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SPECIAL GRASSMAN MANIFOLDS V_3^4 IN THE PROJECTIVE SPACE P_7

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Summary. Some results in the geometry of four-parametric manifolds of three-dimensional spaces in the projective space P_7 are found. The properties of these manifolds with characteristics containing the straight lines are studied. In particular, the properties of the manifolds with the characteristics containing the lines of the tetrahedra are found.

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This paper contains some results in the geometry of four-parametric manifolds of three-dimensional spaces in the projective space P_7. The properties of these manifolds with characteristics containing the straight lines are found. In particular, the properties of the manifolds with the characteristics containing the lines of the tetrahedra are studied.

Let us consider a four-parametric manifold V_3^4 in the projective seven-dimensional space P_7. Let the three-dimensional linear spaces P_3 be the generators of V_3^4. To each space P_3 we associate a frame consisting of independent points $A_i \in P_3$, $\bar{A}_i$; $i = 1, 2, 3, 4$.

The fundamental equations of the moving frame are

$$(1)\qquad \mathrm{d}A_i = \omega_i^s A_s + \varphi_i^{kr}\omega_r \bar{A}_k\,,$$

$$\mathrm{d}\bar{A}_i = \hat{\omega}_i^s A_s + \bar{\omega}_i^s \bar{A}_s\,;\quad i, k, r, s = 1, 2, 3, 4\,;$$

$$(2)\qquad \mathrm{d}\omega_i^k = \omega_i^s \wedge \omega_s^k + \varphi_i^{sr}\omega_r \wedge \hat{\omega}_s^k\,,$$

$$\mathrm{d}(\varphi_i^{kr}\omega_r) = \omega_i^s \wedge \varphi_s^{kr}\omega_r + \varphi_i^{sr}\omega_r \wedge \bar{\omega}_s^k\,,$$

$$\mathrm{d}\hat{\omega}_i^k = \hat{\omega}_i^s \wedge \omega_s^k + \bar{\omega}_i^s \wedge \hat{\omega}_s^k\,,$$

$$\mathrm{d}\bar{\omega}_i^k = \hat{\omega}_i^s \wedge \varphi_s^{kr}\omega_r + \bar{\omega}_i^s \wedge \bar{\omega}_s^k\,;$$

φ_i^{ks} are the functions of the parameters of the frame, $\omega_1, \omega_2, \omega_3, \omega_4$ are independent principal forms.

In all the following calculations we set $i, j, k, r, s = 1, 2, 3, 4$.

The tangent space of V_3^4 at the point $M = x^i A_i$ (the space $T(V_3^4, M)$) is the linear space determined by the points A_i and the points $\mathrm{d}M$. The point M is called the *focal point of* V_3^4 if the dimension of $T(V_3^4, M)$ is less than 7. In this case we get

(3) $$\det(x^i \varphi_i^{k1}, x^i \varphi_i^{k2}, x^i \varphi_i^{k3}, x^i \varphi_i^{k4}) = 0\,.$$

There exist functions $\overset{1}{\varrho}_M, \overset{2}{\varrho}_M, \overset{3}{\varrho}_M, \overset{4}{\varrho}_M$ such that

(4) $$\overset{1}{\varrho}_M x^i \varphi_i^{k1} + \overset{2}{\varrho}_M x^i \varphi_i^{k2} + \overset{3}{\varrho}_M x^i \varphi_i^{k3} + \overset{4}{\varrho}_M x^i \varphi_i^{k4} = 0$$

holds. We shall assume that $\dim T(V_3^4, M) = 6$. There exists $p \in \{1, 2, 3, 4\}$ such that $\overset{p}{\varrho}_M \neq 0$ holds. From (4), (1) we obtain

$$\mathrm{d}M = \sum_{c=1}^{3} x^i \varphi_i^{k, p\oplus c} \bar{A}_k \left[\omega_{p\oplus c} - \frac{\overset{p\oplus c}{\varrho}_M}{\overset{p}{\varrho}_M} \omega_p \right] + (\ldots)\,.$$

$(\ldots)$ is the linear combination of A_1, A_2, A_3, A_4. If $p + c \leqq 4$, then $p \oplus c = p + c$. If $p + c > 4$, then $p \oplus c = (p + c) - 4$.

The form $\varkappa^i \omega_i$ is *the torsal form* corresponding to the focal point $M \in P_3$ if the dimension of the tangent subspace $\varkappa^i \omega_i = 0$ of $T(V_3^4, M)$ is less than 6. The forms

$$\omega_{p\oplus 1} - \frac{\overset{p\oplus 1}{\varrho}_M}{\overset{p}{\varrho}_M} \omega_p\,, \quad \omega_{p\oplus 2} - \frac{\overset{p\oplus 2}{\varrho}_M}{\overset{p}{\varrho}_M} \omega_p\,, \quad \omega_{p\oplus 3} - \frac{\overset{p\oplus 3}{\varrho}_M}{\overset{p}{\varrho}_M} \omega_p$$

are linearly dependent. Consequently,

$$\overset{1}{\varrho}_M \varkappa^1 + \overset{2}{\varrho}_M \varkappa^2 + \overset{3}{\varrho}_M \varkappa^3 + \overset{4}{\varrho}_M \varkappa^4 = 0\,.$$

We shall assume that A_i are the focal points of V_3^4 and $\dim T(V_3^4, A_i) = 6$ for all $i = 1, 2, 3, 4$. For all values of i the rank of the matrices $(\varphi_i^{k1}, \varphi_i^{k2}, \varphi_i^{k3}, \varphi_i^{k4})$ is equal to 3. Generally the spaces $T(V_3^4, A_i)$, $i = 1, 2, 3, 4$, have only the common space (A_1, A_2, A_3, A_4). We shall study the general case.

Let p, q, m, n be different numbers from $\{1, 2, 3, 4\}$. The spaces $T(V_3^4, A_p)$, $T(V_3^4, A_q)$, $T(V_3^4, A_m)$ contain the common linear four-dimensional space. In this space we situate the point $\bar{A}_n$ (for all $n = 1, 2, 3, 4$). Hence

(5) $$\varphi_i^{ir} = 0 \quad (\text{for all } i = 1, 2, 3, 4)\,.$$

According to (4) there exist the functions $\overset{1}{\varrho}_i, \overset{2}{\varrho}_i, \overset{3}{\varrho}_i, \overset{4}{\varrho}_i$ such that

(6) $$\overset{1}{\varrho}_i \varphi_i^{k1} + \overset{2}{\varrho}_i \varphi_i^{k2} + \overset{3}{\varrho}_i \varphi_i^{k3} + \overset{4}{\varrho}_i \varphi_i^{k4} = 0\,,$$

holds. For the coefficients of the torsal forms $\varkappa^s \omega_s$ of A_i we have

(7) $$\varkappa^1 \overset{1}{\varrho}_i + \varkappa^2 \overset{2}{\varrho}_i + \varkappa^3 \overset{3}{\varrho}_i + \varkappa^4 \overset{4}{\varrho}_i = 0\,.$$

We shall assume that the points A_1, A_2, A_3, A_4 have no common torsal form. The system (7) for $i = 1, 2, 3, 4$ has no other solution than

$$\varkappa^1 = \varkappa^2 = \varkappa^3 = \varkappa^4 = 0\,.$$

The points A_p, A_q, A_m have a common torsal form $\tilde{\omega}_n$. The forms $\tilde{\omega}_n$ for $n = 1, 2, 3, 4$ are linearly independent principal forms. It is possible to choose the forms ω_i so that $\omega_i = \tilde{\omega}_i$ holds.

It results that

$$\mathrm{d}A_i = \omega_i^k A_k + \varphi_i^{kr} \omega_r \bar{A}_k , \quad i \neq k , \quad i \neq r . \tag{8}$$

We shall assume that the line (A_p, A_q) belongs to the focal manifold of V_3^4 in the generator space P_3. Then (3) yields

$$\det \left(\varphi_q^{kp}, \varphi_p^{kq}, x^p \varphi_p^{km} + x^q \varphi_q^{km}, x^p \varphi_p^{kn} + x^q \varphi_q^{kn} \right) = 0 \tag{9}$$

for all values of x^p, x^q.

From (9) we obtain

$$\varphi_q^{pp} = 0 , \quad \varphi_p^{qq} = 0 , \tag{10}$$

and at least one of the following conditions

$$\varphi_p^{mq} \varphi_q^{np} - \varphi_p^{nq} \varphi_q^{mp} = 0 , \tag{11}$$

$$\varphi_p^{qm} \varphi_q^{pn} - \varphi_q^{pm} \varphi_p^{qn} = 0 . \tag{12}$$

Let Φ_{pq} be the two-parametric Pfaff manifold of V_3^4 determined by the equations $\omega_m = \omega_n = 0$.

Lemma 1a. *Let the equations* (10), (11) *be satisfied. Then the focal manifold of V_3^4 in P_3 contains the line (A_p, A_q). At all points of (A_p, A_q) the Pfaff manifold Φ_{pq} has the common four-dimensional tangent space.*

Proof. From (10), (8) we easily obtain

$$\begin{aligned} \mathrm{d}(x^p A_p + x^q A_q) &= x^p [\varphi_p^{mq} \bar{A}_m + \varphi_p^{nq} \bar{A}_n] \, \omega_q + \\ &+ x^q [\varphi_q^{mp} \bar{A}_m + \varphi_q^{np} \bar{A}_n] \, \omega_p + x^p [\varphi_p^{sm} \omega_m \bar{A}_s + \varphi_p^{sn} \omega_n \bar{A}_s] + \\ &+ x^q [\varphi_q^{im} \omega_m \bar{A}_i + \varphi_q^{in} \omega_n \bar{A}_i] + (\ldots) , \quad s \neq p , \quad i \neq q . \end{aligned} \tag{13}$$

$(\ldots)$ is the linear combination of A_1, A_2, A_3, A_4. The condition (11) implies

$$\varphi_q^{mp} \bar{A}_m + \varphi_q^{np} \bar{A}_n = \Theta [\varphi_p^{mq} \bar{A}_m + \varphi_p^{nq} \bar{A}_n] .$$

Θ is the function of the parameters of the frame. The tangent space of Φ_{pq} at the points $x^p A_p + x^q A_q$ is given by

$$A_1, A_2, A_3, A_4, \varphi_p^{mq} \bar{A}_m + \varphi_p^{nq} \bar{A}_n .$$

This space is four-dimensional and does not depend on x^p, x^q.

