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BOUNDED SOLUTIONS OF AFFINE STOCHASTIC DIFFERENTIAL EQUATIONS AND STABILITY

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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1. INTRODUCTION

The results in this paper have been stimulated by the ones reported by E. Pardoux at the Workshop on Differential Equations and Control Theory in Iași (September 1984) and by the corresponding results obtained by T. Morozan [6] for the periodic and stationary cases. In his report E. Pardoux considered the equation

$$\mathrm{d}x = (A(t)\,x + f(t))\,\mathrm{d}t + \sum_{j=1}^{m} (B_j(t)\,x + f_j(t))\,\mathrm{d}w_j(t)\,. \tag{1}$$

(Pardoux writes it in the Stratonovich form) with periodic coefficients, assumed that a Liapunov number associated with it is negative and proved that a solution explicitly written by using a variation of constants formula is periodic (has a periodic distribution function), and unique in the sense that any other possible periodic solution has the same distribution (when considered as a process).

T. Morozan performed a detailed study of the problem of existence of periodic solutions and discovered that under a condition which may be interpreted in terms of controllability or under a uniqueness assumption, exponential stability in mean square is necessary and sufficient for existence of periodic solutions. It is the purpose of this paper to extend some of the results of Morozan to the case of bounded coefficients and bounded solutions. We feel it is important to see that the results which in the deterministic case are connected with exponential dichotomy, in the stochastic case hold only for exponentially stable systems. The second part of the paper is concerned with existence of a trajectory of a flow defined on the space of probability laws corresponding to solutions bounded on the whole axis in the deterministic case.

We shall use the following notations. If f is an $\mathbb{R}^n$-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ we denote by $\mathsf{P} \circ f^{-1}$ the law of f, by $\mathsf{E}f = \int f \,\mathrm{d}\mathsf{P}$ the expectation of f and by $\operatorname{cov}(f) = \mathsf{E}((f - \mathsf{E}f)(f - \mathsf{E}f)^*)$ the covariance of f. For a stochastic process $\{x(t)\}_{t\in[a,\infty)}$ and $s \geqq 0$ we denote $x^s(t) = x(t + s) - x(s)$ (the post process after s); if x is a Wiener process then x^s is also a Wiener process.

We denote by M_n the set of all probability measures on $\mathbb{R}^n$ such that $\int |x|^2 \, \mathrm{d}\mu(x) < \infty$; for $\mu \in M_n$, $\mathsf{E}\mu = \int x \, \mathrm{d}\mu(x)$ is the expectation of μ and $\operatorname{cov} \mu = \int (x - E\mu)(x - E\mu)^* \, . \, \mathrm{d}\mu(x)$ is the covariance of μ. We further denote by G_n the set of measures $\mu \in M_n$ such that

$$\int e^{iu^*x} \, \mathrm{d}\mu(x) = e^{iu^*m - \frac{1}{2}u^*Hu} , \quad u \in \mathbb{R}^n , \quad H \geqq 0 ;$$

every $\mu \in G_n$ is called Gaussian and it is known that $\mathsf{E}\mu = m$, $\operatorname{cov}(\mu) = H$.

2. A GENERALIZED LIAPUNOV EQUATION

We shall start with a deterministic object, namely the equation

$$X' = A(t) X + X A^*(t) + \Pi(t, X) + G(t) . \tag{2}$$

A corresponding Riccati equation has been studied by M. Wonham [9]. As in [9], $\Pi(t, \cdot)$ is a linear positive operator and we shall consider the equation on the space $\mathscr{H}$ of symmetric $n \times n$ matrices. In the main application, $\Pi(t, X) = \sum_j B_j(t) X B_j^*(t)$.

Proposition 1. *Consider equation* (2) *for* $G(t) \equiv 0$. *Then the inequality* $X(t_0) \geqq 0$ *implies that* $X(t) \geqq 0$ *for all* $t \geqq t_0$.

