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SWITCHINGS OF OPTIMAL CONTROLS AND THE EQUATION $y^{(4)} + |y|^\alpha \operatorname{sign} y = 0,\ 0 < \alpha < 1$.

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1. INTRODUCTION

It is a well known property of optimal controls in most optimal control problems in which they can be explicitly constructed or characterized that they are piecewise analytic, with a finite number of discontinuity points or points where they do not posesss derivatives (henceforth called switching points). This is somewhat surprising since general existence theorems do not give more than measurable optimal controls.

New results giving bounds on the number of switching points have been recently obtained in the context of regular synthesis of optimal control [2, 6]. The result motivating this paper concerns the optimal control problem given by a linear system

(1) $$\dot{x} = Ax + Bu$$

(A, B analytic), linear control constraints, a target point x and the performance index

(2) $$J = \int_0^T f^0(x, u)\,\mathrm{d}t\,,$$

f being analytic in x, u and having an everywhere positive definite Hessian f^0_{uu}.

For this problem, it follows from [7] that any optimal control has a finite number of switching points on each finite interval.

There is an old example in which the optimal controls have an infinite number of switchings. This is Fuller's problem [5] given by the differential equation

(3) $$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = u\,,$$

the control set

(4) $$U = \{u \mid |u| \leqq 1\} \subset R^1\,,$$

the target point 0 and the performance index

(5) $$J = \int_0^T x_1^2\,\mathrm{d}t\,,$$

*) This work was partly done while the author was visiting at Michigan State University.

T free. Because of the linearity of the system and the convexity of the performance index, for each initial point there is precisely one optimal control for the fixed time problem. It is given by the formula

$$u(t) = -\operatorname{sign} p_2(t) \tag{6}$$

where p_2 is the second component of the adjoint vector and satisfies the differential equation

$$p_2^{(4)} = -\operatorname{sign} p_2 \,. \tag{7}$$

For $u(t)$ given by (6) to be a solution of the free time problem $p_2(t)$ has to vanish identically after a finite time. The equation (7) indeed has such solutions. They are oscillating with the distances of zeros and amplitudes forming geometric progressions with quotients <1. Therefore, the zeros of such a solution have an accumulation point and the solution vanishes to the right of it. The control given by (6) has then an infinite number of switching points. In the state space, the switching points of the response of $u(t)$ lie on a curve consisting of two half-parabolas $x_1 = ax_2^2$, $x_2 < 0$, $x_1 = -ax_2^2$, $x_2 > 0$ for a certain $a > 0$. The details of this analysis which is similar to that of Section 4 of this paper can be found in [5].

It is perhaps interesting to note that the equation (7) appears as the Euler equation of the variational problem of [1] where the existence of solutions with properties mentioned above is also established.

The confrontation of Fuller's problem with the result on finite number of switchings quoted above raises the following question:

Is strict positivity of the Hessian in u of the cost function important to assure a finite number of switchings or can it be replaced by a weaker condition, say strict convexity?

The following analysis of an entire family of optimal control problems with cost functions that are strictly convex in u shows that Fuller's problem is not a singular case and that strict convexity alone is not sufficient to guarantee the finiteness of the number of switchings. It leads to the study of unicity of the solution with zero initial values of the equation

$$y^{(4)} + |y|^\alpha \operatorname{sign} y = 0 \tag{8}$$

(the equation (7) can be considered its limit case for $\alpha = 0$). We establish the existence of a non-zero solution of (8) with zero initial values for each $0 < \alpha < 1$.

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2. A FAMILY OF OPTIMAL CONTROL PROBLEMS

We consider the optimal control problem, given by the system of equations

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = u\,, \tag{9}$$

the control set $U = R^1$, an initial point $x(0) = x^0$ $(x = (x_1, x_2))$, the target point $x = 0$ and the performance index

$$(10) \qquad J(u, T) = \int_0^T \left(\frac{1}{2}x_1^2 + \frac{1}{\gamma}|u|^\gamma\right) dt$$

where $\gamma > 2$, with T free.

Note that in this problem the cost function is strictly convex in u but its second derivative with respect to u vanishes at 0. Because of controllability of the system (9), convexity of the cost function in x, u and its strict convexity in u, the fixed time problem (9), (10) has a unique solution for each fixed T [4, § 3.4]. It follows immediately that if a solution of the free time problem exists it is also unique.

