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AN AVERAGING PRINCIPLE FOR STOCHASTIC EVOLUTION EQUATIONS

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Summary. In the present paper integral continuity theorems for solutions of stochastic partial differential equations of evolution type with small parameter are established. These equations are treated in the framework of the semigroup approach, the equations driven by a Wiener process with nuclear incremental covariance operator, those driven by a cylindrical process and the equations of DaPrato-Zabczyk's type being investigated parallelly. As a preliminary result, a fairly general existence theorem for the equations driven by the cylindrical Wiener process is established.

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From the early sixties some attention was paid to extending the methods of averaging from the ordinary differential equations with a small parameter to the stochastic ones. The first theorem establishing convergence in quadratic mean appeared independently in the papers [5], [15]. An alternative approach concerned with weak convergence of processes was developed by Khas'minskii [6], [7], for recent results see e.g. [13].

The averaging of stochastic partial differential equations of hyperbolic type is dealt with in the paper [10], however, no proofs are included and the results given seem not to be general enough. In the present article, following methods from [15], we establish the method of averaging for semilinear stochastic evolution equations, which are treated in the framework of the semigroup approach to stochastic evolution equations. (See [1], [8].)

Our methods can be applied both to equations driven by an infinite-dimensional Wiener process with nuclear incremental covariance (see [1], [8] for basic definitions) and to those driven by a cylindrical process (the theory of which can be found e.g. in [3], [16]).

In the sequel we will adopt the following assumption:

(I) Let H, Y be separable real Hilbert spaces; let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathsf{P})$ be a stochastic basis; $w(t)$ an $(\mathscr{F}_t)$-adapted Wiener process in Y with a nuclear incremental covariance operator W; $B(t)$ an $(\mathscr{F}_t)$-adapted cylindrical Wiener process in Y. Let $p \geqq 2$.

For any Banach space V, $\mathrm{L}^p(\Omega, V)$ denotes the set of all Bochner measurable V-valued functions with a finite norm $\|f\|_p \equiv (\mathsf{E}\|f\|_V^p)^{1/p}$. The space of all V-valued continuous functions on the segment $[0, T]$ will be denoted by $\mathscr{C}([0, T]; V)$. The norm of the space $\mathrm{L}^p(\Omega)$ will be denoted by $|\cdot|_p$. An $\mathscr{L}(Y, H)$-valued function F, where $\mathscr{L}(Y, H)$ denotes the space of all bounded linear operators from Y to H, will be called measurable if $F(\cdot)\, y$ is Bochner measurable for each $y \in Y$. We will make use of the following estimates.

Lemma 1. *Let G_i: $[0, T] \times \Omega \to \mathscr{L}(Y, H)$, $i = 1, 2$, be $(\mathscr{F}_t)$-adapted and measurable, G_2 with Hilbert-Schmidt values. Then there exists a constant $C(p)$, depedning only on p, such that*

(i) ([8]; Prop. 1.9). If $\int_0^T \|G_1(t)\|_p^p \,\mathrm{d}t < \infty$, then

$$\left\|\int_0^T G_1(t)\,\mathrm{d}w(t)\right\|_p \leqq C(p)\left(\int_0^T |tr\,\{G_1(r)\, WG_1(r)^*\}|_{p/2}\,\mathrm{d}r\right)^{1/2} \leqq$$

$$\leqq C(p)\,(\mathrm{tr}\; W)^{1/2}\, T^{1/2-1/p}\left(\int_0^T \|G_1(r)\|_p^p\,\mathrm{d}r\right)^{1/p}.$$

(ii) ([3]; Prop. 1.3). *If $\int_0^T \mathsf{E}\|G_2(t)\|_{\mathrm{HS}}^p \,\mathrm{d}t < \infty$, then*

$$\left\|\int_0^T G_2(t)\,\mathrm{d}B(t)\right\|_p \leqq C(p)\left(\int_0^T |\,\|G_2(t)\|_{\mathrm{HS}}^2|_{p/2}\,\mathrm{d}t\right)^{1/2},$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of an operator, $\|A\|_{\mathrm{HS}}^2 = \mathrm{tr}(AA^*)$.

Remark. In the quoted papers it is additionally supposed that p is an even integer, but this is unnecessary.

We recall a result on the existence and uniqueness of a mild solution of the equation

$$\begin{aligned} &\mathrm{d}x(t) = (A\,x(t) + a(t, x(t)))\,\mathrm{d}t + b(t, x(t))\,\mathrm{d}w(t)\,, \\ &x(0) = \varphi\,. \end{aligned} \tag{1}$$

Theorem 1 ([8]; Th. 2.1). *Let the assumption (I) be fulfilled, let A: $D(A) \to H$ be an infinitesimal generator of a (C_0)-semigroup $S(t)$ on H, let a: $[0, T] \times H \to H$, b: $[0, T] \times H \to \mathscr{L}(Y, H)$ be measurable functions and let there exist $K > 0$ such that for all $t \in [0, T]$, $x, y \in H$*

$$\|a(t, x) - a(t, y)\| + \|b(t, x) - b(t, y)\| \leqq K\|x - y\|\,, \tag{2}$$

$$\|a(t, 0)\| + \|b(t, 0)\| \leqq K\,. \tag{3}$$

Let $\varphi \in \mathrm{L}^p(\Omega; H)$ be $\mathscr{F}_0$-measurable. Then there exists a unique mild solution of (1) in $\mathscr{C}([0, T]; \mathrm{L}^p(\Omega; H))$.

Remark. A mild solution of (1) is an H-valued $(\mathscr{F}_t)$-adapted measurable process x with $\int_0^T \|x(t)\|^2 \,\mathrm{d}t < \infty$ a.s., satisfying

$$x(t) = S(t)\,\varphi + \int_0^t S(t-s)\,a(s, x(s))\,\mathrm{d}s + \int_0^t S(t-s)\,b(s, x(s))\,\mathrm{d}w(s)$$

a.s. for $t \in [0, T]$. By uniqueness we mean that any two solutions x_1, x_2 are modifications of each other, i.e. for every $t \in [0, T]$ we have $x_1(t) = x_2(t)$ almost surely. We can easily deduce an estimate

$$\sup_{t\in[0,T]} \|x(t)\|_p \leqq C^*(1 + \|\varphi\|_p)\,,$$

where the constant C^* depends only on K, T, p, $\operatorname{tr} W$ and $\sup\{\|S(t)\|, 0 \leqq t \leqq T\}$.

Since an analogous existence theorem appropriate for our purposes for equations driven by a cylindrical Wiener process has not appeared (to our knowledge) in an article published to date, we prove it here. In the course of the proof we will need the following generalized Gronwall's inequality.