Proof. We have the representation formula (see Conti [1])

$$X(t) = C(t, t_0) X(t_0) C^*(t, t_0) + \int_{t_0}^{t} C(t, s) \Pi(s, X(s)) C^*(t, s) \, \mathrm{d}s \tag{3}$$

where $C(t, t_0)$ is the fundamental matrix associated with A.

We may view the above formula as a Volterra integral equation, construct the solution by the usual successive approximations starting with $C(t, t_0) X(t_0) C^*(t, t_0)$ which is positive semidefinite; all the approximations are positive semidefinite and such will be the limit.

If we denote by $X(t, t_0, X_0)$ the solution taking at t_0 the value X_0 we may write $X(t, t_0, X_0) = \mathscr{T}(t, t_0)(X_0)$ where $\mathscr{T}(t, t_0): \mathscr{H} \to \mathscr{H}$ is a linear and positive (Proposition 1) operator; this evolution operator satisfies $\mathscr{T}(t, t_0) = \mathscr{T}(t, s) \mathscr{T}(s, t_0)$ for all t, s, t_0. By direct verification it is seen that we have the representation formula

$$X(t) = \mathscr{T}(t, t_0)(X(t_0)) + \int_{t_0}^{t} \mathscr{T}(t, s)(G(s)) \, \mathrm{d}s . \tag{4}$$

Since the trace of a matrix defines a linear functional we deduce

$$\operatorname{Tr}(X(t)) = \operatorname{Tr}(\mathscr{T}(t, t_0)(X(t_0))) + \int_{t_0}^{t} \operatorname{Tr}(\mathscr{T}(t, s)(G(s))) \, \mathrm{d}s .$$

If $X(t_0) \geqq 0$ and $G(t) \geqq 0$ for all t then $X(t) \geqq 0$, $\mathscr{T}(t, t_0)(X(t_0)) \geqq 0$, $\mathscr{T}(t, s)(G(s)) \geqq$ $\geqq 0$ for $t \geqq s \geqq t_0$ and we deduce that

$$\int_{t_0}^{t} \mathrm{Tr}[\mathscr{T}(t, s)(G(s))] \, \mathrm{d}s \leqq \mathrm{Tr}[X(t)] .$$

However,

$$\|\mathscr{T}(t, s)(G)s))\| = \Lambda(\mathscr{T}(t, s)(G(s))) \leqq \mathrm{Tr}[\mathscr{T}(t, s)(G(s)))]$$

where $\Lambda(H)$ denotes the largest eigenvalue of the symmetric positive semidefinite matrix H. We deduce that if equation (2) has a solution which is positive semidefinite and bounded for $t \geqq 0$ then there exists $c'(t_0) > 0$ such that

$$\int_{t_0}^{t} \|\mathscr{T}(t, s)(G(s))\| \, \mathrm{d}s \leqq c'(t_0).$$

Theorem 1. *Assume $A(\cdot)$ and $\Pi(\cdot, X)$ are continuous and bounded for $t \geqq 0$ and there exist $T > 0$, $\tau \geqq 0$, $\gamma > 0$ such that for all $\sigma \geqq T$ the inequality $\int_{\sigma-\tau}^{\sigma} \mathscr{T}(\sigma, s)(G(s)) \, \mathrm{d}s \geqq \gamma I$ holds and (2) has a bounded positive semidefinite solution defined for $t \geqq T - \tau$. Then the linear equation associated with (2) (obtained for $G \equiv 0$) is exponentially stable.*