The adjoint system of equations for the problem (9), (10) is

$$(11) \qquad \dot{p}_0 = 0\,, \quad \dot{p}_1 = -p_0 x_1\,, \quad \dot{p}_2 = -p_1$$

and the maximum condition for the Hamiltonian is

$$\frac{1}{\gamma}\, p_0|\hat{u}| + p_2\hat{u} = \max_{-1 \leqq u \leqq 1}\left(\frac{1}{\gamma}\, p_0|u|^\gamma + p_2 u\right),$$

from which we immediately find

$$\hat{u} = -\left|\frac{p_2}{p_0}\right|^\alpha \operatorname{sign}\frac{p_2}{p_0}$$

for $p_0 \neq 0$, where $0 < \alpha = 1/(\gamma - 1) < 1$.

Let now $\hat{u}(t)$ be the optimal control for the initial point x^0 and let $\hat{T} < \infty$ be the optimal steering time. This control, extended by $\hat{u}(t) = 0$ for $t \geqq \hat{T}$ is trivially optimal for the problem with fixed T for every $T \geqq \hat{T}$. This implies that there must exist a non-vanishing solution of the adjoint equation p_0, $p_1(t)$, $p_2(t)$ with respect to which this extended control is extremal on $[0, \infty)$. This implies $p_2(t) = 0$ for $t \geqq \hat{T}$, and, by (11), also $p_1(t) = 0$ for $t \geqq \hat{T}$, so $p_1(\hat{T}) = 0$, $p_2(\hat{T}) = 0$ in particular and, consequently, $p_0 \neq 0$. We can therefore choose $p_0 = -1$. Summarizing, we have

Proposition 1. *Assume that the optimal steering time $\hat{T}$ for the problem (9), (10) is finite. Let $u(t)$, $x(t)$ be the (unique) optimal control and trajectory, respectively. Then, there exists a pair of functions $p_1(t)$, $p_2(t)$ such that the following equations are satisfied:*

$$(12) \qquad \dot{x}_1 = x_2\,,$$

$$\dot{x}_2 = u = |p_2|^\alpha \operatorname{sign} p_2\,,$$

$$(13) \qquad \dot{p}_1 = x_1\,,$$

$$\dot{p}_2 = -p_1$$

and

(14) $x_1(\hat{T}) = 0\,, \quad x_2(\hat{T}) = 0\,, \quad p_1(\hat{T}) = 0\,, \quad p_2(\hat{T}) = 0\,, \quad x_1(0) = x_1^0\,,$
$x_2(0) = x_2^0\,.$

Note that because of (14) we can extend $x_1(t)$, $x_2(t)$, $u(t)$, $p_1(t)$, $p_2(t)$ beyond $\hat{T}$ as being equal to 0 for $t \geqq \hat{T}$. Then, for each $T \geqq \hat{T}$, $u(t)$ is an extremal control on $[0, T]$ and steers the system from x^0 to 0.

If the initial state x^0 is not zero then $p_2(t)$, $p_1(t)$ can also not be identically zero. So, we have

Proposition 2. *If for some $x^0 \neq 0$ the optimal steering time T for the optimal control problem* (9), (10) *is finite, then there exists a non-zero solution $y(t)$ of the equation*

$$y^{(4)} + |y|^\alpha \operatorname{sign} y = 0 \tag{15}$$

satisfying

$$y(T) = \dot{y}(T) = \ddot{y}(T) = y^{(3)}(T) = 0\,. \tag{16}$$

The equation (15) is obtained by differentiating three times the last equation of (12) and substituting from the remaining equations.

Conversely, we have

Proposition 3. *Let $p_2(t)$ be a solution of* (15), (16) *such that $\hat{T} = \inf\{T \mid p_2(T) = \dot{p}_2(T) = \ddot{p}_2(T) = p_2^{(3)}(T) = 0\}$. Then, $x_1(t) = -\ddot{p}_2(t)$, $x_2(t) = -p_2^{(3)}(t)$ is the optimal trajectory and $u(t) = |p_2(t)|^\alpha \operatorname{sign} p_2(t)$ is the optimal control for the (free time) problem with the initial state $x^0 = (-\ddot{p}_2(0), -p_2^{(3)}(0))$.*

Proof. By [4, Chapter 3, Theorem 9] the point $(x_0(T), x(T))$ with

$$x_0(T) = J = \int_0^T \left(\frac{1}{2} x_1^2(t) + \frac{1}{\gamma}|u^2(t)|^\gamma\right) \mathrm{d}t$$

is a boundary point of the attainable set in the (x_0, x)-space $\hat{K}(T)$ and $(-1, p_1(T), p_2(T))$ is an outward normal to $\hat{K}(T)$ at the point $(x_0(T), x(T))$ (the attainable set $\hat{K}(T)$ is defined as the set of the responses $(x_0(T), x(T))$ to all possible controls on $[0, T]$). For $T \geqq \hat{T}$ we have $(-1, p_1(T), p_2(T)) = (-1, 0, 0)$ which means that $u(t)$ is the optimal control for the problem (9), (10) with fixed time T for each $T \geqq \hat{T}$ and therefore, also for the free time problem.