Lemma 2 ([1]; Corollary 8.11). *Suppose*

$$f(t) \leqq h(t) + \int_0^t g(t-s)\,f(s)\,\mathrm{d}s$$

with $h \in \mathrm{L}^q([0, T])$, $g \in \mathrm{L}^1([0, T])$ *both positive,* $q \in [1, \infty]$, $f \in \mathrm{L}^1([0, T])$. *Then*

$$f(t) \leqq h(t) + \sum_{n=1}^{\infty} (G^n h)(t)$$

where G is the Volterra operator given by $G\,h(t) = \int_0^t g(t-s)\,h(s)\,\mathrm{d}s$. *The series on the right hand side converges in* $\mathrm{L}^q([0, T])$ *and there exists a constant L, depending only on the function g, such that*

$$\int_0^T \Big|\sum_{n=1}^{\infty} G^n\,h(t)\Big|^q \,\mathrm{d}t \leqq L \int_0^T |h(t)|^q \,\mathrm{d}t\,.$$

In particular, if $h = 0$, then $f = 0$.

Theorem 2. *Let* (I) *be fulfilled, let* $A\colon D(A) \to H$ *be an infinitesimal generator of a* (C_0)*-semigroup* $S(t)$ *on* H *such that* $\int_0^T \|S(t)\|_{\mathrm{HS}}^2 \,\mathrm{d}t < \infty$. *Let* $a\colon [0, T] \times H \to H$, $b\colon [0, T] \times H \to \mathscr{L}(Y, H)$ *be measurable functions satisfying estimates* (2), (3). *Let* $\varphi \in \mathrm{L}^p(\Omega; H)$ *be* $\mathscr{F}_0$*-measurable. Then there exists a unique mild solution of*

$$\mathrm{d}x(t) = (Ax(t) + a(t,x(t)))\mathrm{d}t + b(t,x(t))\mathrm{d}B(t)\,,$$
$$x(0) = \varphi\,.$$

This mild solution lies in the space $\mathscr{C}([0, T]; \mathrm{L}^p(\Omega; H))$. *There exists a constant C^*, depending only on* K, T, p, $\sup\{\|S(t)\|, 0 \leqq t \leqq T\}$ *and on the function* $\|S(\cdot)\|_{\mathrm{HS}}$:

$(0, T] \to R_+$, *such that*

$$\sup_{0 \leqq t \leqq T} \|x(t)\|_p \leqq C^*(1 + \|\varphi\|_p) .$$

Proof. We set $M = \sup \{\|S(t)\|, \ 0 \leqq t \leqq T\}$. Let $\mathscr{C}$ be the closed subspace in $\mathscr{C}([0, T]; \mathrm{L}^p(\Omega; H))$ consisting of the $(\mathscr{F}_t)$-adapted functions, let $\|\cdot\|_{\mathscr{C}}$ be the norm in $\mathscr{C}(\|f\|_{\mathscr{C}} = \sup \{\|f(t)\|_p, \ 0 \leqq t \leqq T\})$, let C_i be constants depending only on K, T, M, p and on the functions $\|S(\cdot)\|_{\mathrm{HS}}$. We introduce an operator

$$\mathfrak{K}x(t) = S(t)\,\varphi + \int_0^t S(t - s)\, a(s, x(s))\, \mathrm{d}s + \int_0^t S(t - s)\, b(s, x(s))\, \mathrm{d}B(s) ,$$

$x \in \mathscr{C}$. For each $t \in [0, T]$ we have $\mathfrak{K}x(t) \in \mathrm{L}^p(\Omega; H)$, since

$$\|\mathfrak{K}x(t)\|_p \leqq \|S(t)\,\varphi\|_p + \int_0^t \|S(t - s)\, a(s, x(s))\|_p\, \mathrm{d}s +$$

$$+ C(p)\left(\int_0^t \left|\, \|S(t - s)\, b(s, x(s))\|_{\mathrm{HS}}^2 \right|_{p/2} \mathrm{d}s\right)^{1/2} \leqq$$

$$\leqq M\|\varphi\|_p + C_1(1 + \|x\|_{\mathscr{C}}) +$$

$$+ C_2(1 + \|x\|_{\mathscr{C}}) \left(\int_0^t \|S(t - s)\|_{\mathrm{HS}}^2\, \mathrm{d}s\right)^{1/2} < \infty .$$

Obviously $\mathfrak{K}x(\cdot)$ is $(\mathscr{F}_t)$-adapted. To prove that $\mathfrak{K}x(\cdot)$ is continuous we compute for $h > 0$

$$\mathfrak{K}x(t + h) - \mathfrak{K}x(t) = [S(h) - I]\, \mathfrak{K}x(t) + \int_t^{t+h} S(t + h - s)\,.$$

$$.\, a(s, x(s))\, \mathrm{d}s + \int_t^{t+h} S(t + h - s)\, b(s, x(s))\, \mathrm{d}B(s) \equiv I_1 + I_2 + I_3 .$$

Using the strong continuity of $S(t)$, the integrability of $\|\mathfrak{K}x(t)\|^p$, and the Lebesgue dominated convergence theorem, we see that $\lim_{h \to 0+} \|I_1\|_p = 0$; further $\|I_2\|_p \leqq$ $\leqq MK(1 + \|x\|_{\mathscr{C}})\, h$ and

$$\|I_3\|_p \leqq C(p)\, K(1 + \|x\|_{\mathscr{C}}) \left(\int_0^h \|S(r)\|_{\mathrm{HS}}^2\, \mathrm{d}r\right)^{1/2}$$

and the term on the right hand side of the inequality tends to 0 as $h \to 0+$. For the continuity from the left, we have $(h > 0)$

$$\mathfrak{K}x(t - h) - \mathfrak{K}x(t) = [S(t - h) - S(t)]\, \varphi +$$

$$+ \int_0^{t-h} [I - S(h)]\, S(t - h - s)\, a(s, x(s))\, \mathrm{d}s +$$

$$+ \int_0^{t-h} [I - S(h)]\, S(t - h - s)\, b(s, x(s))\, \mathrm{d}B(s) -$$

$$- \int_{t-h}^{t} S(t-s)\, a(s, x(s))\, ds -$$

$$- \int_{t-h}^{t} S(t-s)\, b(s, x(s))\, dB(s) \equiv I_4 + \ldots + I_8 .$$

The terms I_4, I_7, I_8 can be estimated in the same way as the analogous terms above; $\|I_5\|_p \leqq M \int_0^T \|[I - S(h)]\, a(s, x(s))\|_p\, ds$ and the dominated convergence theorem can be used. In order to estimate I_6 we choose an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H, concluding

$$\|I_6\|_p \leqq C(p) \Big(\int_0^{t-h} \Big| \sum_{j=1}^{\infty} \|b(s, x(s))^* S(t-h-s)^* [I -$$

$$- S(h)]^* e_j\|^2 \Big|_{p/2} ds \Big)^{1/2} \leqq$$

$$\leqq C(p) \Big(\int_0^{t-h} \sum_{j=1}^{\infty} (\mathsf{E}\|b(s, x(s))^* S(t-h-s)^* [I - S(h)]^* e_j\|^p)^{2/p} ds \Big)^{1/2} .$$