Proof. Since $\mathscr{T}(t, \sigma)$ for $t \geqq \sigma$ is a positive (monotone) linear operator, from the assumption we deduce the inequality

$$\mathscr{T}(t, \sigma)(I) \leqq \frac{1}{\gamma} \int_{\sigma-\tau}^{\sigma} \mathscr{T}(t, s)(G(s)) \, \mathrm{d}s$$

and since the trace is a linear positive functional we have

$$\mathrm{Tr}[\mathscr{T}(t, \sigma)(I)] \leqq \frac{1}{\gamma} \int_{\sigma-\tau}^{\sigma} \mathrm{Tr}[\mathscr{T}(t, s)(G(s))] \, \mathrm{d}s ,$$

hence

$$\Lambda[\mathscr{T}(t, \sigma)(I)] \leqq \frac{n}{\gamma} \int_{\sigma-\tau}^{\sigma} \Lambda[\mathscr{T}(t, s)(G(s))] \, \mathrm{d}s .$$

That means

$$\|\mathscr{T}(t, \gamma)(I)\| \leqq \frac{n}{\gamma} \int_{\sigma-\tau}^{\sigma} \|\mathscr{T}(t, s)(G(s))\| \, \mathrm{d}s$$

and since

$$\|\mathscr{T}(t, \sigma)\| = \sup_{\|H\|=1} \|\mathscr{T}(t, \sigma)(H)\| = \|\mathscr{T}(t, \sigma)(I)\|$$

we deduce that

$$\|\mathscr{T}(t, \sigma)\| \leqq \frac{n}{\gamma} \int_{\sigma-\tau}^{\sigma} \|\mathscr{T}(t, s)(G(s))\| \mathrm{d}s ,$$

hence

$$\int_{t_0}^{t} \|\mathscr{T}(t, \sigma)\| \, d\sigma \leqq \frac{n}{\gamma} \int_{t_0}^{t} \left[\int_{\sigma-\tau}^{\sigma} \|\mathscr{T}(t, s)(G(s))\| \, ds \right] d\sigma \quad \text{for all} \quad t \geqq t_0 \geqq T.$$

We proceed further as in Silverman and Anderson [8] to get

$$\int_{t_0}^{t} \left[\int_{\sigma-\tau}^{\sigma} \|\mathscr{T}(t, s)(G(s))\| \, ds \right] d\sigma \leqq \tau \int_{t_0-\tau}^{t} \|\mathscr{T}(t, \varrho)(G(\varrho))\| \, d\varrho .$$

Since we have assumed that (2) has a bounded positive semidefinite solution defined for $t \geqq T - \tau$ we deduce from the considerations following Proposition 1 that $\int_{T-\tau}^{t} \|\mathscr{T}(t, \varrho)(G(\varrho))\| \, d\varrho \leqq c$. Consequently,

$$\int_{T}^{t} \|\mathscr{T}(t, \sigma)\| \, d\sigma \leqq \frac{c\tau n}{\gamma}, \quad t \geqq T .$$

By the reasoning used in proving the Perron-Bellman Theorem (see for example Conti [1], Th. 2.10.2) we deduce from the above inequality (under the assumption of boundedness of coefficients) that the linear system is exponentially stable, that is,

$$\|\mathscr{T}(t, \sigma)\| \leqq k e^{-\alpha(t-\sigma)}, \quad t \geqq \sigma .$$

Remarks: 1. The above result may be compared with that due to Conti [1], [1], Th. 3.4.1, p. 61 for the case $\Pi(t, X) \equiv 0$, $G(t) \geq \gamma I$ and the equation "adjoint" to equation (2).

2. The uniform positivity condition in the statement implies that the bounded solution must satisfy $X(t) \geqq \gamma I$ for $t \geqq T$.

3. If all the coefficients are defined and bounded on the whole of $\mathbb{R}$, then from $\|\mathscr{T}(t, s)\| \leqq k e^{-\alpha(t-s)}$ we deduce existence of a solution bounded on the whole axis,

$$\tilde{X}(t) = \int_{-\infty}^{t} \mathscr{T}(t, s)(G(s)) \, ds .$$

4. From the representation formula (3) we see that $C(t, t)\, C^*(t, t) \leqq \mathscr{T}(t, t_0)(I)$, hence

$$|C(t, t_0)|^2 \leqq \|\mathscr{T}(t, t_0)\| \leqq k e^{-\alpha(t-t_0)}, \quad t \geqq t_0$$

and under the assumptions of Theorem 1 the linear system associated with A is also exponentially stable.