Let Φ_t be the three-parametric Pfaff manifold of V_3^4 determined by the equation

$$\varphi_p^{qm} \omega_m + \varphi_p^{qn} \omega_n = 0 .$$

Lemma 1b. *Let the equations* (10), (12) *be satisfied. Then the focal manifold of V_3^4 in P_3 contains the line (A_p, A_q). The form $\varphi_p^{qm} \omega_m + \varphi_p^{qn} \omega_n$ is the torsal form* *for all points of* (A_p, A_q). *At all points of* (A_p, A_q) *the Pfaff manifold* Φ_t *has the common tangent space.*

Proof. The condition (12) yields

$$\varphi_q^{pm}\omega_m + \varphi_q^{pn}\omega_n = \hat{\Theta}[\varphi_p^{qm}\omega_m + \varphi_p^{qn}\omega_n],$$

where $\hat{\Theta}$ is the function of the parameters of the frame.

From (13) we obtain

$$\mathrm{d}(x^p A_p + x^q A_q) = (\varphi_p^{qm}\omega_m + \varphi_p^{qn}\omega_n)[x^p \bar{A}_q + \hat{\Theta} x^q \bar{A}_p] + [\ldots].$$

$[\ldots]$ is the linear combination of $A_1, A_2, A_3, A_4, \bar{A}_m, \bar{A}_n$. The form $\varphi_p^{qm}\omega_m + \varphi_p^{qn}\omega_n$ is the torsal form of V_3^4 at all points of (A_p, A_q). At all points of (A_p, A_q) the Pfaff manifold Φ_t has the common tangent space determined by $A_1, A_2, A_3, A_4, \bar{A}_m, \bar{A}_n$.

We shall solve the second case. We shall assume that the equations (10) and (12) hold for all pairs of $p, q \in \{1, 2, 3, 4\}$, $p \neq q$. The focal manifold contains the edges of the tetrahedron $A_1A_2A_3A_4$.

According to (10), (12), the forms

$$\varphi_p^{qr}\omega_r = \varphi_p^{qm}\omega_m + \varphi_p^{qn}\omega_n,$$
$$\varphi_q^{pr}\omega_r = \varphi_q^{pm}\omega_m + \varphi_q^{pn}\omega_n$$

are for each pair (p, q) linearly dependent.

It is possible to use the specification

$$\varphi_p^{qr}\omega_r = \lambda_{pq}\omega_{pq}, \quad \varphi_q^{pr} = \lambda_{qp}\omega_{pq}.$$

The equations of the frame are

$$\text{(14)} \qquad \mathrm{d}A_i = \omega_i^s A_s + \sum_j \lambda_{ij}\omega_{ij}\bar{A}_j,$$
$$\mathrm{d}\bar{A}_i = \hat{\omega}_i^s A_s + \bar{\omega}_i^s \bar{A}_s,$$
$$\omega_{ij} = \omega_{ji}, \quad j \neq i.$$

Using the structure equations of P_7 we obtain from (14)

$$\text{(15)} \qquad (\lambda_{ji}\omega_i^j - \lambda_{ij}\bar{\omega}_j^i) \wedge \omega_{ij} + (\lambda_{ri}\omega_i^r - \lambda_{ir}\bar{\omega}_r^i) \wedge \omega_{ir} + (\lambda_{ki}\omega_i^k - \lambda_{ik}\bar{\omega}_k^i) \wedge \omega_{ik} = 0,$$

$$\text{(16)} \qquad \mathrm{d}\lambda_{ij} \wedge \omega_{ij} + \lambda_{ij}\,\mathrm{d}\omega_{ij} = (\omega_i^i - \bar{\omega}_j^j) \wedge \lambda_{ij}\omega_{ij} + \omega_i^r \wedge \lambda_{rj}\omega_{rj} + \omega_i^k \wedge \lambda_{kj}\omega_{kj} + \lambda_{ir}\omega_{ir} \wedge \bar{\omega}_r^j + \lambda_{ik}\omega_{ik} \wedge \bar{\omega}_k^j,$$

$$\mathrm{d}\lambda_{ji} \wedge \omega_{ij} + \lambda_{ji}\,\mathrm{d}\omega_{ij} = (\omega_j^j - \bar{\omega}_i^i) \wedge \lambda_{ji}\omega_{ij} + \omega_j^r \wedge \lambda_{ri}\omega_{ri} + \omega_j^k\lambda_{ki} \wedge \omega_{ki} + \lambda_{jr}\omega_{jr} \wedge \bar{\omega}_r^i + \lambda_{jk}\omega_{jk} \wedge \bar{\omega}_k^i;$$

i, j, k, r are different numbers of $\{1, 2, 3, 4\}$. The equations (16) imply

$$(17)\qquad [\lambda_{ij}\,d\lambda_{ji} - \lambda_{ji}\,d\lambda_{ij} + \lambda_{ij}\lambda_{ji}(\omega_i^i - \omega_j^j + \bar{\omega}_i^i - \bar{\omega}_j^j)] \wedge \omega_{ij} - $$
$$- (\lambda_{ri}\lambda_{ij}\omega_j^r + \lambda_{ji}\lambda_{ir}\bar{\omega}_r^j) \wedge \omega_{ri} - (\lambda_{ki}\lambda_{ij}\omega_j^k + \lambda_{ji}\lambda_{ik}\bar{\omega}_k^j) \wedge \omega_{ik} +$$
$$+ (\lambda_{rj}\lambda_{ji}\omega_i^r + \lambda_{ij}\lambda_{jr}\bar{\omega}_r^i) \wedge \omega_{jr} + (\lambda_{kj}\lambda_{ji}\omega_i^k + \lambda_{ij}\lambda_{jk}\bar{\omega}_k^i) \wedge \omega_{jk} = 0\,.$$

We denote by $\lambda_{ijk}, \lambda_{ijkr}, \Omega_i^k, \Omega_{ij}^k, \Gamma_{ij}$ the following expressions:

$$\lambda_{ijk} = \lambda_{ik}\lambda_{kj}\lambda_{ji} + \lambda_{jk}\lambda_{ki}\lambda_{ij} = \lambda_{kji}\,,$$
$$\lambda_{ikjr} = \lambda_{ik}\lambda_{kj}\lambda_{jr}\lambda_{ri} - \lambda_{rj}\lambda_{jk}\lambda_{ki}\lambda_{ir} = -\lambda_{rjki}\,,$$
$$(18)\qquad \Omega_i^k = \lambda_{ki}\omega_i^k - \lambda_{ik}\bar{\omega}_k^i\,,$$
$$\Omega_{ij}^k = \lambda_{ji}\lambda_{kj}\omega_i^k + \lambda_{ij}\lambda_{jk}\bar{\omega}_k^i\,,$$
$$\Gamma_{ij} = \lambda_{ij}\,d\lambda_{ji} - \lambda_{ji}\,d\lambda_{ij} + \lambda_{ij}\lambda_{ji}(\omega_i^i - \omega_j^j + \bar{\omega}_i^i - \bar{\omega}_j^j) = -\Gamma_{ji}\,.$$

Using (18) we obtain from (15), (17)

$$(19)\qquad \Omega_i^j \wedge \omega_{ij} + \Omega_i^r \wedge \omega_{ir} + \Omega_i^k \wedge \omega_{ik} = 0\,,$$

$$(20)\qquad \Gamma_{ij} \wedge \omega_{ij} - \Omega_{ji}^r \wedge \omega_{ir} - \Omega_{ji}^k \wedge \omega_{ik} + \Omega_{ij}^r \wedge \omega_{jr} + \Omega_{ij}^k \wedge \omega_{jk} = 0\,.$$

$\Omega_i^j, \Omega_i^r, \Omega_i^k$ are the principal forms.