Now

$$\int_0^{t-h} \sum_{j=1}^{J} \|b(s, x(s))^* S(t-h-s)^* [I - S(h)]^* e_j\|_p^2\, ds \leqq$$

$$\leqq C_3 (1 + \|x\|_{\mathscr{C}})^2 \sum_{j=1}^{J} \|(I - S(h)^*)\, e_j\|^2$$

and for any $J \in \mathbb{N}$ the right hand side term tends to 0 by the strong continuity of the dual semigroup $(S(t)^*)$. Furthermore,

$$\int_0^{t-h} \sum_{j=J+1}^{\infty} \|b(s, x(s))^* (I - S(h))^* S(t-h-s)^* e_j\|_p^2\, ds \leqq$$

$$\leqq C_4 (1 + \|x\|_{\mathscr{C}})^2 \sum_{j=J+1}^{\infty} \int_0^{t-h} \|S(t-h-s)^* e_j\|^2\, ds \leqq$$

$$\leqq C_4 (1 + \|x\|_{\mathscr{C}})^2 \int_0^T \sum_{j=J+1}^{\infty} \|S(r)^* e_j\|^2\, dr$$

and this tends to 0 as $J \to +\infty$. So we have shown that $\mathfrak{K}$ maps $\mathscr{C}$ into itself; further, for $x, y \in \mathscr{C}$ arbitrary we consider the difference

$$\|\mathfrak{K}x(t) - \mathfrak{K}y(t)\|_p \leqq \int_0^t \|S(t-s)\, [a(s, x(s)) - a(s, y(s))]\|_p\, ds +$$

$$+ C(p) \Big(\int_0^t \big| \|S(t-s)\, [b(s, x(s)) - b(s, y(s))]\|_{HS}^2 \big|_{p/2}\, ds \Big)^{1/2} \leqq$$

$$\leqq MK \int_0^t \|x(s) - y(s)\|_p\, ds + C(p)\, K \Big(\int_0^t \|S(t-s)\|_{HS}^2\, \|x(s) -$$

$$- y(s)\|_p^2 \, ds\Big)^{1/2} \leqq MK\sqrt{T}\Big(\int_0^t \|x(s) - y(s)\|_p^2 \, ds\Big)^{1/2} +$$

$$+ C(p)K\Big(\int_0^t \|S(t-s)\|_{HS}^2 \, \|x(s) - y(s)\|_p^2 \, ds\Big)^{1/2},$$

thus

$$\|\mathfrak{K}x(t) - \mathfrak{K}y(t)\|_p^2 \leqq C_5 \int_0^t (1 + \|S(t-s)\|_{HS}^2) \, \|x(s) - y(s)\|_p^2 \, ds. \tag{4}$$

Set $f(\cdot) = C_5(1 + \|S(\cdot)\|_{HS}^2)$, then $f \in L^1([0, T])$ and is positive; define iterated kernels

$$f_n(t) = \int_0^t f(t-s) f_{n-1}(s) \, ds, \quad n > 1,$$

$$f_1(t) = f(t).$$

Now by induction we prove that

$$\|\mathfrak{K}^n x(t) - \mathfrak{K}^n y(t)\|_p^2 \leqq \int_0^t f_n(t-s) \, \|x(s) - y(s)\|_p^2 \, ds. \tag{5}$$

For $n = 1$ this is true by (4). Supposing (5) is valid for $n - 1$ we have

$$\|\mathfrak{K}^n x(t) - \mathfrak{K}^n y(t)\|_p^2 \leqq \int_0^t f(t-s) \| \mathfrak{K}^{n-1}x(s) - \mathfrak{K}^{n-1}y(s)\|_p^2 \, ds \leqq$$

$$\leqq \int_0^t f(t-s) \int_0^s f_{n-1}(s-r) \, \|x(r) - y(r)\|_p^2 \, dr \, ds =$$

$$= \int_0^t \Big(\int_r^t f(t-s) f_{n-1}(s-r) \, ds\Big) \|x(r) - y(r)\|_p^2 \, dr =$$

$$= \int_0^t \Big(\int_0^{t-r} f(t-r-v) f_{n-1}(v) \, dv\Big) \|x(r) - y(r)\|_p^2 \, dr =$$

$$= \int_0^t f_n(t-r) \, \|x(r) - y(r)\|_p^2 \, dr.$$

Let us choose $\beta > 0$ such that $\int_0^T e^{-\beta s} f(s) \, ds \equiv Q < 1$. It follows that $\int_0^T e^{-\beta s} f_n(s) \, ds \leqq Q^n$ (this can be easily seen by induction or found in [1], Lemma 8.10). Thus we have

$$\|\mathfrak{K}^n x(t) - \mathfrak{K}^n y(t)\|_p^2 \leqq \int_0^t e^{-\beta s} f_n(t-s) \, \|x(s) - y(s)\|_p^2 \, ds \leqq$$

$$\leqq e^{\beta t} \int_0^t e^{-\beta(t-s)} f_n(t-s) \, \|x(s) - y(s)\|_p^2 \, ds \leqq$$

$$\leqq e^{\beta t} \int_0^t e^{-\beta(t-s)} f_n(t-s)\, ds \|x - y\|_{\mathscr{C}}^2,$$

that means $\|\mathfrak{K}^n x - \mathfrak{K}^n y\|_{\mathscr{C}}^2 \leqq e^{\beta T} Q^n \|x - y\|_{\mathscr{C}}^2$.

Fix an arbitrary $x_0 \in \mathscr{C}$, then the sequence $\{\mathfrak{K}^n x_0\}_{n=1}^{\infty}$ is Cauchy in $\mathscr{C}$. Let $x \in \mathscr{C}([0, T]; L^p(\Omega; H))$ be its limit, then x is an $(\mathscr{F}_t)$-adapted process satisfying

$$x(t) = S(t)\,\varphi + \int_0^t S(t-s)\, a(s, x(s))\, ds + \int_0^t S(t-s)\, b(s, x(s))\, dB(s),$$

thus being the desired mild solution.

The uniqueness of the mild solution follows easily by Lemma 2. Finally, we can easily obtain the inequality

$$(1 + \|x(t)\|_p)^2 \leqq C_6 \left((1 + \|\varphi\|_p)^2 + \int_0^t f(t-s)\,(1 + \|x(s)\|_p)^2\, ds \right)$$

and Lemma 2 yields the estimate for $\sup \|x(t)\|_p$. Q.E.D.

Following Gikhman's observation [4] we deduce the theorem on averaging from an "integral continuity theorem". First, we will treat the case of the Wiener process the incremental covariance operator of which is nuclear. We denote by $\mathscr{C}(Q, \beta)$ the set of all closed operators $A\colon D(A) \to H$, $\mathrm{cl}(D(A)) = H$, the resolvent $R(A, \xi)$ of which is defined for all $\xi > \beta$ and fulfils the estimate $\|R(A, \xi)^j\| \leqq Q(\xi - \beta)^{-j}$ for all $j \in \mathbb{N}$ and $\xi > \beta$. Each operator in $\mathscr{C}(Q, \beta)$ is an infinitesimal generator of a (C_0)-semigroup. Let us adopt the following assumptions.

(II) Let $A_\alpha \in \mathscr{C}(Q, \beta)$ for $\alpha \in [0, 1]$; let $S_\alpha(t)$ be the semigroup generated by A_α. Suppose that there exists $z \in \mathscr{C}$, $\mathrm{Re}\, z > \beta$, such that $\lim_{\alpha \to 0+} R(A_\alpha, z)\, x = R(A_0, z)\, x$ for all $x \in H$.