3. AFFINE STOCHASTIC EQUATIONS

Consider the Ito equation (1) and denote by $x(t, t_0, x_0)$ the solution defined for $t \geqq t_0 \geqq 0$ by the initial value x_0. Denote

$$m(t, t_0, x_0) = E x(t, t_0, x_0), \quad M(t, t_0, x_0) = \operatorname{cov}(x(t, t_0, x_0)) .$$

Then it is known that m is a solution of

(5) $$x' = A(t)\,x + f(t)\,, \quad x(t_0) = Ex_0$$

and M is a solution of equation (2) with

$$\Pi(t, x) = \sum_j B_j(t)\,X\,B_j^*(t)\,,$$

$$G(t) = \sum_j [B_j(t)\,m(t, t_0, x_0) + f_j(t)]\,[B_j(t)\,m(t, t_0, x_0) + f_j(t)]^*\,.$$

Recall that we have assumed all coefficient functions to be continuous and bounded for $t \geqq 0$. Assume there exists a weakly bounded solution in the sense that $m(t)$ and $M(t)$ are bounded for $t \geqq 0$. Assume further that there exist τ, γ such that

(6) $$\int_{\sigma-\tau}^{\sigma} \mathscr{T}(\sigma, s)\,[(\sum_j B_j(s)\,m(s) + f_j(s))\,(\sum_j B_j(s)\,m(s) + f_j(s))^*]\,\mathrm{d}s \geqq \gamma I$$
$$\text{for all} \quad \sigma \geqq \tau\,.$$

We may apply Theorem 1 with $T = \tau$ since (2) has the solution $M(t)$ which is positive semidefinite and bounded for $t \geqq 0$. We deduce that $\|\mathscr{T}(t, \sigma)\| \leqq k\mathrm{e}^{-\alpha(t-\sigma)}$ and this is equivalent to the fact that the linear system associated with (1) is exponentially stable in mean square [6].

The result may be stated in a form which allows a more transparent interpretation of condition (6). The integral in (6) is the value at σ of the covariance of a solution of (1) starting at $t = \sigma - \tau$ with an initial value having the expectation $m(\sigma - \tau)$ and covariance equal to zero; condition (6) requires for this solution to have a uniformly positive definite covariance after a time interval τ. Since for a deterministic function the covariance is zero, we may say that condition (6) requires that after a time interval of length τ the process starting with a covariance equal to zero becomes essentially stochastic with a uniformly positive definite covariance. Thus we can state

Theorem 2. *Assume $A(\cdot)$, $B_j(\cdot)$ are continuous and bounded and there exist τ and γ such that any solution of* (1) *starting at some $t_0 \geqq 0$ with covariance equal to zero will have after a time τ a covariance which is uniformly positive definite* ($\geqq \gamma I$). *Then if* (1) *has a solution with bounded expectation and bounded covariance, the corresponding linear equation is exponentially stable in mean square.*

Remarks 5. If we assume exponential stability in mean square then the linear equation associated with (2) (and according to Remark 4 also the linear equation associated with (5)) will be exponentially stable, and if f and f_j are bounded then both (2) and (5) will have all solutions bounded for $t \geqq 0$. It follows that all solutions of (1) with f and f_j bounded will be weakly bounded (with bounded mean and covariance).

6. In the special case $B_j = 0$ for all j, assumption (6) is equivalent to the uniform controllability of (A, F), $F = \mathrm{col}\,(f_j)$.

4. EXPONENTIAL STABILITY IN MEAN SQUARE AND MEAN SQUARE BOUNDED SOLUTIONS

In this section we will obtain more information from the assumption of exponential stability in mean square.