We finish this section by observing that by (12) the optimal control is analytic everywhere except of the zeros of $p_2(t)$ where its derivative is discontinuous.

3. THE EQUATION $y^{(4)} + |y|^\alpha \sin y = 0,\ 0 < \alpha < 1$

It follows from Section 2 that for the optimal control problem (9), (10) the equation (15) plays a central role. Without referring to the optimal control problem we establish in this section a family of non-zero solutions of (15) that become identically zero for large t. First, we note that the set of solutions of (15) has a certain homogeneity property expressed by the following lemma which can be proved by direct verification.

Lemma 1. *Let $y(t)$ be a solution of (15). Then, $\lambda\, y(\mu t)$ with $\mu = \lambda^{(\alpha-1)/4}$ is also a solution of (15) for any $\lambda > 0$.*

This homogeneity will be used in a fixed point argument for the proof of existence of the desired solution. In order to carry out the proof we need some more lemmas.

Lemma 2. *Let $y(t)$ be a solution of (15) with $y(0) = 0, \dot{y}(0) = a \geqq 0, \ddot{y}(0) = b \geqq 0$, $y^{(3)}(0) = c \geqq 0$ and assume that*

$$a^2 + b^2 + c^2 \neq 0. \tag{17}$$

Then, the smallest positive zero t_1 of $y(t)$ is well defined and we have

$$a_1 = -\dot{y}(t_1) > 0\,, \quad b_1 = -\ddot{y}(t_1) > 0\,, \quad c_1 = -y^{(3)}(t_1) > 0\,. \tag{18}$$

The functions $y^{(i)}(t)$, $i = 1, 2, 3$ change sign on $[0, t_1]$ at most once.

Proof. From the assumptions on a, b, c it follows immediately that $y(t) > 0$, for $t > 0$ sufficiently small. Let $\tau_i = \sup\{t \geqq 0 \mid y^{(i)}(s) \geqq 0 \text{ for } 0 \leqq s \leqq t\}$, $i = 0, 1, 2, 3$. The functions $y^{(i)}(t)$ are positive and strictly increasing on $[0, \tau_3)$ for $i = 0, 1, 2$. Thus, $y^{(3)}(t)$ is decreasing and concave on $[0, \tau_3)$, so τ_3 is finite. Proceeding similarly we see that $\tau_i \geqq \tau_{i+1}$, $i = 2, 1, 0$, $y^{(j)}(t) \geqq 0$ for $0 \leqq j \leqq i$ while $y^{(j)}(t) < 0$ for $i < j \leqq 3$ on $(\tau_{i+1}, \tau_i]$. This proves the lemma since $t_1 = \tau_0$.

Denote $Q = \{(a, b, c) \in R^3 \mid a + b + c = 1,\ a \geqq 0,\ b \geqq 0,\ c \geqq 0\}$ and $\tilde{Q} = \{0\} \times Q$.

Lemma 3. *There exist constants $q, D > 0$ such that the first zero t_1 of any solution $y(t)$ of (15) with $(y(0), \dot{y}(0), \ddot{y}(0), y^{(3)}(0)) \in \tilde{Q}$ satisfies $1 \leqq t_1 \leqq D$ and $\dot{y}(t_1) \leqq -q$, $\sup_{0 \leqq t \leqq t_1} y(t) \geqq \frac{1}{8}$.*

Proof. Note that

$$y(t_0 + t) = y(t_0) + \dot{y}(t_0)\, t + \tfrac{1}{2}\ddot{y}(t_0)\, t^2 + \tfrac{1}{6}y^{(3)}(t_0)\, t^3 - \tfrac{1}{6} \int_{t_0}^{t_0+t} (t_0 + t - s)^3\, y^\alpha(s)\, ds\,. \tag{19}$$

Denote a, b, c as in Lemma 2 and assume $(a, b, c) \in Q$. For all $t \in [0, \min\{1, t_1\}]$ we have $y(t) \geqq 0$ as well as $t \geqq t^2 \geqq t^3$, so

$$y(t) \leqq at + \frac{b}{2}t^2 + \frac{c}{6}t^3 \leqq (a + b + c)\,t = t\,.$$

Consequently, by (19),

$$y(t) \geqq at + \frac{b}{2}t^2 + \frac{c}{6}t^3 - \frac{1}{6}\int_0^t (t - s)^3 s^\alpha \, ds \geqq \frac{1}{6}t^3\left(1 - \int_0^t s^\alpha \, ds\right) =$$
$$= \frac{1}{6}t^3\left[1 - \frac{1}{\alpha + 1}t^{\alpha+1}\right],$$

which implies $t_1 > 1$.

