and for each a, $\{V_{ab} \mid b \in B_a\}$ a completely $\mathbf{coz}(Y)$-additive σ-discrete cover of Y. It follows immediately from Lemma 1 that these covers are competely $\mathbf{coz}(X * Y)$ additive, and it follows from Lemma 2 that they are σ-discrete in $X * Y$. Hence, they are uniform covers of $\mathbf{m}(X * Y)$.

Proposition 3. *$L \in \mathbf{coz}(X * Y)$ iff L is of the form described in Lemma* 1.

Proof. If $L \in \mathbf{coz}(X * Y)$, then $L \in \mathbf{coz}(\mathbf{m}X * \mathbf{m}Y)$ by Corollary to Proposition 1, and hence L is of that form (consider a sequence in the basis described in the proof of Proposition 1).

Remark. It is easy to show that $L \in \mathbf{coz}(X * Y)$ iff there exists a uniformly continuous f of X into a metric space M (which may be assumed distal), and $L' \in \mathbf{coz}(M * Y)$ such that

$$L = (f \times \mathrm{id}_Y)^{-1} [L'] = \{\langle x, y\rangle \mid \langle fx, y\rangle \in L'\} .$$

§ 3. FORMULATION OF THE MAIN RESULTS

Define by induction $h^0 \mathbf{coz}(X) = \mathbf{coz}(X)$, $h^\alpha \mathbf{coz}(X)$ consists of σ-discrete unions of elements of $\bigcup\{h^\beta \mathbf{coz}(X) \mid \beta < \alpha\}$, and put

$$h\, \mathbf{coz}(X) = \bigcup\{h^\alpha \mathbf{coz}(X)\} .$$

The sets from $h\, \mathbf{coz}(X)$ are called the hyper-cozero sets in X, and those in $h^\alpha \mathbf{coz}(X)$ the hyper-cozero sets of class $\leqq \alpha$. For properties of hyper-cozero- sets we refer to [$F_{2,4}$]. A space X is called coz-normal if any two disjoint closed sets in X can be separated by a coz-function.

Theorem 1. *The following conditions on a space X are equivalent*

(1) *X is paracompact, and $h\, \mathbf{coz}(X) = \mathbf{coz}(X)$.*

(2) *Each open cover of X is a uniform cover of $\mathbf{m}X$.*

(3) *$\mathbf{m}X = \mathbf{t}_f X$, and the topology of X is paracompact.*

(4) *$X * K$ is $\mathbf{coz}$-normal for some compactification K of X.*

Using various descriptions of $\mathbf{m}X$ we can restate (2) to obtain further equivalent conditions – see § 1, e.g.

$(2)_a$ *Each open cover of X is refined by a σ-discrete completely $\mathbf{coz}(X)$-additive cover.*

$(2)_b$ *To each open cover it can be subordinated a partition of unity which is equi-uniform on X (i.e. an l_∞-partition on X).*

Proof of Theorem 1. We shall prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$. If X is paracompact, then by definition each open cover has a σ-discrete refinement consisting of cozero-sets, and if moreover $h\,\mathbf{coz}(X) = \mathbf{coz}(X)$, then this refinement is a uniform cover of $\mathbf{m}X$, and hence $(1) \Rightarrow (2)$. Obviously (2) implies (3). Assume (3) and let K be compact. By a classical theorem, the topology of $X * K$ is paracompact, and hence normal; since $\mathbf{m}(X * K) = \mathbf{m}X * K = \mathbf{t}_f X * K = \mathbf{t}_f(X * K)$, each continuous function is a $\mathbf{coz}$-function on $X * K$, and hence (4) holds. Finally assume (4). Then also X is $\mathbf{coz}$-normal, and hence $\mathbf{coz}(X) = \mathbf{coz}(\mathbf{t}_f X)$ $(\supset h\,\mathbf{coz}(X))$, that means $\mathbf{coz}(X) = h\,\mathbf{coz}(X)$. Paracompactness of X follows from two results

(α) [F_4, 2′] X is paracompact iff for some compactification K of X the space $X \times K$ is $h^1\,\mathbf{coz}(X \times K)$-normal.

(β) $U \in h^1\,\mathbf{coz}(X \times K)$ iff $U = \bigcup\{U_a \times V_a\}$ where $\{U_a\}$ is a σ-discrete family in $\mathbf{coz}(X)$, and $\{V_a\}$ ranges in $\mathbf{coz}(K)$.