For $t \geqq t_0$ define $\mu(t, t_0, \cdot): M_n \to M_n$ by $\mu(t, t_0, \mu_0) = \mathsf{P} \circ [x(t, t_0, x_0)]^{-1}$ where x_0 is a random variable independent of the standard Wiener process w^{t_0} ($w = \operatorname{col}(w_j)$) such that $\mathsf{P} \circ x_0^{-1} = \mu_0$. Since the pathwise uniqueness holds for (1) (cf. [2]) the definition of $\mu(t, t_0, \mu_0)$ is independent of x_0. If $p(s, x, t, A)$ is the transition function associated with (1), then

$$\mu(t, t_0, \mu_0)(A) = \int p(t_0, x, t, A) \, \mathrm{d}\mu(x) . \tag{7}$$

Proposition 2. *The following assertions hold:*

(i) $\mu(t_0, t_0, \mu_0) = \mu_0$;

(ii) $\mu(t, s, \mu(s, t_0, \mu_0)) = \mu(t, t_0, \mu_0)$ *for* $t \geqq s \geqq t_0$;

(iii) $\mu(t, t_0, \cdot)$ *is continuous in the weak topology.*

Proof. (i) is direct from the definition, (ii) is a consequence of (7) and of the Chapman-Kolmogorov relations [2]; for (iii) let $(\mu_k)_k$ converges weakly to μ_0 and let $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ be bounded and continuous; by the Feller property [2] it follows that the function $\psi(x, t_0, t) = \int \varphi(y) \, p(t_0, x, t, \mathrm{d}y)$ is bounded and continuous for fixed t_0, t, hence

$$\int \psi(x, t_0, t) \, \mathrm{d}\mu_k(x) \to \int \psi(x, t_0, t) \, \mathrm{d}\mu_0(x)$$

and since

$$\int \psi(x, t_0, t) \, \mathrm{d}\nu(x) = \int \varphi(y) \, \mathrm{d}\mu(t, t_0, \nu)(y)$$

for every ν we deduce that

$$\int \varphi(y) \, \mathrm{d}\mu(t, t_0, \mu_k)(y) \to \int \varphi(y) \, \mathrm{d}\mu(t, t_0, \mu_0)(y) .$$

Proposition 3. *If the linear system associated to* (1) *is mean square exponentially stable then there exist* $K > 0$, $\tau > 0$ *such that*

$$\int |x|^2 \, \mathrm{d}\mu_0(x) \leqq K \quad \textit{implies} \quad \int |x|^2 \, \mathrm{d}\mu(t, t_0, \mu_0)(x) \leqq K \quad \textit{for} \quad t \geqq t_0 + \tau .$$

Proof. Recall that mean square exponential stability for the linear system associated with (1) implies $\|\mathscr{T}(t, s)\| \leqq k\mathrm{e}^{-\alpha(t-s)}$, $\|C(t, s)\|^2 \leqq k\mathrm{e}^{-\alpha(t-s)}$, $t \geqq s$; consider a solution of (1) and $m(t, t_0, x_0) = \mathsf{E}x(t, t_0, x_0)$, $M(t, t_0, x_0) = \operatorname{cov}(x(t, t_0, x_0))$. Then

$$m(t, t_0, x_0) = C(t, t_0)\,\mathsf{E}x_0 + \int_{t_0}^{t} C(t, s) f(s)\,\mathrm{d}s\,,$$

$$M(t, t_0, x_0) = \mathscr{T}(t, t_0)(\operatorname{cov}(x_0)) + \int_{t_0}^{t} \mathscr{T}(t, s)(G(s))\,\mathrm{d}s$$

with G as in the previous section. We have the standard estimates

$$|m(t, t_0, x_0)|^2 \leqq \beta_1 e^{-\alpha(t-t_0)}\,\mathsf{E}|x_0|^2 + \beta_2\,, \quad \|G(t)\| \leqq \beta_3 e^{-\alpha(t-t_0)}\,\mathsf{E}|x_0|^2 + \beta_4$$

and we deduce that

$$\|M(t, t_0, x_0)\| \leqq k e^{-\alpha(t-t_0)}\|\operatorname{cov}(x_0)\| + \beta_5 e^{-\alpha(t-t_0)}(t - t_0)\,\mathsf{E}|x_0|^2 + \beta_6\,.$$