Now, assume $0 \leqq y(t) < \frac{1}{8}$ for $0 \leqq t \leqq 1$. From (19) it follows that

$$y(1) \geqq a + \frac{b}{2} + \frac{c}{6} - \frac{1}{24\,.\,8^\alpha} \geqq \frac{1}{6} - \frac{1}{24} \geqq \frac{1}{8}\,.$$

This contradiction shows that there is a $0 \leqq \sigma_1 \leqq 1$ such that $y(\sigma_1) = \frac{1}{8}$, $\dot{y}(\sigma_1) \geqq 0$, $0 \leqq y(t) \leqq \frac{1}{8}$ for $0 \leqq t \leqq \sigma_1$. Assume $t \geqq 1$ is such that $y(s) \geqq \frac{1}{8}$ for $\sigma_1 \leqq s \leqq t$. Then, we have

$$y(t) \leqq t^3 - \frac{1}{8^\alpha}\frac{(t - 1)^4}{24}\,. \tag{20}$$

Since the right-hand side of (20) tends to $-\infty$ as $t \to \infty$ and is independent of a, b, c, it follows that there is a σ_2 such that $y(\sigma_2) = \frac{1}{8}$, $y(t) \geqq \frac{1}{8}$ for $\sigma_1 \leqq t \leqq \sigma_2$, $y(t) < \frac{1}{8}$ for $t \geqq \sigma_2$ near σ_2 and that σ_2 has an upper bound K independent of $(a, b, c) \in Q$. From the boundedness of σ_2 and Lemma 2 it follows that there is an $L > 0$ independent of $(a, b, c) \in Q$ such that $0 \geqq y^{(i)}(\sigma_2) \geqq -L$ for $i = 1, 2, 3$. Then, using (19), we obtain

$$y(\sigma_2 + t) \geqq \frac{1}{8} - Lt - \frac{1}{8^\alpha}\frac{t^4}{24} \geqq \frac{1}{8} - (2L + 1)\,t\,.$$

Thus, using (19) again, we have

$$\dot{y}\left(\sigma_2 + \frac{1}{8(2L + 1)}\right) \leqq -\frac{1}{2}\int_0^{1/[8(2L+1)]}\left[\frac{1}{8(2L + 1)} - s\right]^2\left[\frac{1}{8} - (2L + 1)\,s\right]^\alpha ds\,.$$

The integral on the right-hand side is positive and we denote its value by $2q$. Since $\dot{y}$ is decreasing on $[\sigma_2, t_1]$, we have $\dot{y}(t) \leqq -q$ on $[\sigma_2 + 1/[8(2L + 1)], t_1]$. Furthermore, for $\sigma_2 + 1/[8(2L + 1)] \leqq t \leqq t_1$ we have

$$y(t) \leqq \frac{1}{8} - q\left(t - \sigma_2 - \frac{1}{8(2L + 1)}\right)$$

from which it follows that we can take $D = K + 1/[8(2L + 1)] + 1/8q$. This completes the proof of Lemma 3.

For $(a, b, c) \in Q$ define $F(a, b, c)$ to be the set of all (a_1, b_1, c_1) which are related to (a, b, c) as in Lemma 2. Further, denote $Q_1 = F(Q)$, $\tilde{Q}_1 = \{0\} \times Q_1$. From Lemma 3 we immediately obtain

Corollary 4. *The sets Q_1 and $\tilde{Q}_1$ are bounded.*

Consider now the system of differential equations

$$\dot{z}_1 = z_2,\ \dot{z}_2 = z_3,\ \dot{z}_3 = z_4,\ \dot{z}_4 = -|z_1|^\alpha \operatorname{sign} z_1 . \tag{21}$$

Lemma 4. *The set Q_1 is a compact subset of the set*

$$R^3_+ = \{z = (z_1, z_2, z_3) \in R^3 \mid z_i > 0,\ i = 1, 2, 3\} .$$

Proof. The inclusion $Q_1 \subset R^3_+$ follows from Lemma 2. By Corollary 1, Q_1 is bounded. Thus, it suffices to prove that Q_1 (or, $\tilde{Q}_1$) is closed.