Four of the six forms ω_{ij} $(i \neq j,\ i, j \in \{1, 2, 3, 4\},\ \omega_{ij} = \omega_{ji})$ are linearly independent. We shall assume that $\omega_{12}, \omega_{23}, \omega_{34}, \omega_{41}$ are the independent forms. For the remaining forms we obtain

$$(21)\qquad \omega_{13} = \varkappa^{12}\omega_{12} + \varkappa^{23}\omega_{23} + \varkappa^{34}\omega_{34} + \varkappa^{41}\omega_{41}\,,$$
$$\omega_{24} = \varrho^{12}\omega_{12} + \varrho^{23}\omega_{23} + \varrho^{34}\omega_{34} + \varrho^{41}\omega_{41}\,.$$

The equations of the frame have the following form:

$$(22)\qquad dA_1 = \omega_1^s A_s + \lambda_{12}\omega_{12}\bar{A}_2 + \lambda_{13}(\varkappa^{12}\omega_{12} + \varkappa^{23}\omega_{23} + \varkappa^{34}\omega_{34} +$$
$$+ \varkappa^{41}\omega_{41})\,\bar{A}_3 + \lambda_{14}\omega_{41}\bar{A}_4\,,$$
$$dA_2 = \omega_2^s A_s + \lambda_{21}\omega_{12}\bar{A}_1 + \lambda_{23}\omega_{23}\bar{A}_3 + \lambda_{24}(\varrho^{12}\omega_{12} + \varrho^{23}\omega_{23} +$$
$$+ \varrho^{34}\omega_{34} + \varrho^{41}\omega_{41})\,\bar{A}_4\,,$$
$$dA_3 = \omega_3^s A_s + \lambda_{31}(\varkappa^{12}\omega_{12} + \varkappa^{23}\omega_{23} + \varkappa^{34}\omega_{34} + \varkappa^{41}\omega_{41})\,\bar{A}_1 +$$
$$+ \lambda_{32}\omega_{23}\bar{A}_2 + \lambda_{34}\omega_{34}\bar{A}_4\,,$$
$$dA_4 = \omega_4^s A_s + \lambda_{41}\omega_{41}\bar{A}_1 + \lambda_{42}(\varrho^{12}\omega_{12} + \varrho^{23}\omega_{23} + \varrho^{34}\omega_{34} +$$
$$+ \varrho^{41}\omega_{41})\,\bar{A}_2 + \lambda_{43}\omega_{34}\bar{A}_3\,,$$
$$d\bar{A}_i = \hat{\omega}_i^s A_s + \bar{\omega}_i^s \bar{A}_s\,;\quad s, i = 1, 2, 3, 4\,.$$

We find the equation of the characteristic of V in the space P_3. Let $M = x^i A_i$ $(i = 1, 2, 3, 4)$ be a point of P_3. Using (22) we obtain the following equation of the characteristic:

$$\begin{vmatrix} x^2\lambda_{21} + x^3\lambda_{31}\varkappa^{12} & x^1\lambda_{12} + x^4\lambda_{42}\varrho^{12} & x^1\lambda_{13}\varkappa^{12} & x^2\lambda_{24}\varrho^{12} \\ x^3\lambda_{31}\varkappa^{23} & x^3\lambda_{32} + x^4\lambda_{42}\varrho^{23} & x^1\lambda_{13}\varkappa^{23} + x^2\lambda_{23} & x^2\lambda_{24}\varrho^{23} \\ x^3\lambda_{31}\varkappa^{34} & x^4\lambda_{42}\varrho^{34} & x^1\lambda_{13}\varkappa^{34} + x^4\lambda_{43} & x^2\lambda_{24}\varrho^{34} + x^3\lambda_{34} \\ x^3\lambda_{31}\varkappa^{41} + x^4\lambda_{41} & x^4\lambda_{42}\varrho^{41} & x^1\lambda_{13}\varkappa^{41} & x^2\lambda_{24}\varrho^{41} + x^1\lambda_{14} \end{vmatrix} = 0.$$

This equation implies

$$\begin{aligned}(23)\qquad & (x^1)^2\, x^2x^3\varkappa^{34}\lambda_{14}\lambda_{123} + (x^1)^2\, x^2x^4(\varrho^{23}\varkappa^{34} - \varrho^{34}\varkappa^{23})\,\lambda_{13}\lambda_{124} - \\ & - (x^1)^2\, x^3x^4\varkappa^{23}\lambda_{12}\lambda_{143} - (x^2)^2\, x^1x^3(\varrho^{34}\varkappa^{41} - \varrho^{41}\varkappa^{34})\,\lambda_{24}\lambda_{123} - \\ & - (x^2)^2\, x^1x^4\varrho^{34}\lambda_{23}\lambda_{142} + (x^2)^2\, x^3x^4\varrho^{41}\lambda_{21}\lambda_{234} - (x^3)^2\, x^1x^2\varkappa^{41}\lambda_{34}\lambda_{132} + \\ & + (x^3)^2\, x^1x^4\varkappa^{12}\lambda_{32}\lambda_{143} - (x^3)^2\, x^2x^4(\varrho^{12}\varkappa^{41} - \varrho^{41}\varkappa^{12})\,\lambda_{31}\lambda_{234} + \\ & + (x^4)^2\, x^1x^2\varrho^{23}\lambda_{43}\lambda_{142} - (x^4)^2\, x^1x^3(\varrho^{12}\varkappa^{23} - \varrho^{23}\varkappa^{12})\,\lambda_{42}\lambda_{143} - \\ & - (x^4)^2\, x^2x^3\varrho^{12}\lambda_{41}\lambda_{243} + x^1x^2x^3x^4[-(\varrho^{12}\varkappa^{34} - \varrho^{34}\varkappa^{12})\,\lambda_{1324} - \\ & - (\varrho^{23}\varkappa^{41} - \varrho^{41}\varkappa^{23})\,\lambda_{1342} - \lambda_{1234}] = 0\,.\end{aligned}$$

Let $i,, j, k, r$ be different numbers of $\{1, 2, 3, 4\}$. If $\lambda_{ijk} = 0$ holds, then the plane $x^r = 0$ belongs to the characteristic. We shall assume that $\lambda_{ijk} \neq 0$ for all triples i, j, k.

According to (18) the forms $\Omega_i^k, \Omega_{ij}^k$ for fixed i, j, k are linearly independent combinations of $\omega_i^k, \bar{\omega}_k^i$ $(i \neq k)$.

From (18) we obtain

$$(24)\qquad \Omega_{ir}^k = A_{ir}^k\Omega_i^k + B_{ir}^k\Omega_{ij}^k\,,$$

$$A_{ir}^k = \frac{\lambda_{ijkr}}{\lambda_{kij}}\,,\quad B_{ir}^k = \frac{\lambda_{kir}}{\lambda_{kij}}\,.$$

Each of the planes (A_i, A_j, A_k) intersects the focal manifold of V_3^4 in S_3 in three lines (A_i, A_j), (A_j, A_k), (A_k, A_i) and in the line p_r.

These lines have the equations

$$(25a)\qquad x^1\varkappa^{34}\lambda_{14} - x^2(\varrho^{34}\varkappa^{41} - \varrho^{41}\varkappa^{34})\,\lambda_{24} - x^3\varkappa^{41}\lambda_{34} = 0\,,\quad x^4 = 0\,;$$

$$(25b)\qquad x^2\varrho^{41}\lambda_{21} - x^3(\varrho^{12}\varkappa^{41} - \varrho^{41}\varkappa^{12})\,\lambda_{31} - x^4\varrho^{12}\lambda_{41} = 0\,,\quad x^1 = 0\,;$$

$$(25c)\qquad -x^1\varkappa^{23}\lambda_{12} + x^3\varkappa^{12}\lambda_{32} - x^4(\varrho^{12}\varkappa^{23} - \varrho^{23}\varkappa^{12})\,\lambda_{42} = 0\,,\quad x^2 = 0\,;$$

$$(25d)\qquad x^1(\varrho^{23}\varkappa^{34} - \varrho^{34}\varkappa^{23})\,\lambda_{13} - x^2\varrho^{34}\lambda_{23} + x^4\varrho^{23}\lambda_{43} = 0\,,\quad x^3 = 0\,.$$

Lemma 2. *Let i, j, k, r be different numbers from $\{1, 2, 3, 4\}$. The forms $\omega_{ij}, \omega_{ik}, \omega_{jk}$ are torsal corresponding to all points of p_r.*

Proof. Let $M_4 = x^1A_1 + x^2A_2 + x^3A_3$ be a point of P_3. Then

$$
\begin{aligned}
dM_4 = {} & \bar{A}_1[x^2\lambda_{21}\omega_{12} + x^3\lambda_{31}(\varkappa^{12}\omega_{12} + \varkappa^{23}\omega_{23} + \varkappa^{34}\omega_{34} + \varkappa^{41}\omega_{41})] + \\
& + \bar{A}_2[x^1\lambda_{12}\omega_{12} + x^3\lambda_{32}\omega_{23}] + \\
& + \bar{A}_3[x^1\lambda_{13}(\varkappa^{12}\omega_{12} + \varkappa^{23}\omega_{23} + \varkappa^{34}\omega_{34} + \varkappa^{41}\omega_{41}) + x^2\lambda_{23}\omega_{23}] + \\
& + \bar{A}_4[x^1\lambda_{14}\omega_{41} + x^2\lambda_{24}(\varrho^{12}\omega_{12} + \varrho^{23}\omega_{23} + \varrho^{34}\omega_{34} + \varrho^{41}\omega_{41}) + \\
& + x^3\lambda_{34}\omega_{34}] + (\dots)\,.
\end{aligned}
$$

$(\dots)$ is the linear combination of A_1, A_2, A_3, A_4. The forms $\omega_{12}, \omega_{23}, \omega_{13} = \varkappa^{12}\omega_{12} + \varkappa^{23}\omega_{23} + \varkappa^{34}\omega_{34} + \varkappa^{41}\omega_{41}$ are the torsal forms corresponding to M_4 if the forms

$$
\omega_{12}, \omega_{23}, \omega_{13}, x^1\lambda_{14}\omega_{41} + x^2\lambda_{24}(\varrho^{12}\omega_{12} + \varrho^{23}\omega_{23} + \varrho^{34}\omega_{34} + \varrho^{41}\omega_{41}) + x^3\lambda_{34}\omega_{34}
$$

are linearly dependent. This comes true if (25a) holds. The point M_4 lies on the line p_4.

For the lines p_1, p_2, p_3 the proof is similar.