On the other hand,

$$M(t, t_0, x_0) = \mathsf{E}x(t, t_0, x_0)\,x^*(t, t_0, x_0) - m(t, t_0, x_0)\,m^*(t, t_0, x_0)$$

and we deduce

$$\mathsf{E}|x(t, t_0, x_0)|^2 \leqq \beta_7 e^{-\hat{\alpha}(t-t_0)}\mathsf{E}|x_0|^2 + \beta_8\,, \quad 0 < \hat{\alpha} < \alpha\,.$$

We take $K = 2\beta_8$, τ such that $\beta_7 e^{-\hat{\alpha}\tau} \leqq \frac{1}{2}$ and deduce that $\mathsf{E}|x_0|^2 \leqq K$ and $t > t_0 + \tau$ imply $\mathsf{E}|x(t, t_0, x_0)|^2 \leqq K$. Let now μ_0 be such that $\int |x|^2\,\mathrm{d}\mu_0(x) \leqq K$, let x_0 be independent of w^{t_0} and such that $\mathsf{P} \circ x_0^{-1} = \mu_0$, let $x(t, t_0, x_0)$ be the corresponding solution; then

$$\int |x|^2\,\mathrm{d}\mu(t, t_0, \mu_0)(x) = \mathsf{E}|x(t, t_0, x_0)|^2 \leqq K \quad \text{for} \quad t \geqq t_0 + \tau$$

and the proposition is proved.

Remark 7. Instead of the expectation and covariance we could have used in the above proof a Liapunov function argument, which holds in more general, nonlinear situations. Assume there exists $V: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ of class C^1 in the first argument and C^2 in the second. Assume V has the properties

(i) $\alpha|x|^2 \leqq V(t, x) \leqq \beta|x|^2$, $\alpha > 0$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$;
(ii) $(\mathscr{L}V)(t, x) \leqq -\gamma V(t, x)$, $t \in \mathbb{R}$, $|x| \geqq r_0$;
(iii) $(\mathscr{L}V)(t, x) \leqq c$, $t \in \mathbb{R}$, $|x| \leqq r_0$.

Then for $\delta > c + \gamma\beta r_0^2$ we have $(\mathscr{L}V)(t, x) \leqq -\gamma V(t, x) + \delta$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and Ito's formula gives

$$\mathsf{E}V(t, x(t, t_0, x_0)) - \mathsf{E}V(t_0, x_0) = \int_{t_0}^{t} \mathsf{E}(\mathscr{L}V)(\tau, x(\tau, t_0, x_0))\,\mathrm{d}\tau\,.$$

We deduce

$$\mathsf{E}V(t, x(t, t_0, x_0)) \leqq e^{-\gamma(t-t_0)}\mathsf{E}V(t_0, x_0) + \frac{\delta}{\gamma}$$

and hence $\mathsf{E}|x(t, t_0, x_0)|^2 \leqq \beta_1, e^{-\gamma(t-t_0)}\mathsf{E}|x_0|^2 + \beta_2, t \geqq t_0$

In the linear case considered $V(t, x) = \int_t^\infty \mathsf{E}|y(s, t, x)|^2 \, ds = x^* P(t) x$ where y is the solution of the linear equation associated with (1); we have $(\mathscr{L}V)(t, x) = -|x|^2 + 2x^* b(t) + r(t)$, $b(t) = P(t) f(t) \sum_j B_j^*(t) P(t) B_j(t)$, $r(t) = \sum_j f_j^*(t) P(t) f_j(t)$.