Let $v^k \in \tilde{Q}_1$, $v^k \to v^0$. Then, there are $w^k \in \tilde{Q}$ and solutions $z^k(t)$ of (21) with $z^k(0) = w^k$ such that $z^k(t_1^k) = v^k$, where t_1^k is the first positive zero of z_1^k. By Lemma 3, passing to a subsequence, we may assume $w^k \to w^0 \in \tilde{Q}$, $t_1^k \to t_1^0$. By [3, Theorem I.2.4] there is a solution $z^0(t)$ of (21) such that $z^0(0) = w^0$ and $z^k \to z^0$ uniformly on $[0, D]$, which yields $z^0(t_1^0) = v^0$, $z_1^0(t_1^0) = 0$, $z_1^0(t) \geqq 0$ on $[0, t_1^0]$. By Lemmas 2, 3, $z_1^k(t)$ have a single local maximum on $[0, t_1^k]$ with value $\geqq 1/8$ from which it follows that $z_1^0(t)$ cannot have a zero on $(0, t_1^0)$. This completes the proof.

Lemma 5. *The map F is single-valued over $Q_0 = \{a, b, c \in Q \mid a > 0\}$.*

Proof. By [3, Theorem I.2.4] it suffices to be proved that if $a > 0$ then the solution $z(t)$ of (21) with $z_1(0) = 0$, $z_2(0) = a$, $z_3(0) = b$, $z_4(0) = c$ is unique up to the first positive zero of z_1.

Assume there are two such solutions z^1, z^2. Denote $w(t) = \sum_{i=1}^{4} |z_i^1(t) - z_i^2(t)|$. By the mean value theorem we have

$$\frac{dw}{dt}(t) \leqq q(t)\, w , \quad w(0) = 0 ,$$

where $q(t) = 3 + \alpha \sup_{0 \leqq \vartheta \leqq 1} |\vartheta z_1^1(t) + (1 - \vartheta)\, z_1^2(t)|^{\alpha - 1}$.

Since $a > 0$, there exists an $\varepsilon > 0$ such that $z_1^i(t) \geqq \frac{1}{2}at$ for $t \in [0, \varepsilon]$, $i = 1, 2$. Thus,

$$q(t) \leqq 3 + \alpha\left(\frac{a}{2}t\right)^{\alpha - 1} . \tag{22}$$

Using Gronwall's inequality we have

$$w(t) = w(0) \exp\left\{\int_0^t q(s)\, ds\right\}$$

for $t \in [0, \varepsilon]$. Since by (22) q is integrable, we have $w(t) = 0$ for $t \in [0, \varepsilon]$. Obviously, w remains zero until z_1 vanishes.

It follows from Lemmas 1, 5 that F has the following homogeneity property over Q_0:

If $(a_1, b_1, c_1) = F(a, b, c)$ then

$$F(\lambda\mu a, \lambda\mu^2 b, \lambda\mu^3 c) = (\lambda\mu a_1, \lambda\mu^2 b_1, \lambda\mu^3 c_1) \tag{23}$$

with λ and μ related as in Lemma 1.

Assume now that we find a point $(a^*, b^*, c^*) \in Q_0$ such that

$$F(a^*, b^*, c^*) = (\lambda^*\mu^* a^*, \lambda^*\mu^{*2} b^*, \lambda^*\mu^{*3} c^*), \tag{24}$$

$\mu^* = \lambda^{*(\alpha-1)/4}$; denote by $y(t, a, b, c)$ the solution of (15) with $y(0) = 0$, $\dot{y}(0) = a$. $\ddot{y}(0) = b$, $y^{(3)}(0) = c$, $y^*(t) = y(t, a^*, b^*, c^*)$ and by t_i^* the i-th positive zero of y^*, From (23), (24) it follows immediately that

$$y^*(t_1^* + t) = -\lambda^* y^*(\mu^* t) \quad \text{for} \quad t \in [0, t_2^* - t_1^*], \tag{25}$$

$t_2^* - t_1^* = \mu^{*-1} t_1^*$ and, by induction,

$$y^*(t_i^* + t) = (-1)^i \lambda^{*i} y(\mu^{*i} t) \quad \text{for} \quad t \in [0, t_{i+1}^* - t_i^*], \tag{26}$$

$t_{i+1}^* - t_i^* = \mu^{*-i} t_1^*$. Obviously, (26) holds also for i negative if by t_{-i}, $i > 0$ we understand the i-th negative zero of y^*.

It follows that y^* is an oscillatory solution of (15) the amplitudes and distances of zeros of which form geometric progressions with quotienst λ and μ^{-1}, respectively.