We shall study the invariants of V and the existence theorem for V. The frame of V is given by the equations (22). The coefficients by $\bar{A}_i$, $i = 1, 2, 3, 4$, in the first four equations of the frame satisfy the equations (19), (20), (21) and the exterior differentiation of (21). In accordance with (24) we obtain the following system:

$$
\begin{aligned}
(26)\quad & \Omega_1^2 \wedge \omega_{12} + \Omega_1^3 \wedge (\varkappa^{12}\omega_{12} + \varkappa^{23}\omega_{23} + \varkappa^{34}\omega_{34} + \varkappa^{41}\omega_{41}) + \Omega_1^4 \wedge \omega_{41} = 0\,, \\
& \Omega_2^1 \wedge \omega_{12} + \Omega_2^3 \wedge \omega_{23} + \Omega_2^4 \wedge (\varrho^{12}\omega_{12} + \varrho^{23}\omega_{23} + \varrho^{34}\omega_{34} + \\
& + \varrho^{41}\omega_{41}) = 0\,, \\
& \Omega_3^1 \wedge (\varkappa^{12}\omega_{12} + \varkappa^{23}\omega_{23} + \varkappa^{34}\omega_{34} + \varkappa^{41}\omega_{41}) + \Omega_3^2 \wedge \omega_{23} + \Omega_3^4 \wedge \omega_{34} = 0\,, \\
& \Omega_4^1 \wedge \omega_{41} + \Omega_4^2 \wedge (\varrho^{12}\omega_{12} + \varrho^{23}\omega_{23} + \varrho^{34}\omega_{34} + \varrho^{41}\omega_{41}) + \Omega_4^3 \wedge \omega_{34} = 0\,,
\end{aligned}
$$

$$
\begin{aligned}
(27)\quad & (\Gamma_{12} - \varkappa^{12}\Omega_{21}^3 + \varrho^{12}\Omega_{12}^4) \wedge \omega_{12} + (-\varkappa^{23}\Omega_{21}^3 + \Omega_{12}^3 + \varrho^{23}\Omega_{12}^4) \wedge \omega_{23} + \\
& + (-\varkappa^{34}\Omega_{21}^3 + \varrho^{34}\Omega_{12}^4) \wedge \omega_{34} + (-\varkappa^{41}\Omega_{21}^3 - \Omega_{21}^4 + \varrho^{41}\Omega_{12}^4) \wedge \omega_{41} = 0\,, \\
& (\Omega_{41}^2 + \varkappa^{12}\Omega_{41}^3 - \varrho^{12}\Omega_{14}^2) \wedge \omega_{12} + (\varkappa^{23}\Omega_{41}^3 - \varrho^{23}\Omega_{14}^2) \wedge \omega_{23} + \\
& + (\varkappa^{34}\Omega_{41}^3 - \varrho^{34}\Omega_{14}^2 - \Omega_{14}^3) \wedge \omega_{34} + (\Gamma_{41} + \varkappa^{41}\Omega_{41}^3 - \varrho^{41}\Omega_{14}^2) \wedge \omega_{41} = 0\,, \\
& (-\Omega_{32}^1 - \varrho^{12}\Omega_{32}^4 + \varkappa^{12}\Omega_{23}^1) \wedge \omega_{12} + (\Gamma_{23} - \varrho^{23}\Omega_{32}^4 + \varkappa^{23}\Omega_{23}^1) \wedge \omega_{23} + \\
& + (-\varrho^{34}\Omega_{32}^4 + \varkappa^{34}\Omega_{23}^1 + \Omega_{23}^4) \wedge \omega_{34} + (-\varrho^{41}\Omega_{32}^4 + \varkappa^{41}\Omega_{23}^1) \wedge \omega_{41} = 0\,, \\
& (-\varkappa^{12}\Omega_{43}^1 + \varrho^{12}\Omega_{34}^2) \wedge \omega_{12} + (-\varkappa^{23}\Omega_{43}^1 - \Omega_{43}^2 + \varrho^{23}\Omega_{34}^2) \wedge \omega_{23} + \\
& + (\Gamma_{34} - \varkappa^{34}\Omega_{43}^1 + \varrho^{34}\Omega_{34}^2) \wedge \omega_{34} + \\
& + (-\varkappa^{41}\Omega_{43}^1 + \Omega_{34}^1 + \varrho^{41}\Omega_{34}^2) \wedge \omega_{41} = 0\,;
\end{aligned}
$$

$$
\begin{aligned}
(28)\quad & (\varkappa^{12}\Gamma_{13} - A_{31}^2\Omega_3^2 - B_{31}^2\Omega_{34}^2) \wedge \omega_{12} + (\varkappa^{23}\Gamma_{13} + A_{13}^2\Omega_1^2 + B_{13}^2\Omega_{14}^2) \wedge \omega_{23} + \\
& + (\varkappa^{34}\Gamma_{13} + A_{13}^4\Omega_1^4 + B_{13}^4\Omega_{12}^4) \wedge \omega_{34} + \\
& + (\varkappa^{41}\Gamma_{13} - A_{31}^4\Omega_3^4 - B_{31}^4\Omega_{32}^4) \wedge \omega_{41} = 0\,,
\end{aligned}
$$

$$(\varrho^{12}\Gamma_{24} - A^1_{42}\Omega^1_4 - B^1_{42}\Omega^1_{43}) \wedge \omega_{12} + (\varrho^{23}\Gamma_{24} - A^3_{42}\Omega^3_4 - B^3_{42}\Omega^3_{41}) \wedge \omega_{23} +$$
$$+ (\varrho^{34}\Gamma_{24} + A^3_{24}\Omega^3_2 + B^3_{24}\Omega^3_{21}) \wedge \omega_{34} +$$
$$+ (\varrho^{41}\Gamma_{24} + A^1_{24}\Omega^1_2 + B^1_{24}\Omega^1_{23}) \wedge \omega_{41} = 0 ;$$

(29) $$\left[d\varkappa^{12} + \varkappa^{12}\left(-\frac{d\lambda_{12}}{\lambda_{12}} + \omega^1_1 - \bar{\omega}^2_2 + \frac{d\lambda_{13}}{\lambda_{13}} - \omega^1_1 + \bar{\omega}^3_3\right) + (\ldots)\right] \wedge \omega_{12} +$$
$$+ \left[d\varkappa^{23} + \varkappa^{23}\left(-\frac{d\lambda_{32}}{\lambda_{32}} + \omega^3_3 - \bar{\omega}^2_2 + \frac{d\lambda_{13}}{\lambda_{13}} - \omega^1_1 + \bar{\omega}^3_3\right) + (\ldots)\right] \wedge$$
$$\wedge\, \omega_{23} + \left[d\varkappa^{34} + \varkappa^{34}\left(-\frac{d\lambda_{34}}{\lambda_{34}} + \omega^3_3 - \bar{\omega}^4_4 + \frac{d\lambda_{13}}{\lambda_{13}} - \omega^1_1 + \bar{\omega}^3_3\right) +\right.$$
$$\left. + (\ldots)\right] \wedge \omega_{34} + \left[d\varkappa^{41} + \varkappa^{41}\left(-\frac{d\lambda_{14}}{\lambda_{14}} + \omega^1_1 - \bar{\omega}^4_4 + \frac{d\lambda_{13}}{\lambda_{13}} - \omega^1_1 +\right.\right.$$
$$\left.\left. + \bar{\omega}^3_3\right) + (\ldots)\right] \wedge \omega_{41} = 0,$$

$$\left[d\varrho^{12} + \varrho^{12}\left(-\frac{d\lambda_{21}}{\lambda_{21}} + \omega^2_2 - \bar{\omega}^1_1 + \frac{d\lambda_{24}}{\lambda_{24}} - \omega^2_2 + \bar{\omega}^4_4\right) + (\ldots)\right] \wedge \omega_{12} +$$
$$+ \left[d\varrho^{23} + \varrho^{23}\left(-\frac{d\lambda_{23}}{\lambda_{23}} + \omega^2_2 - \bar{\omega}^3_3 + \frac{d\lambda_{24}}{\lambda_{24}} - \omega^2_2 + \bar{\omega}^4_4\right) + (\ldots)\right] \wedge$$
$$\wedge\, \omega_{23} + \left[d\varrho^{34} + \varrho^{34}\left(-\frac{d\lambda_{43}}{\lambda_{43}} + \omega^4_4 - \bar{\omega}^3_3 + \frac{d\lambda_{24}}{\lambda_{24}} - \omega^2_2 + \bar{\omega}^4_4\right) +\right.$$
$$\left. + (\ldots)\right] \wedge \omega_{34} + \left[d\varrho^{41} + \varrho^{41}\left(-\frac{d\lambda_{41}}{\lambda_{41}} + \omega^4_4 - \bar{\omega}^1_1 + \frac{d\lambda_{24}}{\lambda_{24}} - \omega^2_2 +\right.\right.$$
$$\left.\left. + \bar{\omega}^4_4\right) + (\ldots)\right] \wedge \omega_{41} = 0 .$$

The expressions (...) depend only on the forms $\omega^k_i, \bar{\omega}^k_i$, $i, k = 1, 2, 3, 4$, $i \neq k$.