Proposition 4. *If the linear system associated with* (1) *is mean square exponentially stable and τ, K are as in Proposition 3 then the family $\{\mu(t, t_0, \mu_0)\}_{t \geqq t_0 + \tau}$ with $\int |x|^2 \, d\mu_0(x) \leqq K$ is relatively compact in the weak topology.*

Proof. We have

$$\mu(t, t_0, \mu_0)(\{|x| \geqq r\}) \leqq \frac{1}{r^2} \int |x|^2 \, d\mu(t, t_0, \mu_0)(x) \leqq \frac{K}{r^2}$$

for $t \geqq t_0 + \tau$, hence $\lim_{r \to \infty} \mu(t, t_0, \mu_0)(\{|x| \geqq r\}) = 0$ uniformly in $t \geqq t_0 + \tau$; the conclusion follows from the theorem of Prohorov [7], [3].

We shall now use an idea of J. Kurzweil [4] to deduce

Theorem 3. *Assume the linear equation associated with* (1) *is mean square exponentially stable. Then there exist $K > 0$ and a unique function $\hat{\mu}: \mathbb{R} \to M_n$ such that*

$$\sup_t \int |x|^2 \, d\hat{\mu}(t)(x) \leqq K, \tag{8}$$

$$\mu(t, t_0, \hat{\mu}(t_0)) = \hat{\mu}(t) \quad \text{for} \quad t \geqq t_0. \tag{9}$$

Proof. Fix $t \in \mathbb{R}$ and consider a sequence $t_j \to -\infty$. Let K, τ be the constants from Proposition 3. Let ε_0 be the Dirac measure concentrated at zero; from Proposition 4 we deduce that $\{\mu(t, t_j, \varepsilon_0)\}_j$ is relatively compact for $j \geqq j_0$, hence there exists a subsequence $(t_{j_k})_k$ such that $\mu(t, t_{j_k}, \varepsilon_0)$ converges weakly to an element $\hat{\mu}(t) \in M_n$. We have

$$\int |x|^2 \, d\hat{\mu}(t)(x) = \lim_{N \to \infty} \int \min(|x|^2, N) \, d\hat{\mu}(t)(x) =$$

$$= \lim_{N \to \infty} \lim_{k \to \infty} \int \min(|x|^2, N) \, d\mu(t, t_{j_k}, \varepsilon_0)(x) \leqq \sup_k \int |x|^2 \, d\mu(t, t_{j_k}, \varepsilon_0)(x) \leqq K.$$

From the identity $\mu(t, t_0, \mu(t_0, t_{j_k}, \varepsilon_0)) = \mu(t, t_{j_k}, \varepsilon_0)$ for $k \geqq k(t_0)$ and from Proposition 2(iii) we obtain $\mu(t, t_0, \hat{\mu}(t_0)) = \hat{\mu}(t)$.

Let now $\tilde{\mu} \in M_n$ with (8), (9). Choose $\hat{x}_0$ and $\tilde{x}_0$ independent of w^{t_0} and such that $\mathsf{P} \circ \hat{x}_0^{-1} = \hat{\mu}(t_0)$, $\mathsf{P} \circ \tilde{x}_0^{-1} = \tilde{\mu}(t_0)$; consider the corresponding solutions defined for $t \geqq t_0$.

Then for $u \in \mathbb{R}^n$ we have

$$\left|\int e^{iu^*x} \, d\hat{\mu}(t)(x) - \int e^{iu^*x} \, d\tilde{\mu}(t)(x)\right| = \left|\int e^{iu^*x} \, d\mu(t, t_0, \hat{\mu}(t_0))(x) - \right.$$

$$\left. - \int e^{iu^*x} \, d\mu(t, t_0, \tilde{\mu}(t_0))(x)\right| = \left|\mathsf{E}e^{iu^*x(t,t_0,\hat{x}_0)} - \mathsf{E}e^{iu^*x(t,t_0,\tilde{x}_0)}\right| \leqq$$

$$\leqq |u| \, \mathsf{E}|x(t, t_0, \hat{x}_0) - x(t, t_0, \tilde{x}_0)| \leqq |u| \left(E|x(t, t_0, \hat{x}_0) - x(t, t_0, \tilde{x}_0)|^2\right)^{1/2}.$$