Note that according to Kato's theorem ([9]; Th. IX. 2.16) the hypothesis (II) implies $\lim_{\alpha \to 0+} S_\alpha(t)\, x = S_0(t)\, x$ for each $x \in H$ uniformly in t from compact intervals.

(III) Let $a_\alpha\colon \mathbb{R}_+ \times H \to H$, $b_\alpha\colon \mathbb{R}_+ \times H \to \mathscr{L}(Y, H)$, $\alpha \in [0, 1]$, be measurable functions satisfying the Lipschitz condition: there exists $K > 0$ such that for all $x, y \in H$, $\alpha \in [0, 1]$, $t \in R_+$

$$\|a_\alpha(t, x) - a_\alpha(t, y)\| + \|b_\alpha(t, x) - b_\alpha(t, y)\| \leqq K \|x - y\|, \tag{6}$$

$$\|a_\alpha(t, 0)\| + \|b_\alpha(t, 0)\| \leqq K. \tag{7}$$

(IV) Let $\varphi_\alpha \in L^p(\Omega; H)$ be $\mathscr{F}_0$-measurable, $\alpha \in [0, 1]$.

(V) Suppose that there exists $\Delta_0 > 0$ such that for all $x \in H$ and $t_1, t_2 \in \mathbb{R}_+$ such that $0 \leqq t_1 \leqq t_2 \leqq t_1 + \Delta_0$ we have

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} S_\alpha(t_2 - s)\,[a_\alpha(s, x) - a_0(s, x)]\, ds = 0, \tag{8}$$

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} (\mathrm{tr}\, \{[b_\alpha(s, x) - b_0(s, x)]\, W [b_\alpha(s, x) - b_0(s, x)]^*\})^{p/2}\, ds = 0. \tag{9}$$

Theorem 3. *Let the assumptions* (I), (II), (III), (IV), (V) *be fulfilled. Suppose that* $\lim_{\alpha \to 0+} \|\varphi_\alpha - \varphi_0\|_p = 0$. *Denote by* x_α *the mild solutions of the equations*

$$\text{(10)} \qquad \begin{aligned} dx_\alpha(t) &= (A_\alpha x_\alpha(t) + a_\alpha(t, x_\alpha(t)))\, dt + b_\alpha(t, x_\alpha(t))\, dw(t)\,, \\ x_\alpha(0) &= \varphi_\alpha\,. \end{aligned}$$

Then for every $T > 0$,

$$\lim_{\alpha \to 0+} \sup_{t \in [0,T]} \|x_\alpha(t) - x_0(t)\|_p = 0\,.$$

Remark. If T is fixed a priori we may assume that (6), (7) are fulfilled only for $t \in [0, T]$ and (8), (9) for $0 \leqq t_1 \leqq t_2 \leqq T$, $t_2 - t_1 < \Delta_0$.

Proof. Fix $T > 0$, $\eta > 0$. Set C_i for constants depending only on $K, T, p, \operatorname{tr} W$, φ_0 and on $M = \sup\{\|S_\alpha(t)\|,\ \alpha \in [0, 1],\ t \in [0, T]\}$. (Obviously $M < \infty$ since $\|S_\alpha(t)\| \leqq Q e^{\beta t}$.)

If $\bar{x}_\alpha$ is a mild solution of the problem

$$\begin{aligned} d\bar{x}_\alpha(t) &= (A_\alpha \bar{x}_\alpha(t) + a_\alpha(t, \bar{x}_\alpha(t)))\, dt + b_\alpha(t, \bar{x}_\alpha(t))\, dw(t)\,, \\ \bar{x}_\alpha(0) &= \varphi_0\,, \end{aligned}$$

then

$$\begin{aligned} \bar{x}_\alpha(t) - x_\alpha(t) &= S_\alpha(t)\,[\varphi_0 - \varphi_\alpha] + \\ &+ \int_0^t S_\alpha(t - s)\,[a_\alpha(s, \bar{x}_\alpha(s)) - a_\alpha(s, x_\alpha(s))]\, ds + \\ &+ \int_0^t S_\alpha(t - s)\,[b_\alpha(s, \bar{x}_\alpha(s)) - b_\alpha(s, x_\alpha(s))]\, dw(s)\,, \quad t \in [0, T]\,, \end{aligned}$$

hence using (6) we have

$$\|\bar{x}_\alpha(t) - x_\alpha(t)\|_p^p \leqq C_1 \left(\|\varphi_\alpha - \varphi_0\|_p^p + \int_0^t \|\bar{x}_\alpha(s) - x_\alpha(s)\|_p^p\, ds \right);$$

applying Gronwall's inequality we obtain

$$\lim_{\alpha \to 0+} \sup_{t \in [0,T]} \|\bar{x}_\alpha(t) - x_\alpha(t)\|_p = 0\,,$$

so we may (and will) assume that $\varphi_\alpha = \varphi_0$, $\alpha > 0$.

Theorem 1 implies that $x_0 \in \mathscr{C}([0, T]; L^p(\Omega; H))$. Using this uniform continuity we can choose a partition $\{\tau_i\}_{i=0}^N$ of the interval $[0, T]$ such that for $i = 1, \ldots, N$ and $t \in [\tau_{i-1}, \tau_i]$ we have

$$\text{(11)} \qquad \|x_0(t) - x_0(\tau_{i-1})\|_p < \eta\,.$$

We may further assume that $\max\{\tau_i - \tau_{i-1},\ i = 1, \ldots, N\} \leqq \min(\eta, \Delta_0)$. Set $\tau(t) = \max\{i, \tau_i \leqq t\}$, $\sigma(t) = \tau_{\tau(t)}$. We have

$$x_\alpha(t) - x_0(t) = (S_\alpha(t) - S_0(t))\,\varphi_0 +$$

$$+ \int_0^t \{S_\alpha(t-s)\,a_\alpha(s, x_\alpha(s)) - S_0(t-s)\,a_0(s, x_0(s))\}\,ds +$$

$$+ \int_0^t \{S_\alpha(t-s)\,b_\alpha(s, x_\alpha(s)) - S_0(t-s)\,b_0(s, x_0(s))\}\,dw(s) \equiv$$

$$\equiv R_1 + R_2 + R_3\,.$$

As we have remarked, the hypothesis (II) implies

$$\lim_{\alpha\to 0+} \sup_{t\in[0,T]} \|[S_\alpha(t) - S_0(t)]\,\varphi_0(\omega)\| = 0 \quad \text{for almost}$$

every $\omega \in \Omega$, hence the Lebesgue dominated convergence theorem ensures the existence of $\alpha_1 > 0$ such that

$$\sup_{t\in[0,T]} \|[S_\alpha(t) - S_0(t)]\,\varphi_0\|_p \leqq \eta \quad \text{for all} \quad \alpha \in (0, \alpha_1]\,.$$