Lemma 6. *If* $(a^*, b^*, c^*) \in Q_0$, λ^* *is a solution of* (24) *for* $0 < \alpha < 1$, *then* $\lambda^* > 1$.

Proof. Integrating by parts twice and using (23) (stars dropped at $y, a, b, t_i, \lambda, \mu$) we obtain

$$0 < \int_0^{t_2} \ddot{y}^2(t)\,\mathrm{d}t = \dot{y}(t_2)\,\ddot{y}(t_2) - ab + \int_0^{t_2} y(t)\,y^{(4)}(t)\,\mathrm{d}t =$$

$$= ab(\lambda^4\mu^6 - 1) - \int_0^{t_1} y^{\alpha+1}(t)\,\mathrm{d}t + \int_0^{t_2 - t_1} y^{\alpha+1}(t_1 + t)\,\mathrm{d}t =$$

$$= ab(\lambda^{\frac{1}{4}(10+6\alpha)} - 1) + \int_0^{t_1} y^{\alpha+1}(t)\,(\mu^{-1}\lambda^{\alpha+1} - 1)\,\mathrm{d}t =$$

$$= ab(\lambda^{\frac{1}{4}(10+6\alpha)} - 1) + (\lambda^{(3\alpha+5)/4} - 1)\int_0^{t_1} y^{\alpha+1}(t)\,\mathrm{d}t\,.$$

The last inequality can hold true only if $\lambda > 1$ which proves the lemma.

Denote $\bar{y}(t) = y^*(-t)$. From the invariance of the equation (15) with respect to the change of the time direction it follows that $\bar{y}(t)$ is also a solution of (15). The ratios of two consecutive amplitudes and distances of zeros of $\bar{y}$ are now λ^{*-1} and μ^*, respectively. This means that the zeros of $\bar{y}$ accumulate at a point $\bar{t} > 0$ and we have

$$\lim_{t \nearrow \bar{t}} \bar{y}^{(i)}(t) = 0 \quad \text{for} \quad i = 0, 1, 2, 3 .$$

Consequently, $\bar{y}(t)$ becomes identically zero for $t \geqq \bar{t}$. Due to Lemma 1 it is obvious that these properties of $\bar{y}$ are shared by an entire one-parameter family of solutions, namely $y_\lambda(t) = \lambda\, \bar{y}(\mu t)$, $\lambda \neq 0$, $\mu = \alpha^{\frac{1}{4}(\alpha-1)}$.

Summarizing, we have

Proposition 4. *Let* $0 < \alpha < 1$ *and let* $(a^*, b^*, c^*) \in Q_0$ *satisfy* (24). *Then, for each* $\lambda \neq 0$, $y_\lambda(t) = \lambda\, \bar{y}(\mu t) = \lambda y(-\mu t, a^*. b^*, c^*)$ *is a solution of* (15) *such that*

$$t^\lambda = \inf \{t > 0 | \; y_\lambda^{(i)}(t) = 0 \; for \; i = 0, 1, 2, 3\}$$

is finite positive and y_λ *has an infinite number of zeros in each left neighbourhood of* t^λ.

In virtue of this proposition, in order to establish the existence of a non-zero solution of (15), (16) it suffices to prove that there exists an $(a^*, b^*, c^*) \in Q_0$ such that (24) holds true.

Lemma 7. *Let* $0 < \alpha < 1$. *Then, there exists an* (a^*, b^*, c^*) *such that* (24) *holds true.*

Proof. Denote $\varrho = \lambda\mu = \lambda^{(\alpha+3)/4}$. Then, we have $\lambda\mu^2 = \varrho^{(2\alpha+2)/(\alpha+3)}$, $\lambda\mu^3 = \varrho^{(3\alpha+1)/(\alpha+3)}$. The homogeneity property (23) can now be rewritten as

$$F(\varrho a, \varrho^{(2\alpha+2)/(\alpha+3)} b, \; \varrho^{(3\alpha+1)/(\alpha+3)} c) = (\varrho a_1, \; \varrho^{(2\alpha+2)/(\alpha+3)} b_1, \; \varrho^{(3\alpha+1)/(\alpha+3)} c_1) . \tag{27}$$

We have to prove that there exists a point $(a^*, b^*, c^*) \in Q_0$ such that

$$F(a^*, b^*, c^*) = (\varrho^* a^*, \; \varrho^{*(2\alpha+2)/(\alpha+3)} b^*, \; \varrho^{*(3\alpha+1)/(\alpha+3)} c^*) \tag{28}$$

for some ϱ^*.