But $x(t, t_0, \hat{x}_0) - x(t, t_0, \tilde{x}_0)$ is a solution of the linear equation associated with (1) with the initial value $\hat{x}_0 - \tilde{x}_0$ and the mean square exponential stability yields

$$\mathsf{E}|x(t, t_0, \hat{x}_0) - x(t, t_0, \tilde{x}_0)|^2 \leqq C e^{-\alpha(t-t_0)} E|\hat{x}_0 - \tilde{x}_0|^2.$$

Further we have

$$\mathsf{E}|\hat{x}_0|^2 = \int |x|^2 \, d\hat{\mu}(t_0)(x) \leqq K$$

and the same for $E|\tilde{x}_0|^2$, hence

$$\left(\mathsf{E}|x(t, t_0, \hat{x}_0) - x(t, t_0, \tilde{x}_0)|^2\right)^{1/2} \leqq 2\sqrt{(KC)} \, e^{-(\alpha/2)(t-t_0)}$$

and

$$\left|\int e^{iu^*x} \, d\hat{\mu}(t)(x) - \int e^{iu^*x} \, d\tilde{\mu}(t)(x)\right| \leqq 2\sqrt{(KC)} \, e^{-(\alpha/2)(t-t_0)}|u|.$$

Now we may let $t_0 \to -\infty$ to deduce that

$$\int e^{iu^*x} \, d\tilde{\mu}(t) = \int e^{iu^*x} \, d\hat{\mu}(t),$$

hence $\hat{\mu}(t) = \tilde{\mu}(t)$ and the theorem is proved.

Remarks: 8. The property of $\hat{\mu}$ to define a trajectory of the flow may be written as $\int p(t_0, x, t, A) \, d\hat{\mu}(t_0)(x) = \hat{\mu}(t)(A)$ and can be interpreted as an invariance property.

Let us stress that $\hat{\mu}$ is what Paul-André Meyer [5] calls after Neveu a "loi d'entrée normée" which is nontrivial just because it is defined on the whole of $\mathbb{R}$ (a set without a first element).

9. The boundedness property (8) implies that $\hat{m}(t) = \mathsf{E}\hat{\mu}(t)$ $M(t) = \operatorname{cov} \hat{\mu}(t)$ are bounded solutions of (5) and (2), respectively, with G as in Section 3.

10. If the coefficients are θ-periodic then

$$p(t_0 + \theta, x, t + \theta, A) = p(t_0, x, t, A) \quad \text{for all} \quad t_0, t, x, A$$

and this directly implies that $\hat{\mu}(t + \theta) = \hat{\mu}(t)$, as in Morozan [6] a periodic solution (unique in the sense of the probability law) will exist for (1). Analogous conclusions may be obtained in the almost periodic case.

11. Consider the special case $B_j(t) \equiv 0$ for all j. The standard representation formula yields that if x_0 is Gaussian then $x(t, t_0, x_0)$ is Gaussian and equivalently if μ_0 is in G_n then $\mu(t, t_0, \mu_0)$ is in G_n and we have a flow in G_n. We may introduce in G a distance

$$d_G(\mu_1, \mu_2) = |\mathsf{E}\mu_1 - \mathsf{E}\mu_2| + \|\operatorname{cov}(\mu_1) - \operatorname{cov}(\mu_2)\|$$

and G_n becomes a complete metric space. From Proposition 3 we deduce that there exist R, τ such that if $d_G(\mu_0, \varepsilon_0) \leqq K$ then $d_G(\mu(t, t_0, \mu_0), \varepsilon_0) \leqq K$ for $t \geqq t_0 + \tau$.

We may now obtain the result from Theorem 3 with $\hat{\mu}(t) \in G_n$. The use of Proposition 4 in the proof (based on the Prohorov theorem) we may replace in this special case by a direct argument using the completeness of G_n.

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