The system characteristic of (26), (27), (28), (29) contains 12 forms Ω^k_i, $i, k = 1, 2, 3, 4$, $i \neq k$, 12 forms

(30) $$\Gamma_{12} + [\ldots],\quad \Gamma_{23} + [\ldots],\quad \Gamma_{34} + [\ldots],\quad \Gamma_{41} + [\ldots],$$
$$d\varkappa^{12} + \{\ldots\},\quad d\varkappa^{23} + \{\ldots\},\quad d\varkappa^{34} + \{\ldots\},\quad d\varkappa^{41} + \{\ldots\},$$
$$d\varrho^{12} + \{\ldots\},\quad d\varrho^{23} + \{\ldots\},\quad d\varrho^{34} + \{\ldots\},\quad d\varrho^{41} + \{\ldots\}$$

and the linear combinations of 14 forms

(31) $$\Omega^3_{21}, \Omega^4_{12}, \Omega^4_{21}, \Omega^3_{41}, \Omega^2_{14}, \Omega^3_{14}, \Omega^4_{32}, \Omega^1_{23}, \Omega^1_{32}, \Omega^1_{43}, \Omega^2_{34}, \Omega^2_{43}, \Gamma_{13}, \Gamma_{24} .$$

(The equations (24) are fulfilled.) The expressions [...] depend on the forms (31), the expressions {...} depend on $\omega^i_i, \bar{\omega}^i_i, d\lambda_{ik}$, $i, k = 1, 2, 3, 4$, $i \neq k$.

The first twelve forms (31) and the forms Ω^k_i, $i, k = 1, 2, 3, 4$, $i \neq k$, are independent combinations of ω^k_i, $\overline{\omega}^k_i$.

For simplicity of calculation we shall use the following short record of the equations (27), (28), (29):

$$(32) \qquad \begin{aligned} &P^i_1 \wedge \omega_{12} + P^i_2 \wedge \omega_{23} + P^i_3 \wedge \omega_{34} + P^i_4 \wedge \omega_{41} = 0\,, \quad i = 1, 2, 3, 4\,;\\ &Q^c_1 \wedge \omega_{12} + Q^c_2 \wedge \omega_{23} + Q^c_3 \wedge \omega_{34} + Q^c_4 \wedge \omega_{41} = 0\,, \quad c = 1, 2\,;\\ &R^c_1 \wedge \omega_{12} + R^c_2 \wedge \omega_{23} + R^c_3 \wedge \omega_{34} + R^c_4 \wedge \omega_{41} = 0\,, \quad c = 1, 2\,. \end{aligned}$$

The forms

$$(33) \qquad P^1_3, P^1_4, P^2_2, P^2_3, P^3_4, P^3_1, P^4_1, P^4_2, Q^1_1, Q^1_3, Q^1_4, Q^2_2, Q^2_3, Q^2_4$$

are linear combinations of the forms (31) and the forms Ω^k_i, $i, k = 1, 2, 3, 4$, $i \neq k$. The determinant of the coefficients of the forms (31) in these combinations is equal to

$$-2\varkappa^{12}\varkappa^{34}\varkappa^{41}\varrho^{23}\varrho^{34}\varrho^{41}\,.$$

This expression is in the general case not equal to zero, the forms (33) are independent forms of (31) and Ω^k_i.

The forms $P^1_2, P^2_1, P^3_3, P^4_4$. Q^1_2, Q^2_1 are linear combinations of the forms (33) and the forms Ω^k_i.

We obtain the following result:

$$(34) \qquad P^1_2 = \frac{\varrho^{23}}{\varrho^{34}} P^1_3 + \frac{\varrho^{34}}{\varrho^{23}} B^3_{12} P^2_2 - B^3_{12} P^2_3 + \frac{B^3_{12}}{B^3_{42}} \left(\frac{\varrho^{34}}{\varrho^{23}} \varkappa^{23} - \varkappa^{34} \right) Q^2_2 + {}$$
$$+ \frac{1}{B^3_{24}} \left(-\varkappa^{23} + \varkappa^{34} \frac{\varrho^{23}}{\varrho^{34}} \right) Q^2_3 + (\ldots)\,,$$

$$P^2_1 = \frac{\varrho^{12}}{\varrho^{23}} P^2_2 + \frac{1}{\varrho^{41}} \frac{\varkappa^{12}}{\varkappa^{41}} B^4_{31} \frac{B^2_{41}}{B^2_{31}} \left(\varrho^{12} \frac{\varkappa^{23}}{\varkappa^{12}} - \varrho^{23} \right) P^3_4 + \frac{\varkappa^{23}}{\varkappa^{12}} B^2_{41} P^4_1 - {}$$
$$- B^2_{41} P^4_2 + \frac{B^2_{41}}{B^2_{31}} \left(\varrho^{12} \frac{\varkappa^{23}}{\varkappa^{12}} - \varrho^{23} \right) Q^1_1 - \frac{\varkappa^{12}}{\varkappa^{41}} \frac{B^2_{41}}{B^2_{31}} \left(\varrho^{12} \frac{\varkappa^{23}}{\varkappa^{12}} - \varrho^{23} \right) Q^1_4 - {}$$
$$- \frac{1}{B^3_{42}} \left(\varkappa^{12} - \varkappa^{23} \frac{\varrho^{12}}{\varrho^{23}} \right) Q^2_2 - \frac{\varkappa^{12}}{\varrho^{41}} \frac{B^4_{31}}{B^1_{24}} \cdot \frac{B^2_{41}}{B^2_{31}} \left(\varrho^{12} \frac{\varkappa^{23}}{\varkappa^{12}} - \varrho^{23} \right) Q^2_4 + (\ldots)\,,$$

$$P^3_3 = \frac{\varkappa^{41}}{\varkappa^{34}} B^4_{23} P^1_3 - B^4_{23} P^1_4 + \frac{\varkappa^{34}}{\varkappa^{41}} P^3_4 + \frac{B^4_{23}}{B^4_{13}} \left(\varrho^{41} - \varrho^{34} \frac{\varkappa^{41}}{\varkappa^{34}} \right) Q^1_3 + {}$$
$$+ \frac{1}{B^4_{31}} \left(\varrho^{34} - \varrho^{41} \frac{\varkappa^{34}}{\varkappa^{41}} \right) Q^1_4 + (\ldots)\,,$$

$$P^4_4 = - \frac{1}{\varkappa^{34}} \frac{\varrho^{41}}{\varrho^{34}} \frac{B^1_{34}}{B^1_{24}} B^3_{24} \left(\varkappa^{12} - \varkappa^{41} \frac{\varrho^{12}}{\varrho^{41}} \right) P^1_3 + B^1_{34} \frac{\varrho^{12}}{\varrho^{41}} P^3_4 - B^1_{34} P^3_1 + {}$$

$$+\frac{\varkappa^{41}}{\varkappa^{12}} P_1^4 + \frac{1}{B_{31}^2}\left(\varrho^{12}\frac{\varkappa^{41}}{\varkappa^{12}} - \varrho^{41}\right) Q_1^1 - \frac{1}{B_{31}^2}\frac{\varkappa^{12}}{\varkappa^{34}}\left(\varrho^{12}\frac{\varkappa^{41}}{\varkappa^{12}} - \varrho^{41}\right) Q_3^1 -$$

$$-\frac{\varrho^{41}}{\varrho^{34}}\frac{B_{34}^1}{B_{24}^1}\left(\varkappa^{12} - \varkappa^{41}\frac{\varrho^{12}}{\varrho^{41}}\right) Q_3^2 + \frac{B_{34}^1}{B_{24}^1}\left(\varkappa^{12} - \varkappa^{41}\frac{\varrho^{12}}{\varrho^{41}}\right) Q_4^2 + (\ldots),$$

$$Q_2^1 = -\frac{1}{\varrho^{23}} B_{13}^2 P_2^2 - \frac{1}{\varrho^{41}}\frac{\varkappa^{23}}{\varkappa^{41}} B_{31}^4 P_4^3 + \frac{\varkappa^{23}}{\varkappa^{41}} Q_4^1 - \frac{\varkappa^{23}}{\varrho^{23}}\frac{B_{13}^2}{B_{42}^3} Q_2^2 +$$

$$+\frac{\varkappa^{23}}{\varrho^{41}}\frac{B_{31}^4}{B_{24}^1} Q_4^2 + (\ldots),$$

$$Q_1^2 = \frac{1}{\varkappa^{34}}\frac{\varrho^{12}}{\varrho^{34}} B_{24}^3 P_3^1 + \frac{1}{\varkappa^{12}} B_{42}^1 P_1^4 + \frac{\varrho^{12}}{\varkappa^{12}}\frac{B_{42}^1}{B_{31}^2} Q_1^1 - \frac{\varrho^{12}}{\varkappa^{34}}\frac{B_{24}^3}{B_{13}^4} Q_3^1 +$$

$$+\frac{\varrho^{12}}{\varrho^{34}} Q_3^2 + (\ldots).$$

The expressions $(\ldots)$ are linear combinations of Ω_i^k.

The system characteristic of (26)–(29) contains $q = 38$ forms Ω_i^k ($i \neq k$, $i, k = 1, 2, 3, 4$), (30) and (33).

For the integral elements of (26)–(29) we easily obtain

$$s_1 = s_2 = 12\,.$$

Let us study the integral element ε_3. We use the polar system to the system (26), (27), (28).