Further,

$$R_2 = \int_{\sigma(t)}^t \{S_\alpha(t-s)\,a_\alpha(s, x_\alpha(s)) - S_0(t-s)\,a_0(s, x_0(s))\}\,ds +$$

$$+ \int_0^{\sigma(t)} S_\alpha(t-s)\,[a_\alpha(s, x_\alpha(s)) - a_\alpha(s, x_0(s))]\,ds +$$

$$+ \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} S_\alpha(t-s)\,[a_\alpha(s, x_0(s)) - a_\alpha(s, x_0(\tau_{i-1}))]\,ds +$$

$$+ \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} S_\alpha(t-s)\,[a_\alpha(s, x_0(\tau_{i-1})) - a_0(s, x_0(\tau_{i-1}))]\,ds +$$

$$+ \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} [S_\alpha(t-s) - S_0(t-s)]\,a_0(s, x_0(\tau_{i-1}))\,ds +$$

$$+ \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} S_0(t-s)\,[a_0(s, x_0(\tau_{i-1})) - a_0(s, x_0(s))]\,ds \equiv I_1 + \ldots + I_6\,.$$

Now, using (7) and the estimate stated after Theorem 1, we obtain

$$\|I_1\|_p \leqq M \int_{\sigma(t)}^t (\|a_\alpha(s, x_\alpha(s))\|_p + \|a_0(s, x_0(s))\|_p)\,ds \leqq$$

$$\leqq 2MK \int_{\sigma(t)}^t (1 + \|x_0(s)\|_p + \|x_\alpha(s)\|_p)\,ds \leqq C_2(1 +$$

$$+ \|\varphi_0\|_p)\,(t - \sigma(t)) \leqq C_3\eta\,;$$

$$\|I_2\|_p \leqq \sigma(t)^{(p-1)/p} \left(\int_0^t \|S_\alpha(t-s)\,[a_\alpha(s, x_\alpha(s)) -\right.$$

$$- a_\alpha(s, x_0(s))] \|_p^p ds \Big)^{1/p} \leqq$$

$$\leqq T^{(p-1)/p} MK \Big(\int_0^t \|x_\alpha(s) - x_0(s) \|_p^p ds \Big)^{1/p} .$$

According to (11) we can estimate

$$\|I_3\|_p \leqq \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} \|S_\alpha(t-s) [a_\alpha(s, x_0(s)) - a_\alpha(s, x_0(\tau_{i-1}))]\|_p ds \leqq$$

$$\leqq MK \sum_{i=1}^{N} \int_{\tau_{i-1}}^{\tau_i} \|x_0(s) - x_0(\tau_{i-1})\|_p ds \leqq C_4 \eta ;$$

by analogy $\|I_6\|_p \leqq C_4 \eta$.

Further,

$$\|I_5\|_p \leqq \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} \|[S_\alpha(t-s) - S_0(t-s)] a_0(s, x_0(\tau_{i-1}))\|_p ds \leqq$$

$$\leqq \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} \Big| \sup_{r \in [0,T]} \|[S_\alpha(r) - S_0(r)] a_0(s, x_0(\tau_{i-1}))\| \Big|_p ds ;$$

but for any $s \in [\tau_{i-1}, \tau_i]$ and almost every $\omega \in \Omega$

$$\lim_{\alpha \to 0+} \sup_{r \in [0,T]} \|[S_\alpha(r) - S_0(r)] a_0(s, x_0(\tau_{i-1})(\omega))\| = 0 ,$$

and we use again the dominated convergence theorem of Lebesgue to obtain $\alpha_2 > 0$ such that for $\alpha \in (0, \alpha_2]$ we have $\|I_5\|_p < \eta$. Finally,

$$\|I_4\|_p \leqq M \sum_{i=1}^{\tau(t)} \Big\| \int_{\tau_{i-1}}^{\tau_i} S_\alpha(\tau_i - s) [a_\alpha(s, x_0(\tau_{i-1})) - a_0(s, x_0(\tau_{i-1}))] ds \Big\|_p ,$$

hence, using the assumption (8) and the Lebesgue dominated convergence theorem, we can find $\alpha_3 > 0$ such that $\|I_4\|_p \leqq \eta$ for $\alpha \in (0, \alpha_3]$. Now we need to estimate the term R_3.

$$R_3 = \int_{\sigma(t)}^t \{S_\alpha(t-s) b_\alpha(s, x_\alpha(s)) - S_0(t-s) b_0(s, x_0(s))\} dw(s) +$$

$$+ \int_0^{\sigma(t)} S_\alpha(t-s) [b_\alpha(s, x_\alpha(s)) - b_\alpha(s, x_0(s))] dw(s) +$$

$$+ \int_0^{\sigma(t)} \{S_\alpha(t-s) b_\alpha(s, x_0(s)) - S_0(t-s) b_0(s, x_0(s))\} dw(s) \equiv$$

$$= K_1 + K_2 + K_3 .$$

Employing Lemma 1 we can easily deduce

$$\|K_1\|_p \leqq C(p)\,(\operatorname{tr} W)^{1/2}\,(t-\sigma(t))^{1/2-1/p}\,.$$

$$\cdot\left(\int_{\sigma(t)}^{t}\|S_\alpha(t-s)\,b_\alpha(s,x_\alpha(s))-S_0(t-s)\,b_0(s,x_0(s))\|_p^p\,\mathrm{d}s\right)^{1/p}\leqq$$

$$\leqq C_5(t-\sigma(t))^{1/2}\leqq C_5\eta^{1/2}\,;$$

$$\|K_2\|_p \leqq C_6\left(\int_0^t\|x_\alpha(s)-x_0(s)\|_p^p\,\mathrm{d}s\right)^{1/p}.$$

Set now $Z(s)=S_\alpha(t-s)\,b_\alpha(s,x_0(s))-S_0(t-s)\,b_0(s,x_0(s))$; using the estimate (7) we can easily prove $\sup\{\|Z(s)\|_p,\ 0\leqq s\leqq t\leqq T\}\leqq C_7$, where the constant C_7 does not depend on α. We have, by Lemma 1,

$$\|K_3\|_p \leqq C(p)\left(\int_0^{\sigma(t)}|\operatorname{tr}\{Z(s)\,WZ(s)^*\}|_{p/2}\,\mathrm{d}s\right)^{1/2}\leqq$$

$$\leqq C(p)\left(\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}|\operatorname{tr}\{S_\alpha(t-s)\,[b_\alpha(s,x_0(s))-\right.$$

$$-\,b_\alpha(s,x_0(\tau_{i-1}))]\,WZ(s)^*\}|_{p/2}\,\mathrm{d}s\,+$$

$$+\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}|\operatorname{tr}\{S_\alpha(t-s)\,[b_\alpha(s,x_0(\tau_{i-1}))-$$

$$-\,b_0(s,x_0(\tau_{i-1}))]\,WZ(s)^*\}|_{p/2}\,\mathrm{d}s\,+$$

$$+\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}|\operatorname{tr}\{[S_\alpha(t-s)-S_0(t-s)]\,b_0(s,x_0(\tau_{i-1}))\,WZ(s)^*\}|_{p/2}\,\mathrm{d}s\,+$$

$$+\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}|\operatorname{tr}\{S_0(t-s)\,[b_0(s,x_0(\tau_{i-1}))-$$

$$\left.-\,b_0(s,x_0(s))]\,WZ(s)^*\}|_{p/2}\,\mathrm{d}s\right)^{1/2}\equiv$$

$$\equiv C(p)\,(J_1+\ldots+J_4)^{1/2}\,.$$

We now proceed to estimate the terms $J_1,\ldots,J_4$.