Indeed, by (β) and Proposition 3 of § 2, each cozero-set in $X * K$ is in $h^1\,\mathbf{coz}(X \times K)$, and hence $\mathbf{coz}$-normality of $X * K$ implies $h^1\,\mathbf{coz}$-normality of $X \times K$.

By the same method we obtain immediately:

Proposition 4. *The spaces satisfying Theorem 1 are just the* $\mathbf{coz}$*-normal paracompact spaces. If X satisfies Theorem 1 then so does $X * K$ for each compact space K.*

Remark. If K is an uncountable compact space, and if Y is K with the uniformly discrete uniformity then $X = Y \times K$ does not satisfy Theorem 1 because $\mathbf{coz}(X) \neq h\,\mathbf{coz}(X)$.

Theorem 2. *The following conditions on a space X are equivalent:*

(a) *Each finitely additive open cover of X is a uniform cover of* $\mathbf{m}X$.

(b) *For some, and then each, compactification K of X the following holds: for each compact $C \subset K \setminus X$ there exists a uniform cover $\mathscr{V}$ of $\mathbf{m}X$ such that $\overline{V}^K \cap C = \emptyset$ for each V in $\mathscr{V}$.*

(c) *For some, and then each, compactification K of X the following holds: for each compact $C \subset K \setminus X$ there exists a* $\mathbf{coz}$*-function f on $X * K$ which is 1 on Δ_X and 0 on $X \times C$.*

(d) *For some, and then each, compactification K of X, and each compact $C \subset K \setminus X$ there exists $G \in \mathbf{coz}(X * K)$ with $\Delta_X \subset G \subset X \times K \setminus X \times C$.*

Clearly each of the conditions (a) and (b) says exactly that

(e) $\mathbf{m}X$ *is uniformly paracompact.*

Examples. Let D be a non-void set, and consider on $X = \omega_1 \times D$ the uniformity which has the following covers $\mathscr{U}(\alpha)$, $\alpha < \omega_1$, for a basis:

$$\mathscr{U}(\alpha) = \{(\beta) \times D \mid \beta \geqq \alpha\} \cup \{(\langle \gamma, d \rangle) \mid \gamma < \alpha,\ d \in D\}\,.$$

Clearly X is topologically discrete, and moreover, the first Ginsburg-Isbell derivative of X is the discrete uniformity on X. Hence X is λ-paracompact (i.e., λX is paracompact); actually $\lambda K = \mathbf{t}_f X$ and $\mathbf{t}_f X$ is discrete. One can easily show that X is metric-fine (any uniformly continuous mapping into a metric space is constant on each $(\alpha) \times D$ with α large enough). Next, X is paracompact iff D is countable, X satisfies Theorem 2 iff D is finite, and finally, X satisfies Theorem 1 iff D is a singleton. The proof of these properties is straightforward.

It is easily seen from (a) that each space X satisfying Theorem 2 is paracompact, and moreover

Proposition 5. *A space X satisfies* (a) *in Theorem* 1 *iff X is paracompact and each σ-discrete cover of X consisting of cozero sets has a refinement of the form $\{U_a \cap V_{ab} \mid a \in A,\ b \in B_a\}$ such that U_a is a completely* **coz**-*additive σ-discrete cover, and for each a $\{V_{ab} \mid b \in B_a\}$ is a finite cover of U_a consisting of cozero-sets.*

In the proof of Theorem 2 we shall not use any properties of paracompact spaces.

§ 4. PROOF OF THEOREM 2

As we noticed Condition (a) as well as Condition (b) is equivalent to uniform paracompactness of $\mathbf{m}X$. We shall prove (b) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (b) for both "some" and "any".

(b) $\Rightarrow$ (d). Let K be a compactification of X with the property in (b). Given a compact $C \subset K \setminus X$, choose a uniform cover $\mathscr{V}$ of $\mathbf{m}X$ with $\bar{V}^K \cap C = \emptyset$ for each V in $\mathscr{V}$. We may assume that $\mathscr{V}$ is a σ-discrete completely $\mathbf{coz}(X)$-additive cover. For each $V \in \mathscr{V}$ choose a cozero set $W(V)$ in K such that $\bar{V} \subset W(V) \subset K \setminus C$. By Lemma 1 of § 2