Let $u_{12}^s, u_{23}^s, u_{34}^s, u_{41}^s$, $s = 1, 2, 3, 4$, be the vectors of the values of the forms $\omega_{12}, \omega_{23}, \omega_{34}, \omega_{41}$. We denote by c_m^{pn} the coefficients of P_m^n ($m, n = 1, 2, 3, 4$) on the right hand sides of the equations (34). Similarly we denote by $\bar{c}_m^{qn}$ ($p, q = 1, 2, 3, 4, 5, 6$) the coefficients of Q_m^n on the right hand sides of these equations. Using the equations (34) we get:

(35a)
$$u_{12}^s\Omega_1^2 + (\varkappa^{12}u_{12}^s + \varkappa^{23}u_{23}^s + \varkappa^{34}u_{34}^s + \varkappa^{41}u_{41}^s)\,\Omega_1^3 + u_{41}^s\Omega_1^4 = 0\,,$$
$$u_{12}^s\Omega_2^1 + u_{23}^s\Omega_2^3 + (\varrho^{12}u_{12}^s + \varrho^{23}u_{23}^s + \varrho^{34}u_{34}^s + \varrho^{41}u_{41}^s)\,\Omega_2^4 = 0\,,$$
$$(\varkappa^{12}u_{12}^s + \varkappa^{23}u_{23}^s + \varkappa^{34}u_{34}^s + \varkappa^{41}u_{41}^s)\,\Omega_3^1 + u_{23}^s\Omega_3^2 + u_{34}^s\Omega_3^4 = 0\,,$$
$$u_{41}^s\Omega_4^1 + (\varrho^{12}u_{12}^s + \varrho^{23}u_{23}^s + \varrho^{34}u_{34}^s + \varrho^{41}u_{41}^s)\,\Omega_4^2 + u_{34}^s\Omega_4^3 = 0\,,$$

(35b)
$$u_{12}^sP_1^1 + u_{23}^s[c_2^{12}P_2^2 + c_3^{12}P_3^2 + \bar{c}_2^{12}Q_2^2 + \bar{c}_3^{12}Q_3^2 + (\ldots)] +$$
$$+ (u_{34}^s + c_3^{11}u_{23}^s)\,P_3^1 + u_{41}^sP_4^1 = 0\,,$$
$$u_{12}^s[c_4^{23}P_4^3 + c_1^{24}P_1^4 + c_2^{24}P_2^4 + \bar{c}_1^{21}Q_1^1 + \bar{c}_4^{21}Q_4^1 + \bar{c}_4^{22}Q_4^2 + \bar{c}_4^{22}Q_4^2 +$$
$$+ (\ldots)] + (u_{23}^s + c_2^{22}u_{12}^s)\,P_2^2 + u_{34}^sP_3^2 + u_{41}^sP_4^2 = 0\,,$$
$$u_{12}^sP_1^3 + u_{23}^sP_2^3 + u_{34}^s[c_3^{31}P_3^1 + c_4^{31}P_4^1 + \bar{c}_3^{31}Q_3^1 + \bar{c}_4^{31}Q_4^1 + (\ldots)] +$$
$$+ (u_{41}^s + c_4^{33}u_{34}^s)\,P_4^3 = 0\,,$$

$$(u_{12}^s + c_1^{44}u_{41}^s)\,P_1^4 + u_{23}^s P_2^4 + u_{34}^s P_3^4 + u_{41}^s[c_3^{41}P_3^1 + c_4^{43}P_4^3 + c_1^{43}P_1^3 +$$
$$+ \bar{c}_1^{41}Q_1^1 + \bar{c}_3^{41}Q_3^1 + \bar{c}_3^{42}Q_3^2 + \bar{c}_4^{42}Q_4^2 + (\ldots)] = 0\,,$$
$$u_{12}^s Q_1^1 + u_{23}^s[c_2^{52}P_2^2 + c_4^{53}P_4^3 + \bar{c}_2^{52}Q_2^2 + \bar{c}_4^{52}Q_4^2 + (\ldots)] +$$
$$+ u_{34}^s Q_3^1 + (u_{41}^s + \bar{c}_4^{51}u_{23}^s)\,Q_4^1 = 0\,,$$
$$u_{12}^s[c_3^{61}P_3^1 + c_1^{64}P_1^4 + \bar{c}_1^{61}Q_1^1 + \bar{c}_3^{61}Q_3^1 + (\ldots)] + u_{23}^s Q_2^2 +$$
$$+ (u_{34}^s + \bar{c}_3^{62}u_{12}^s)\,Q_3^2 + u_{41}^s Q_4^2 = 0\,.$$

The 30 equations (35a), (35b) contain 30 independent forms. If these equations are linearly independent, then there exists only the trivial solution of (35a), (35b).

By $\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}$ we denote the following determinants:

$$\Delta_{12} = (u_{23}^s, u_{34}^s, u_{41}^s)\,, \quad \Delta_{23} = (u_{34}^s, u_{41}^s, u_{12}^s)\,, \quad \Delta_{34} = (u_{41}^s, u_{12}^s, u_{23}^s)\,,$$
$$\Delta_{41} = (u_{12}^s, u_{23}^s, u_{34}^s)\,.$$

Generally, the equations (35a) have the solution $\Omega_i^k = 0$, $i \neq k$. Another solution exists in the following special cases:

$$\varkappa^{23}\Delta_{34} + \varkappa^{34}\Delta_{23} = 0\,, \quad \varrho^{34}\Delta_{41} + \varrho^{41}\Delta_{34} = 0\,, \quad \varkappa^{12}\Delta_{41} + \varkappa^{41}\Delta_{12} = 0\,,$$
$$\varrho^{12}\Delta_{23} + \varrho^{23}\Delta_{12} = 0\,.$$

By using $\Omega_i^k = 0$, $i \neq k$, it is possible to find the solution of each triple of equations (35b). We obtain

$$P_1^1 = \varkappa_1\Delta_{12}\,, \quad c_2^{12}P_2^2 + c_3^{12}P_3^2 + \bar{c}_2^{12}Q_2^2 + \bar{c}_3^{12}Q_3^2 = -\varkappa_1\Delta_{23} - \varkappa_1 c_3^{11}\Delta_{34}\,,$$
$$P_3^1 = \varkappa_1\Delta_{34}\,, \quad P_4^1 = -\varkappa_1\Delta_{41}\,,$$
$$c_4^{23}P_4^3 + c_1^{24}P_1^4 + c_2^{24}P_2^4 + \bar{c}_1^{21}Q_1^1 + \bar{c}_4^{21}Q_4^1 + \bar{c}_2^{22}Q_2^2 + \bar{c}_4^{22}Q_4^2 =$$
$$= \varkappa_2\Delta_{12} + \varkappa_2 c_2^{22}\Delta_{23}\,, \quad P_2^2 = -\varkappa_2\Delta_{23}\,, \quad P_3^2 = \varkappa_2\Delta_{34}\,, \quad P_4^2 = -\varkappa_2\Delta_{41}\,,$$
$$P_1^3 = \varkappa_3\Delta_{12}\,, \quad P_2^3 = -\varkappa_3\Delta_{23}\,, \quad c_3^{31}P_3^1 + c_4^{31}P_4^1 + \bar{c}_3^{31}Q_3^1 + \bar{c}_4^{31}Q_4^1 =$$
$$= \varkappa_3\Delta_{34} + \varkappa_3 c_4^{33}\Delta_{41}\,, \quad P_4^3 = -\varkappa_3\Delta_{41}\,,$$
$$P_1^4 = \varkappa_4\Delta_{12}\,, \quad P_2^4 = -\varkappa_4\Delta_{23}\,, \quad P_3^4 = \varkappa_4\Delta_{34}\,, \quad c_3^{41}P_3^1 + c_4^{43}P_4^3 + c_1^{43}P_1^3 +$$
$$+ \bar{c}_1^{41}Q_1^1 + \bar{c}_3^{41}Q_3^1 + \bar{c}_3^{42}Q_3^2 + \bar{c}_4^{42}Q_4^2 = -\varkappa_4\Delta_{41} - \varkappa_4 c_1^{44}\Delta_{12}\,,$$
$$Q_1^1 = \varkappa_5\Delta_{12}\,, \quad c_2^{52}P_2^2 + c_4^{53}P_4^3 + \bar{c}_2^{52}Q_2^2 + \bar{c}_4^{52}Q_4^2 = -\varkappa_5\Delta_{23} + \varkappa_5\bar{c}_4^{51}\Delta_{41}\,,$$
$$Q_3^1 = \varkappa_5\Delta_{34}\,, \quad Q_4^1 = -\varkappa_5\Delta_{41}\,,$$
$$c_3^{61}P_3^1 + c_1^{64}P_1^4 + \bar{c}_1^{61}Q_1^1 + \bar{c}_3^{61}Q_3^1 = \varkappa_6\Delta_{12} - \varkappa_6\bar{c}_3^{62}\Delta_{34}\,,$$
$$Q_2^2 = -\varkappa_6\Delta_{23}\,, \quad Q_3^2 = \varkappa_6\Delta_{34}\,, \quad Q_4^2 = -\varkappa_6\Delta_{41}\,.$$