$$J_1 \leqq \operatorname{tr} W\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}|\,\|S_\alpha(t-s)\,[b_\alpha(s,x_0(s))-$$

$$-\,b_\alpha(s,x_0(\tau_{i-1}))]\|\,\|Z(s)\|\,|_{p/2}\,\mathrm{d}s\leqq$$

$$\leqq \operatorname{tr} W\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}\|S_\alpha(t-s)\,[b_\alpha(s,x_0(s))-b_\alpha(s,x_0(\tau_{i-1}))]\|_p\,\|Z(s)\|_p\,\mathrm{d}s\leqq$$

$$\leqq \operatorname{tr} WMKC_7\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}\|x_0(s)-x_0(\tau_{i-1})\|_p\,\mathrm{d}s\leqq C_8\eta\,,$$

analogously $J_4 \leqq C_8\eta$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of Y consisting of eigenvectors of the covariance operator W, i.e. $W = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$. Then relying on the formula $\operatorname{tr}(UVS^*) = \operatorname{tr}(S^*UV)$ valid for any $S, U \in \mathscr{L}(Y, H)$ and $V \in \mathscr{L}(Y)$, V nuclear, we obtain

$$J_3 = \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} |\sum_{j=1}^{\infty} \langle Z(s)^* [S_\alpha(t-s) - S_0(t-s)] \,.$$
$$. \, b_0(s, x_0(\tau_{i-1}))\, We_j, e_j\rangle|_{p/2} \, \mathrm{d}s \leqq$$
$$\leqq \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} \sum_{j=1}^{\infty} |\langle Z(s)^* [S_\alpha(t-s) - S_0(t-s)] \,.$$
$$. \, b_0(s, x_0(\tau_{i-1}))\, We_j, e_j\rangle|_{p/2} \, \mathrm{d}s \leqq$$
$$\leqq \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} \sum_{j=1}^{\infty} \lambda_j \|[S_\alpha(t-s) - S_0(t-s)] \,.$$
$$. \, b_0(s, x_0(\tau_{i-1}))\, e_j\|_p \, \|Z(s)\, e_j\|_p \, \mathrm{d}s \leqq$$
$$\leqq C_7 \sum_{i=1}^{N} \int_{\tau_{i-1}}^{\tau_i} \sum_{j=1}^{\infty} \lambda_j \|[S_\alpha(t-s) - S_0(t-s)]\, b_0(s, x_0(\tau_{i-1}))\, e_j\|_p \, \mathrm{d}s \,.$$

For arbitrary $i = 1, \ldots, N$

$$\int_{\tau_{i-1}}^{\tau_i} \sum_{j=1}^{J} \lambda_j \|[S_\alpha(t-s) - S_0(t-s)]\, b_0(s, x_0(\tau_{i-1}))\, e_j\|_p \, \mathrm{d}s \leqq$$
$$\leqq \sum_{j=1}^{J} \lambda_j \int_{\tau_{i-1}}^{\tau_i} | \sup_{r \in [0,T]} \|[S_\alpha(r) - S_0(r)]\, b_0(s, x_0(\tau_{i-1}))\, e_j\| \,|_p \, \mathrm{d}s \,,$$

and for arbitrary $J \in \mathbb{N}$ the term on the right hand side of the inequality tends to 0 as $\alpha \to 0+$. Further,

$$\sum_{j=J+1}^{\infty} \lambda_j \int_{\tau_{i-1}}^{\tau_i} \|[S_\alpha(t-s) - S_0(t-s)]\, b_0(s, x_0(\tau_{i-1}))\, e_j\|_p \, \mathrm{d}s \leqq C_9 \sum_{j=J+1}^{\infty} \lambda_j \,,$$

and this is arbitrarily small for J sufficiently large. But this means that there exists $\alpha_4 > 0$ such that $\alpha \in (0, \alpha_4]$ implies $J_3 < \eta$.

Set $Q_i(s) = S_\alpha(t-s)\,[b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1}))]$; the estimate $\sup\{\|Q_i(s)\|,\ i = 1, \ldots, N;\ s \in [\tau_{i-1}, \tau_i]\} \leqq C_{10}$ can be easily checked. Now

$$J_2 = \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} |\operatorname{tr}\{Q_i(s)\, W[S_\alpha(t-s)\, b_\alpha(s, x_0(s)) -$$
$$- S_0(t-s)\, b_0(s, x_0(s))]^*\}|_{p/2} \, \mathrm{d}s \leqq$$
$$\leqq \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} |\operatorname{tr}\{Q_i(s)\, W[S_\alpha(t-s)\,[b_\alpha(s, x_0(s)) -$$
$$- b_\alpha(s, x_0(\tau_{i-1}))]]^*\}|_{p/2} \, \mathrm{d}s +$$

$$+\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}|\mathrm{tr}\{Q_i(s)\,W\,Q_i(s)^*\}|_{p/2}\,\mathrm{d}s\,+$$

$$+\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}|\mathrm{tr}\{Q_i(s)\,W[[S_\alpha(t-s)-S_0(t-s)]\,.$$

$$.\,b_0(s,x_0(\tau_{i-1}))]^*\}|_{p/2}\,\mathrm{d}s\,+$$

$$+\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}|\mathrm{tr}\{Q_i(s)\,W[S_0(t-s)\,[b_0(s,x_0(\tau_{i-1}))-$$

$$-\,b_0(s,x_0(s))]]^*\}|_{p/2}\,\mathrm{d}s\equiv J_5+\ldots+J_8\,.$$

As before we can obtain $J_5 + J_8 \leqq C_{11}\eta$ and there exists $\alpha_5 > 0$ such that $J_7 \leqq \eta$ whenever $\alpha \in (0, \alpha_5]$.

Further, setting $U_i(s) = b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1}))$, we have

$$J_6 \leqq M^2 \sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i}|\mathrm{tr}\{U_i(s)\,W U_i(s)^*\}|_{p/2}\,\mathrm{d}s \leqq$$

$$\leqq M^2\sum_{i=1}^{N}(\tau_i-\tau_{i-1})^{(p-2)/p}\left(\mathsf{E}\int_{\tau_{i-1}}^{\tau_i}(\mathrm{tr}\{U_i(s)\,W U_i(s)^*\})^{p/2}\,\mathrm{d}s\right)^{2/p}\leqq$$

$$\leqq M^2 T^{(p-2)/p}\sum_{i=1}^{N}\left(\mathsf{E}\int_{\tau_{i-1}}^{\tau_i}(\mathrm{tr}\{U_i(s)\,W U_i(s)^*\})^{p/2}\,\mathrm{d}s\right)^{2/p}.$$

By the assumption (9) for all $i = 1, \ldots, N$ and almost all $\omega \in \Omega$

$$\lim_{\alpha\to 0+}\int_{\tau_{i-1}}^{\tau_i}(\mathrm{tr}\{U_i(s)(\omega)\,W U_i(s)(\omega)^*\})^{p/2}\,\mathrm{d}s = 0\,,$$

hence applying the dominated convergence theorem we obtain $\alpha_6 > 0$ such that $J_6 \leqq \eta$ for every $\alpha \in (0, \alpha_6]$.