$$G = \bigcup\{V \times W(V) \mid V \in \mathscr{V}\}$$

is a cozero-set in $X * K$, and clearly

$$\Delta_X \subset G \subset X \times K \setminus X \times C\,.$$

(d) $\Rightarrow$ (c). Let K be a compactification of X with the property in (d). Given a compact $C \subset K \setminus X$, choose a G with the property in (d). By Proposition 3 G can be written in the union of a family $\{U_a \times V_a \mid a \in A\}$ such that $\{U_a\}$ is a σ-discrete completely $\mathbf{coz}(X)$-additive family, and V_a ranges in $\mathbf{coz}(K)$. We may assume that $\bar{V}_a^K \cap C = \emptyset$ for each a. Indeed, G can be written as the union of a sequence $\{G_n\}$ of cozero sets such that $\bar{G}_n \subset G$ for each n; if we take the families for each G_n as above, and put them together, we obtain such a family with the additional property. By 1.3.(h) we can take an (l_1)-partition of unity $\{h_a\}$ on $\mathbf{m}X$ such that $\mathbf{coz}(h_a) = U_a$.

Since $\overline{V}_a^K \cap C = \emptyset$, we can take a uniformly continuous function g_a on K such that $0 \leqq g_a \leqq 1$, g_a is 1 on V_a and 0 on C. Put

$$f\langle x, y\rangle = \Sigma\{f_a x \,.\, g_a y\}$$

Clearly f is uniformly continuous on $\mathbf{m}X * K$, and since $\mathbf{m}X * K = \mathbf{m}(X * K)$, f is uniformly continuous on $\mathbf{m}(X * K)$, and hence a $\mathbf{coz}$-function on $X * K$. Clearly f is 1 on Δ_X and 0 on $X \times C$.

(c) $\Rightarrow$ (b). Let K be a compactification with the property in (c). Given $C \subset K \setminus X$ compact choose f with the property in (c). Again f is uniformly continuous on $\mathbf{m}(X * K)$, hence on $\mathbf{m}X * K$. Put

$$d\langle x, y\rangle = \sup\{|f\langle x, k\rangle - f\langle y, k\rangle| \mid k \in K\}.$$

By definition of the semi-uniform product d is a uniformly continuous pseudometric on $\mathbf{m}X$. Now if S is any ball in $\langle X, d\rangle$ of radius $\leqq \frac{1}{3}$, then $f \geqq \frac{1}{3}$ on $S \times S$ because for $\langle x, y\rangle \in S \times S$

$$|f\langle x, y\rangle - f\langle y, y\rangle| \leqq \tfrac{1}{3}$$

and $f\langle y, y\rangle = 1$. Hence $\bar{S} \times \bar{S} \cap (X \times C) = \emptyset$, and hence $\bar{S} \cap C = \emptyset$. Thus the cover $\mathscr{V}$ of X consisting of all balls of radius $\frac{1}{3}$ is the required uniform cover of $\mathbf{m}X$.

5. CONCLUDING REMARKS

In a subsequent paper [F–F] it is shown that

A. $X = \mathbf{t}_f X$ and the topology of X is paracompact iff for some, and then any, compactification K of X the space $X * K$ is normal (i.e. disjoint closed sets are separated by uniformly continuous functions).

B. X is uniformly paracompact iff for some, and then any, compactification K of X the diagonal Δ_X and each $X \times C$, $C \subset K \setminus X$ compact are separated by a uniformly continuous function on $X * K$.

If we apply A to the proof of Theorem 1 we obtain immediately that (3) and (4) are equivalent. If we apply B to the proof of Theorem 2 we obtain immediately that (b) is equivalent to (c). Of course, we must use Proposition 2 of § 2 ($\mathbf{m}X * K = \mathbf{m}(X * K)$) and $\mathbf{coz}(X) = \mathbf{coz}(\mathbf{m}X)$. This seems to be typical for aplications of general results A and B in concrete situations.

If $\mathscr{M}_f$ denotes the measure-fine coreflection it would be interesting to find a description of spaces X such that $\mathscr{M}_f X = X$, and the topology is paracompact in terms of functional analytic properties of the space $\mathbf{M}_u(X)$ of uniform measures on X, or completenes of some space of closed subsets.

For a survey see the author's contribution to the proceedings of the International topological conference in Leningrad 1982. The dissertation of J.Fried (Mathematical Institute of the Czechoslovak Academy of Sciences 1983) contains nice results on

$\mathfrak{m}$-paracompact spaces, and a nice notion of $\mathscr{C}$-paracompact spaces. Also spaces satisfying Theorem 2 are characterized similarly to the Corson's characterization of paracompact topological spaces in terms of clustering of certain filters.

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