The values $\varkappa_i$, $i = 1, 2, 3, 4$, are linearly dependent. We obtain

$$(\Delta_{23} + c_3^{11}\Delta_{34})\,\varkappa_1 + (-c_2^{12}\Delta_{23} + c_3^{12}\Delta_{34})\,\varkappa_2 + (-\bar{c}_2^{12}\Delta_{23} + \bar{c}_3^{12}\Delta_{34})\,\varkappa_6 = 0\,,$$

$$(-\Delta_{12} - c_2^{22}\Delta_{23})\,\varkappa_2 + (-c_4^{23}\Delta_{41})\,\varkappa_3 + (c_1^{24}\Delta_{12} - c_2^{24}\Delta_{23})\,\varkappa_4 +$$
$$+ (\bar{c}_1^{21}\Delta_{12} - \bar{c}_4^{21}\Delta_{41})\,\varkappa_5 + (-\bar{c}_2^{22}\Delta_{23} - \bar{c}_4^{22}\Delta_{41})\,\varkappa_6 = 0\,,$$
$$(c_3^{31}\Delta_{34} - c_4^{31}\Delta_{41})\,\varkappa_1 + (-\Delta_{34} - c_4^{33}\Delta_{41})\,\varkappa_3 + (\bar{c}_3^{31}\Delta_{34} - \bar{c}_4^{51}\Delta_{41})\,\varkappa_5 = 0\,,$$
$$c_3^{41}\Delta_{34}\varkappa_1 + (c_1^{43}\Delta_{12} - c_4^{43}\Delta_{41})\,\varkappa_3 + (\Delta_{41} + c_1^{44}\Delta_{12})\,\varkappa_4 +$$
$$+ (\bar{c}_1^{41}\Delta_{12} + \bar{c}_3^{41}\Delta_{34})\,\varkappa_5 + (\bar{c}_3^{42}\Delta_{34} - \bar{c}_4^{42}\Delta_{41})\,\varkappa_6 = 0\,,$$
$$-c_2^{52}\Delta_{23}\varkappa_2 - c_4^{53}\Delta_{41}\varkappa_3 + (\Delta_{23} - \bar{c}_4^{51}\Delta_{41})\,\varkappa_5 + (-\bar{c}_2^{52}\Delta_{23} - \bar{c}_4^{52}\Delta_{41})\,\varkappa_6 = 0\,,$$
$$c_3^{61}\Delta_{34}\varkappa_1 + c_1^{64}\Delta_{12}\varkappa_4 + (\bar{c}_1^{61}\Delta_{12} + \bar{c}_3^{61}\Delta_{34})\,\varkappa_5 + (-\Delta_{12} + \bar{c}_3^{62}\Delta_{34})\,\varkappa_6 = 0\,.$$

The determinant D of the coefficients of $\varkappa_i$, $i = 1, 2, 3, 4$, satisfies the equation

$$D = 4\varrho^{12}\varkappa^{23}(\varkappa^{12}\varkappa^{23}\varkappa^{34}\varkappa^{41}\varrho^{12}\varrho^{23}\varrho^{34}\varrho^{41})^{-1}\,(\varrho^{34}\Delta_{23} + \varrho^{23}\Delta_{34})\,.$$
$$.\,(\varkappa^{23}\Delta_{12} + \varkappa^{12}\Delta_{23})\,(\varkappa^{41}\Delta_{34} + \varkappa^{34}\Delta_{41})\,(\varrho^{12}\Delta_{41} + \varrho^{41}\Delta_{12})\,.$$
$$.\,[-(-\varrho^{41}\varkappa^{12} + \varrho^{12}\varkappa^{41})\,\Delta_{23}\Delta_{34} + (-\varrho^{23}\varkappa^{12} + \varrho^{12}\varkappa^{23})\,\Delta_{34}\Delta_{41} +$$
$$+ (-\varrho^{41}\varkappa^{34} + \varrho^{34}\varkappa^{41})\,\Delta_{23}\Delta_{12} - (-\varrho^{23}\varkappa^{34} + \varrho^{34}\varkappa^{23})\,\Delta_{12}\Delta_{41}]\,.$$

This expression is equal to zero only for some special choices of u_{12}^s, u_{23}^s. u_{34}^s, u_{41}^s. In the general case the system has the trivial solution $\varkappa_i = 0$, $i = 1, 2, \ldots, 6$, only. The equations (35a), (35b) are linearly independent. Each of the equations of (29) contains 4 forms of the system (26)–(29), these forms are not contained in the other equations. The polar system adjoint to ε_3 of (26)–(29) contains 36 linearly independent equations.

For the characters of the system (26)–(29) we obtain

$$s_1 = s_2 = s_3 = 12\,, \quad s_4 = 38 - 36 = 2\,.$$

By using Cartan's Lemma in the equations (26) it is possible to calculate Ω_i^k $(i, k = 1, 2, 3, 4,\ i \neq k)$ as linear combinations of the forms ω_{ik}. We denote the 24 coefficients of these combinations by

$$\alpha_{gh}^i\,; \quad \alpha_{gh}^i = \alpha_{hg}^i\,; \quad g, h \in \{1, 2, 3\}\,.$$

Similarly, using Cartan's Lemma for the equations (32) we obtain

$$\begin{aligned}
(36)\qquad & P_r^i = \beta_{r1}^i\omega_{12} + \beta_{r2}^i\omega_{23} + \beta_{r3}^i\omega_{34} + \beta_{r4}^i\omega_{41}\,,\\
& Q_r^c = \gamma_{r1}^c\omega_{12} + \gamma_{r2}^c\omega_{23} + \gamma_{r3}^c\omega_{34} + \gamma_{r4}^c\omega_{41}\,,\\
& R_r^c = \psi_{r1}^c\omega_{12} + \psi_{r2}^c\omega_{23} + \psi_{r3}^c\omega_{34} + \psi_{r4}^c\omega_{41}\,,\\
& i = 1, 2, 3, 4\,, \quad c = 1, 2, \quad r = 1, 2, 3, 4\,,\\
& \beta_{rs}^i = \beta_{sr}^i\,, \quad \gamma_{rs}^c = \gamma_{sr}^c\,, \quad \psi_{rs}^c = \psi_{sr}^c\,, \quad s = 1, 2, 3, 4\,.
\end{aligned}$$

Substituting these results into (34) and comparing the coefficients of ω_{12}, ω_{23}, ω_{34}, ω_{41} we get

$$(37)\quad -\beta^1_{21}+(\ldots)=0\,,\quad -\beta^1_{22}+(\ldots)=0\,,\quad -B^3_{12}\beta^2_{33}+(\ldots)=0\,,$$
$$-B^3_{12}\beta^2_{34}+(\ldots)=0\,,$$
$$-\beta^2_{11}+(\ldots)=0\,,\quad -B^2_{41}\beta^4_{22}+(\ldots)=0\,,\quad -B^2_{41}\beta^4_{23}+(\ldots)=0\,,$$
$$-\beta^2_{14}+(\ldots)=0\,,$$
$$-B^4_{23}\beta^1_{41}+(\ldots)=0\,,\quad -\beta^3_{32}+(\ldots)=0\,,\quad -\beta^3_{33}+(\ldots)=0\,,$$
$$-B^4_{23}\beta^1_{44}+(\ldots)=0\,,$$
$$-B^1_{34}\beta^3_{11}+(\ldots)=0\,,\quad -B^1_{34}\beta^3_{12}+(\ldots)=0\,,\quad -\beta^4_{43}+(\ldots)=0\,,$$
$$-\beta^4_{44}+(\ldots)=0\,,$$
$$-\gamma^1_{22}+[\ldots]=0\,,\quad \frac{\varkappa^{23}}{\varkappa^{41}}\gamma^1_{44}+[\ldots]=0\,,\quad -\gamma^2_{11}+[\ldots]=0\,,$$
$$\frac{\varrho^{12}}{\varrho^{34}}\gamma^2_{33}+[\ldots]=0\,,$$
$$-\gamma^1_{12}+\frac{\varkappa^{23}}{\varkappa^{41}}\gamma^1_{14}-\frac{\varkappa^{23}}{\varrho^{23}}\lambda\gamma^2_{12}+\frac{\varkappa^{23}}{\varrho^{41}}\lambda\gamma^2_{14}+\{\ldots\}=0\,,$$
$$-\gamma^1_{23}+\frac{\varkappa^{23}}{\varkappa^{41}}\gamma^1_{34}-\frac{\varkappa^{23}}{\varrho^{23}}\lambda\gamma^2_{23}+\frac{\varkappa^{23}}{\varrho^{41}}\lambda\gamma^2_{34}+\{\ldots\}=0\,,$$
$$\frac{\varrho^{12}}{\varkappa^{12}}(\lambda)^{-1}\gamma^1_{12}-\frac{\varrho^{12}}{\varkappa^{34}}(\lambda)^{-1}\gamma^1_{23}-\gamma^2_{12}+\frac{\varrho^{12}}{\varrho^{34}}\gamma^2_{23}+\{\ldots\}=0\,,$$
$$\frac{\varrho^{12}}{\varkappa^{12}}(\lambda)^{-1}\gamma^1_{14}-\frac{\varrho^{12}}{\varkappa^{34}}(\lambda)^{-1}\gamma^1_{34}-\gamma^2_{14}+\frac{\varrho^{12}}{\varrho^{34}}\gamma^2_{34}+\{\ldots\}=0\,.$$
$$\left(\lambda=\frac{\lambda_{123}\lambda_{341}}{\lambda_{234}\lambda_{412}}\right).$$

Each of the parameters $\beta^1_{21}, \beta^1_{22}, \beta^2_{33}, \beta^2_{34}, \beta^2_{11}, \beta^4_{22}, \beta^4_{23}, \beta^2_{14}, \beta^1_{41}, \beta^3_{32}, \beta^3_{33}, \beta^1_{44}, \beta^3_{11}, \beta^3_{12}, \beta^4_{43}, \beta^4_{44}$ is contained in one equation of (37) only. These parameters are not contained in the expressions (...). The expressions [...], {...} do not contain the parameters $\gamma^1_{22}, \gamma^1_{44}, \gamma^2_{11}, \gamma^2_{33}$. {...} do not contain $\gamma^1_{12}, \gamma^1_{14}, \gamma^1_{23}, \gamma^1_{34}, \gamma^2_{12}, \gamma^2_{14}, \gamma^2_{23}, \gamma^2_{34}$. The equations (37) form a system of 24 independent linear equations of the coefficients of (36) and α^i_{gh}.

The integral element ε_4 of (26)–(29) depends on $N = 24 + 40 + 20 + 20 - 24 = 80$ parameters. We have $s_1 + 2s_2 + 3s_3 + 4s_4 = 80 = N$. The system (26)–(29) or (26), (32) is involutive, $s_4 = 2$.