Combining all the deduced estimates, we have for α small enough

$$\|R_3\|_p \leqq C_{12}\left(\eta^{1/2} + \left(\int_0^t \|x_\alpha(s) - x_0(s)\|_p^p\,\mathrm{d}s\right)^{1/p}\right).$$

The proof is complete, for we have found a constant C_{13} depending only on T, M, K, $\mathrm{tr}W$, p, $\|\varphi_0\|_p$ such that for an arbitrary $\eta > 0$ there exists $\alpha_0 > 0$ such that for $\alpha \in (0, \alpha_0]$

$$\|x_\alpha(t) - x_0(t)\|_p \leqq C_{13}\left(\eta + \eta^{1/2} + \left(\int_0^t \|x_\alpha(s) - x_0(s)\|_p^p\,\mathrm{d}s\right)^{1/p}\right).$$

It remains only to use Gronwall's lemma. Q.E.D.

With Theorem 3 available we can establish the averaging theorem without any difficulties. We will need the following assumptions.

(IIa) Let $A\colon D(A) \to H$ be an infinitesimal generator of a (C_0)-semigroup $S(t)$ in $\mathscr{L}(H)$.

(IIIa) Let $a: \mathbb{R}_+ \times H \to H$, $b: \mathbb{R}_+ \times H \to \mathscr{L}(Y, H)$ be measurable functions satisfying the estimates (2), (3) whenever $t \geqq 0$, $x, y \in H$.

(Va) Suppose that there exist Lipschitz functions $\bar{a}: H \to H$, $\bar{b}: H \to \mathscr{L}(Y, H)$ such that for some $\Delta_0 > 0$ and all $x \in H$ and $t_1, t_2 \in \mathbb{R}_+$ such that $0 \leqq t_1 \leqq t_2 \leqq \leqq t_1 + \Delta_0$ we have

$$\lim_{\varepsilon \to 0+} \int_{t_1}^{t_2} S(t_2 - r)\left[a\left(\frac{r}{\varepsilon}, x\right) - \bar{a}(x)\right] \mathrm{d}r = 0 , \tag{12}$$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T (\operatorname{tr}\{[b(r, x) - \bar{b}(x)]\, W[b(r, x) - \bar{b}(x)]^*\})^{p/2} \mathrm{d}r = 0 . \tag{13}$$

Set $w_\varepsilon(t) = \varepsilon^{-1/2} w(\varepsilon t)$, $w_\varepsilon(t)$ is also a Wiener process with the incremental covariance operator W.

Theorem 4. *Suppose that the assumptions* (I), (IIa), (IIIa), (IV), (Va) *are fulfilled. Let* $x_\varepsilon(t)$, $\varepsilon > 0$, *be the mild solutions of the equations*

$$\mathrm{d}x_\varepsilon(t) = \varepsilon(A\, x_\varepsilon(t) + a(t, x_\varepsilon(t)))\, \mathrm{d}t + \varepsilon^{1/2} b(t, x_\varepsilon(t))\, \mathrm{d}w_\varepsilon(t) ,$$
$$x_\varepsilon(0) = \varphi_\varepsilon .$$

Let $y(t)$ *denote a mild solution of*

$$\mathrm{d}y(t) = (Ay(t) + \bar{a}(y(t)))\, \mathrm{d}t + \bar{b}(y(t))\, \mathrm{d}w(t) ,$$
$$y(0) = \varphi_0 .$$

If $\lim_{\varepsilon \to 0+} \|\varphi_\varepsilon - \varphi_0\|_p = 0$ *then for any* $T > 0$

$$\lim_{\varepsilon \to 0+} \sup_{t \in [0,T]} \left\| x_\varepsilon\left(\frac{t}{\varepsilon}\right) - y(t) \right\|_p = 0 .$$

Proof. The operator εA is the infinitesimal generator of the semigroup $\{S(\varepsilon t), t \geqq 0\}$, hence

$$x_\varepsilon(t) = S(\varepsilon t)\, \varphi_\varepsilon + \varepsilon \int_0^t S(\varepsilon(t - s))\, a(s, x_\varepsilon(s))\, \mathrm{d}s + $$
$$+ \varepsilon^{1/2} \int_0^t S(\varepsilon(t - s))\, b(s, x_\varepsilon(s))\, \mathrm{d}w_\varepsilon(s) .$$

Setting $\hat{x}_\varepsilon(t) = x_\varepsilon(t/\varepsilon)$ we obtain

$$\hat{x}_\varepsilon(t) = S(t)\, \varphi_\varepsilon + \int_0^t S(t - s)\, a\left(\frac{s}{\varepsilon}, \hat{x}_\varepsilon(s)\right) \mathrm{d}s + $$
$$+ \int_0^t S(t - s)\, b\left(\frac{s}{\varepsilon}, \hat{x}_\varepsilon(s)\right) \mathrm{d}w(s) ,$$

i.e. $\hat{x}_\varepsilon(t)$ is a mild solution of

$$\mathrm{d}\hat{x}_\varepsilon(t) = \left(A\hat{x}_\varepsilon(t) + a\left(\frac{t}{\varepsilon}, \hat{x}_\varepsilon(t)\right)\right)\mathrm{d}t + b\left(\frac{t}{\varepsilon}, \hat{x}_\varepsilon(t)\right)\mathrm{d}w(t),$$
$$\hat{x}_\varepsilon(0) = \varphi_\varepsilon$$

It follows that we can estimate $\|\hat{x}_\varepsilon(t) - y(t)\|_p$ using Theorem 3, if we set $a_\varepsilon(t, x) = a(t/\varepsilon, x)$; $b_\varepsilon(t, x) = b(t/\varepsilon, x)$, $\varepsilon > 0$, and $a_0(t, x) = \bar{a}(x)$, $b_0(t, x) = \bar{b}(x)$. It remains to show that under the hypothesis (13) the assumption (9) is fulfilled, but we have

$$\int_{t_1}^{t_2}\left(\operatorname{tr}\left\{\left[b\left(\frac{t}{\varepsilon}, x\right) - \bar{b}(x)\right] W\left[b\left(\frac{t}{\varepsilon}, x\right) - \bar{b}(x)\right]^*\right\}\right)^{p/2}\mathrm{d}t =$$
$$= (t_2 - t_1)\frac{\varepsilon}{t_2 - t_1}\int_{t_1/\varepsilon}^{t_2/\varepsilon}(\operatorname{tr}\{[b(v, x) - \bar{b}(x)]\,.$$
$$.\, W[b(v, x) - \bar{b}(x)]^*\})^{p/2}\,\mathrm{d}v,$$

further, (13) implies

$$\lim_{T\to\infty}\frac{1}{T}\int_{\beta T}^{\beta T + T}(\operatorname{tr}\{[b(s, x) - \bar{b}(x)]\, W[b(s, x) - \bar{b}(x)]^*\})^{p/2}\,\mathrm{d}s = 0$$

for each $\beta \geqq 0$ and the result follows. Q.E.D.