We make the change of variables $H: (u, v, w) \mapsto (a, b, c)$ defined by $a = u$, $b = v^{(2\alpha+2)/(\alpha+3)}$, $c = w^{(3\alpha+1)/(\alpha+3)}$. Obviously, $H: Q \to Q$ is a homeomorphism that maps Q_0 onto Q_0. By Lemma 5, $G = H^{-1}FH$ is continuous on Q_0. We can rewrite (27), (28) as

$$G(\varrho u, \varrho v, \varrho w) = \varrho G(u, v, w) \tag{29}$$

and

$$G(u^*, v^*, w^*) = (\varrho^* u^*, \varrho^* v^*, \varrho^* w^*) , \tag{30}$$

respectively, where $(a^*, b^*, c^*) = H(u^*, v^*, w^*)$.

Denote by G_1, G_2, G_3 the components of G and $|G| = G_1 + G_2 + G_3$, $\tilde{G} = |G|^{-1} G$. By Lemma 4, $\tilde{G}(Q)$ is a compact subset of Q_0. Let P be a convex compact set such that $\tilde{G}(Q) \subset P \subset Q_0$. We have $\tilde{G}(P) \subset \tilde{G}(Q) \subset P$. Since G is continuous in P, by Brouwer's fixed point theorem it has a fixed point $(u^*, v^*, w^*) \in P \subset Q_0$. The point (u^*, v^*, w^*) together with $\varrho^* = |G(u^*, v^*, w^*)|$ obviously solves (30) and $(a^*, b^*, c^*) = H(u^*, v^*, w^*)$, $\lambda^* = \varrho^{*4/(\alpha+3)}$ is a solution of (24).

Thus we have

Corollary 2. *For each* $0 < \alpha < 1$ *there is a non-zero solution of* (15) *that becomes identically zero for t sufficiently large as well as a non-zero solution that becomes identically zero for t sufficiently small.*

4. SYNTHESIS OF THE OPTIMAL CONTROL PROBLEM

Consider now the family of solutions $y_\lambda(t)$ of (15) given by Proposition 4. Let $\{t_i^\lambda\}$, $i = 1, 2, 3, \ldots$ be the zeros of y_λ. From (23), (24) it follows that we have

$$\ddot{y}_\lambda(t_i^\lambda) = (-1)^i \varrho^{i(2\alpha+2)/(\alpha+3)} (\varrho^*)^{-i(2\alpha+2)/(\alpha+3)} b^* ,$$

$$y^{(3)}(t_i^\lambda) = (-1)^{i+1} \varrho^{i(3\alpha+1)/(\alpha+3)} (\varrho^*)^{-i(3\alpha+1)/(\alpha+3)} c^*$$

$(\varrho = \lambda\mu)$ which implies

$$\left(\frac{\ddot{y}_\lambda(t_i^\lambda)}{b^*}\right)^{3\alpha+1} = \left(\frac{y_\lambda^{(3)}(t_i^\lambda)}{c^*}\right)^{2\alpha+2} , \quad \ddot{y}_\lambda(t_i^\lambda)\, y_\lambda^{(3)}(t_i^\lambda) < 0 . \tag{31}$$

Returning to the optimal control problem (9), (10) we identify y and its derivatives with the state and adjoint variables by $y = p_2$, $\dot{y} = -p_1$, $\ddot{y} = -x_1$, $y^{(3)} = -x_2$ (cf. (13)). By Proposition 2, each y_λ generates the optimal trajectory $x_1^\lambda(t) = -\ddot{y}^\lambda(t)$, $x_2^\lambda(t) = -y_\lambda^{(3)}(t)$ for the initial point $(\lambda b^*, -\lambda c^*)$ and the optimal control $u^\lambda(t) = |y_\lambda(t)|^\alpha \operatorname{sign} y(t)$ with infinitely many switching points t_i^λ, $i > 0$. It follows from (31) that for all λ all the switching points of the optimal trajectory lie on the curve