Lemma 3. *The manifold V depends on two functions of four parameters.*

The forms (33) and Ω^k_i are principal forms. It is possible to calculate the forms (31) as functions of the forms (33) and Ω^k_i. The forms (31) are principal forms, too.

According to (18), (27), (24), (28), the forms Ω_{ir}^k $(i, k, r = 1, 2, 3, 4,\ i \neq k,\ k \neq r,\ i \neq r)$, $\omega_i^k, \bar{\omega}_i^k$ $(i, k = 1, 2, 3, 4,\ i \neq k)$ and the forms $\Gamma_{12}, \Gamma_{23}, \Gamma_{34}, \Gamma_{41}, \Gamma_{13}, \Gamma_{24}$ are principal forms.

Let i, j, k, r be different numbers from $\{1, 2, 3, 4\}$. The forms

$$\frac{1}{\lambda_{ij}\lambda_{ji}} \Gamma_{ij} + \frac{1}{\lambda_{jk}\lambda_{kj}} \Gamma_{jk} + \frac{1}{\lambda_{ki}\lambda_{ik}} \Gamma_{ki} = \frac{\lambda_{ij}\lambda_{jk}\lambda_{ki}}{\lambda_{ik}\lambda_{kj}\lambda_{ji}} \mathrm{d} \frac{\lambda_{ik}\lambda_{kj}\lambda_{ji}}{\lambda_{ij}\lambda_{jk}\lambda_{ki}}$$

$$\frac{1}{\lambda_{ij}\lambda_{ji}} \Gamma_{ij} + \frac{1}{\lambda_{jk}\lambda_{kj}} \Gamma_{jk} + \frac{1}{\lambda_{kr}\lambda_{rk}} \Gamma_{kr} + \frac{1}{\lambda_{ri}\lambda_{ir}} \Gamma_{ri} =$$

$$= \frac{\lambda_{ij}\lambda_{jk}\lambda_{kr}\lambda_{ri}}{\lambda_{ir}\lambda_{rk}\lambda_{kj}\lambda_{ji}} \mathrm{d} \frac{\lambda_{ir}\lambda_{rk}\lambda_{kj}\lambda_{ji}}{\lambda_{ij}\lambda_{jk}\lambda_{kr}\lambda_{ri}}$$

are principal forms. The functions

$$J_{ijk} = \frac{\lambda_{ik}\lambda_{kj}\lambda_{ji}}{\lambda_{ij}\lambda_{jk}\lambda_{ki}}, \quad J_{ijkr} = \frac{\lambda_{ir}\lambda_{rk}\lambda_{kj}\lambda_{ji}}{\lambda_{ij}\lambda_{jk}\lambda_{kr}\lambda_{ri}}$$

are invariants of V. We have

$$J_{ijk} \,.\, J_{kri} = J_{ijkr} \,.$$

We shall characterize geometrically the invariant J_{ijk}. The equations $\omega_{ir} = = \omega_{jr} = \omega_{kr} = 0$ determine on V the one-parametric Pfaff system Φ_r. This system satisfies

$$\omega_{ij} = l_{ij}\omega\,, \quad \omega_{jk} = l_{jk}\omega\,, \quad \omega_{ik} = l_{ik}\omega\,.$$

ω is a principal form, l_{ij}, l_{jk}, l_{ik} are functions of the parameters considered. For this system we have

$$\begin{aligned}
\mathrm{d}A_i &= \omega_i^s A_s + (\lambda_{ij} l_{ij} \bar{A}_j + \lambda_{ik} l_{ik} \bar{A}_k)\, \omega\,,\\
\mathrm{d}A_j &= \omega_j^s A_s + (\lambda_{ji} l_{ij} \bar{A}_i + \lambda_{jk} l_{jk} \bar{A}_k)\, \omega\,,\\
\mathrm{d}A_k &= \omega_k^s A_s + (\lambda_{ki} l_{ik} \bar{A}_i + \lambda_{kj} l_{jk} \bar{A}_j)\, \omega\,, \quad s = 1, 2, 3, 4\,.
\end{aligned}$$

In the five-dimensional space $(A_1, A_2, A_3, A_4, \bar{A}_j, \bar{A}_k)$ we shall define four-dimensional spaces T_1, T_2, T_3, T_4 with the common three-dimensional space (A_1, A_2, A_3, A_4):

$$\begin{aligned}
(38) \qquad T_1 &= T(V, A_i) \cap T(V, A_k) \cap T(V, A_r)\,,\\
T_2 &= T(V, A_i) \cap T(V, A_r) \cap T(V, A_j)\,,\\
T_3 &= T(\Phi_r, A_i)\,,\\
T_4 &= [T(\Phi_r, A_j) \cup T(\Phi_r, A_k)] \cap [T(V, A_i) \cap T(V, A_r)]\,.
\end{aligned}$$

From (38) we obtain

$$\begin{aligned}
(39) \qquad &\bar{A}_j \subset T_1\,, \quad \bar{A}_k \subset T_2\,, \quad \lambda_{ij} l_{ij} \bar{A}_j + \lambda_{ik} l_{ik} \bar{A}_k \subset T_3\,,\\
&\lambda_{ji}\lambda_{kj} l_{ij} l_{jk} \bar{A}_j - \lambda_{jk}\lambda_{ki} l_{ik} l_{jk} \bar{A}_k \subset T_4\,.
\end{aligned}$$

Lemma 4. *The anharmonic ratio of T_1, T_2, T_3, T_4 is equal to $-J_{ijk}$.*

Proof. In accordance with (39) it suffices to calculate the anharmonic ratio of the points

$$\bar{A}_j, \bar{A}_k, \lambda_{ij} l_{ij} \bar{A}_j + \lambda_{ik} l_{ik} \bar{A}_k, \lambda_{ji} \lambda_{kj} l_{ij} l_{jk} \bar{A}_j - \lambda_{jk} \lambda_{ki} l_{ik} l_{jk} \bar{A}_k .$$

This anharmonic ratio is equal to $-J_{ijk}$.

From the equations (29) we obtain that the forms

$$\frac{d\varkappa^{12}}{\varkappa^{12}} - \frac{d\varkappa^{23}}{\varkappa^{23}} + \frac{d\varkappa^{34}}{\varkappa^{34}} - \frac{d\varkappa^{41}}{\varkappa^{41}} - \frac{d\lambda_{12}}{\lambda_{12}} + \frac{d\lambda_{32}}{\lambda_{32}} - \frac{d\lambda_{34}}{\lambda_{34}} + \frac{d\lambda_{14}}{\lambda_{14}} = \\ = \frac{\varkappa^{23}\varkappa^{41}\lambda_{12}\lambda_{34}}{\varkappa^{12}\varkappa^{34}\lambda_{32}\lambda_{14}} d \frac{\varkappa^{12}\varkappa^{34}\lambda_{32}\lambda_{14}}{\varkappa^{23}\varkappa^{41}\lambda_{12}\lambda_{34}},$$

$$\frac{d\varrho^{12}}{\varrho^{12}} - \frac{d\varrho^{23}}{\varrho^{23}} + \frac{d\varrho^{34}}{\varrho^{34}} - \frac{d\varrho^{41}}{\varrho^{41}} - \frac{d\lambda_{21}}{\lambda_{21}} + \frac{d\lambda_{23}}{\lambda_{23}} - \frac{d\lambda_{43}}{\lambda_{43}} + \frac{d\lambda_{41}}{\lambda_{41}} = \\ = \frac{\varrho^{23}\varrho^{41}\lambda_{21}\lambda_{43}}{\varrho^{12}\varrho^{34}\lambda_{23}\lambda_{41}} d \frac{\varrho^{12}\varrho^{34}\lambda_{23}\lambda_{41}}{\varrho^{23}\varrho^{41}\lambda_{21}\lambda_{43}}$$

are principal forms.

The functions

$$J_{13} = \frac{\varkappa^{12}\varkappa^{34}\lambda_{32}\lambda_{14}}{\varkappa^{23}\varkappa^{41}\lambda_{12}\lambda_{34}}, \quad J_{24} = \frac{\varrho^{12}\varrho^{34}\lambda_{23}\lambda_{41}}{\varrho^{23}\varrho^{41}\lambda_{21}\lambda_{43}}$$

are invariants.

Lemma 5. *The anharmonic ratio of A_1, A_3, $p_4 \cap (A_1, A_3)$, $p_2 \cap (A_1, A_3)$ is equal to J_{13}, the anharmonic ratio of A_2, A_4, $p_3 \cap (A_2, A_4)$, $p_1 \cap (A_2, A_4)$ is equal to J_{24}.*

Proof follows from (25 a – d).

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Souhrn

SPECIÁLNÍ GRASSMANOVY VARIETY V_3^4 V PROJEKTIVNÍM PROSTORU P_7

JOSEF VALA

Jsou nalezeny některé výsledky v geometrii čtyřparametrických variet třírozměrných prostorů v projektivním prostoru P_7. Jsou studovány vlastnosti těchto variet, jejichž charakteristiky obsahují přímky. Zvláště jsou nalezeny vlastnosti variet, jejichž charakteristiky obsahují přímky tetraedrů.

Резюме

СПЕЦИАЛЬНЫЕ МНОГООБРАЗИЯ ГРАСМАНА V_3^4 В ПРОЕКТИВНОМ ПРОСТРАНСТВЕ P_7

JOSEF VALA

Найдены некоторые результаты в геометрии четырепараметрического многообразия трехмерных пространств в проективном пространстве P_7. Рассматриваются свойства этих многообразий, которых характеристики содержают прямые. Особенно найденын свойства таких многообразий, характеристики которых содержают ребра тетраэдров.

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