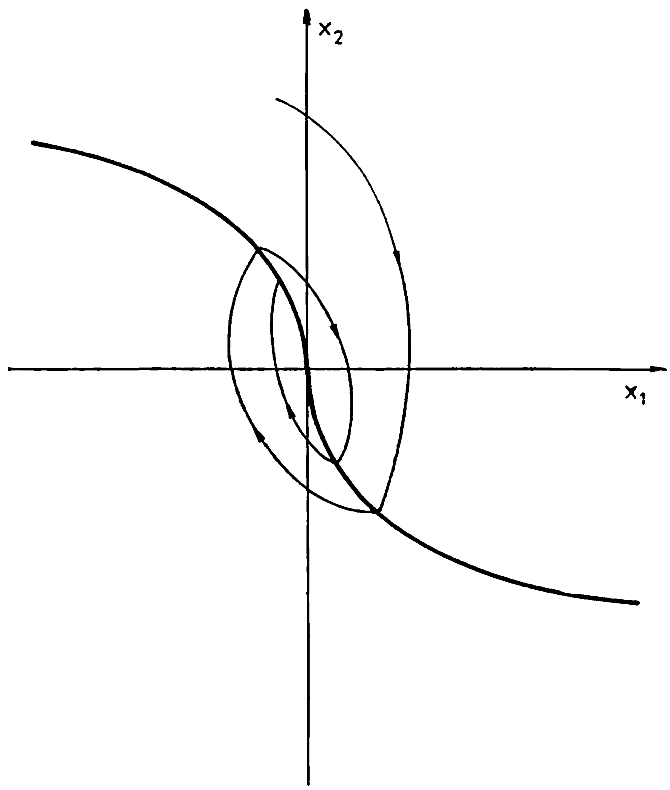
$$\left|\frac{x_1}{b}\right|^{3\alpha+1} = \left|\frac{x_2}{c}\right|^{2\alpha+2} , \quad \operatorname{sign} x_1 = -\operatorname{sign} x_2 \tag{32}$$

As in Fuller's problem, each optimal trajectory intersects the curve (32) infinitely many times until it reaches the target point 0 in a finite time. It is now obvious that through each point in the (x_1, x_2) – plane there is precisely one trajectory of the family $(x_1^\lambda(t), x_2^\lambda(t))$. Therefore, as an immediate consequence of Proposition 2 we obtain the following propositions:

Proposition 5. *For each point* $x^0 \in R^2$ *there exist* λ, τ *such that* $(x_1^\lambda(t + \tau), x_2^\lambda(t + \tau))$ *is the optimal trajectory of* x^0.

Since the optimal trajectory for a given initial point x^0 is unique, from Propositions 2, 3 we obtain

(Fig. 1).



Proposition 6. *For each non-zero solution $y(t)$ of (15) which is identically zero for sufficiently large t there exist λ, μ such that $y(t) = y_\lambda(t + \tau)$.*

This proposition is of some interest for the unicity problem of the solution with zero intial values for the equation (15) itself. It shows that knowing one non-zero solution $y^*(t)$ that vanishes in a finite time, all the other such solutions can be obtained from this one by time shifts, reversion of time and simultaneous coupled linear changes of the y and t variables. In other words, the family of all solutions vanishing in a finite time is the smallest family of functions containing y^* and closed with respect to time shifts, reversion of time and simultaneous multiplications of y and t by $\mu = \lambda^{(\alpha-1)/4}$.

5. CONCLUDING REMARKS

1. Besides the fact that the optimal controls have infinitely many switchings the family of problems (9), (10) is interesting for another reason. It is well known that

for the linear-quadratic problem ($\gamma = 2$) the optimal steering time is infinite, or, more precisely, the optimal cost for the fixed time problem is strictly decreasing with the length of the time interval on $[0, \infty)$. It would be interesting to establish finiteness of the optimal steering time for a larger class of problems. By "common sense" one could expect a comparison theorem saying that if we have two problems with the same dynamics and cost functions $\int_0^T (f(x) + g_i(u))\,dt$, $i = 1, 2$, $f(x) \geqq 0$, $0 < g_1(u) \leqq g_2(u)$ for $u \neq 0$, $g_1(0) = g_2(0) = 0$ then the optimal steering time for the first problem should not exceed the optimal steering time for the second one.

2. Using the same method of proof the results of Section 3 can be extended to equations

$$y^{(4k)} + |y|^\alpha \operatorname{sign} y = 0\,.$$

The place where the proof fails for equations of other orders is Lemma 6.

3. While in the existence proof of the non-zero solution of (15), (16) there was no need of referring to the motivating optimal control problem (9), (10) we have no idea how Proposition 6, characterizing the family of all such solutions, could be proved without using the properties of optimal controls in (9), (10).

4. This paper was written in 1981. Due to the difficulties of communication of the authors caused by their geographical distance it was not submitted for some time during which the paper [8] appeared. In this paper the same equation is treated in a different way and with a different motivation. We believe that our paper is still of some interest for two reasons: First, the proof of existence of the nontrivial solution is different from that of [8] and does not refer to any underlying minimization problem and second, because of its control theory motivation and implications.

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Fig. 1