Remark. If $\dim H < \infty$ and $w(t)$ is a standard Wiener process in H then the assumption (13) is equivalent to the assumption

$$(14) \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T\|b(t, x) - \bar{b}(x)\|^2\,\mathrm{d}t = 0, \quad x \in H,$$

which was adopted in [5], [15]. For H infinite-dimensional, however, (14) is stronger then (13), as the following trivial example shows: $H = \ell^2$, $b(t, x) = B_n$ for $t \in [n - 1, n)$ and arbitrary $x \in H$, where

$$B_n: \ell^2 \to \ell^2, \quad (y_1, y_2, \ldots) \mapsto (0, \ldots, 0, y_{n+1}, y_{n+2}, \ldots).$$

Let $w(t)$ be a Wiener process in ℓ^2 with a nuclear covariance operator $W = \sum_{n=1}^{\infty}\lambda_n e_n \otimes e_n$, where $\{e_n\}_{n=1}^{\infty}$ is the standard orthonormal basis of ℓ^2. Setting $\bar{b} \equiv 0$ we have for $N \in \mathbb{N}$

$$\frac{1}{N}\int_0^N \operatorname{tr}\{b(t, x)\, Wb(t, x)^*\}\,\mathrm{d}t = \frac{1}{N}\sum_{i=1}^{N}\operatorname{tr}\{B_i W B_i^*\} =$$
$$= \frac{1}{N}\sum_{k=1}^{N}(k - 1)\lambda_k + \sum_{k=N+1}^{\infty}\lambda_k,$$

the first term converging to 0 according to Kronecker's lemma, the second as a remainder of a convergent series. On the other hand,

$$\frac{1}{N}\int_0^N \|B(t,x)\|^2 \, dt = 1 .$$

It is worth noticing that in the papers [5], [15] the authors have shown that it does not suffice to assume only

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T b(t,x)\, dt = \bar{b}(x) .$$

Analogously to the averaging theorems for both the ordinary differential equations and the stochastic ones in a finite-dimensional space one would like to replace the assumption (12) by a more natural hypothesis

$$(15) \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^T a(t,x)\, dt = \bar{a}(x) , \quad x \in H .$$

(See [15].) If $\dim H < \infty$ then (12) and (15) are equivalent as can be easily shown by integrating by parts. The following lemma states that for a wide class of equations (roughly speaking, for parabolic ones) this equivalence holds also in the infinite-dimensional case.

Lemma 3. *Let H be a separable Hilbert space over $\mathbb{R}$. Let $a: \mathbb{R}_+ \times H \to H$ be a function such that for any $x \in H$*

(i) $a(.,x) \in L^1_{\mathrm{loc}}([0,\infty); H)$;

$$\sup\{\|a(s,x)\|,\ s \in (0,\infty)\} \equiv K_x < \infty ,$$

$$(16) \quad \text{(ii)} \lim_{T\to\infty} \frac{1}{T}\int_0^T a(s,x)\, ds \equiv \bar{a}(x) \ \textit{exists.}$$

If $S(t)$ is a holomorphic semigroup in $\mathscr{L}(H)$, then for any $x \in H$ and $t_1, t_2 \in \mathbb{R}_+$ such that $0 \leqq t_1 \leqq t_2 < \infty$ we have

$$\lim_{\varepsilon\to 0+} \int_{t_1}^{t_2} S(t_2 - s)\left[a\left(\frac{s}{\varepsilon}, x\right) - \bar{a}(x)\right] ds = 0 .$$

Proof. We may assume $\bar{a} \equiv 0$ without loss of generality. Let us choose $x \in H$, $0 \leqq t_1 \leqq t_2 \leqq T < \infty$ arbitrary. By the holomorphicity of the semigroup $S(t)$ we have $S(\cdot) \in \mathscr{C}^\infty((0,T]; \mathscr{L}(H))$, moreover there exists a constant Q such that $\|(d/ds)\, S(s)\| \leqq Qs^{-1}$, $s \in (0,T]$. (For the definition and basic properties of holomorphic semigroups see e.g. [9], § IX.1.6.) Set

$$A(t,x) = \int_0^t a(s,x)\, ds \, ; \quad M = \sup\{\|S(t)\|,\ 0 \leqq t \leqq T\} .$$

Under the hypothesis (16)

$$\text{(17)} \qquad \lim_{\varepsilon \to 0+} \varepsilon A\left(\frac{t}{\varepsilon}, x\right) = \lim_{\varepsilon \to 0+} \varepsilon \int_0^{t/\varepsilon} a(s, x)\, \mathrm{d}s = 0$$

for each $t \geqq 0$. Let $\delta > 0$. Setting $c = t_2 - t_1$ we have

$$\int_{t_1}^{t_2} S(t_2 - s)\, a\left(\frac{s}{\varepsilon}, x\right) \mathrm{d}s = \int_0^c (1 - s^\delta)\, S(s)\, a(\varepsilon^{-1}(t_2 - s), x)\, \mathrm{d}s +$$

$$+ \int_0^c s^\delta\, S(s)\, a(\varepsilon^{-1}(t_2 - s), x)\, \mathrm{d}s \equiv J_1 + J_2 .$$

Now $\|J_1\| \leqq MK_x \int_0^c (1 - s^\delta)\, \mathrm{d}s = MK_x c(1 - (\delta + 1)^{-1} c^\delta)$, and so J_1 tends to 0 as $\delta \to 0+$. The proof will be complete if we show that $\lim_{\varepsilon \to 0+} J_2 = 0$ for arbitrary $\delta > 0$. In order to see this we first realize that the function $s \to s^\delta S(s)\, A(\varepsilon^{-1}(t_2 - s), x)$ is absolutely continuous on $[g, c]$ for every $g > 0$ (this fact can be easily checked by using the definition of the absolute continuity). Consequently

$$\text{(18)} \qquad [s^\delta\, S(s)\, A(\varepsilon^{-1}(t_2 - s), x)]_g^c = \int_g^c \frac{\mathrm{d}}{\mathrm{d}s} (s^\delta\, S(s)\, A(\varepsilon^{-1}(t_2 - s), x))\, \mathrm{d}s .$$

Furthermore, for almost all $s \in (0, c]$

$$\left\| \frac{\mathrm{d}}{\mathrm{d}s} (s^\delta\, S(s)\, A(\varepsilon^{-1}(t_2 - s), x)) \right\| \leqq \|s^\delta\, S(s)\| \left\| \frac{1}{\varepsilon}\, a(\varepsilon^{-1}(t_2 - s), x) \right\| +$$

$$+ \left\| \delta s^{\delta - 1}\, S(s) + s^\delta \frac{\mathrm{d}}{\mathrm{d}s}\, S(s) \right\| \|A(\varepsilon^{-1}(t_2 - s), x)\| \leqq \frac{1}{\varepsilon}\, c^\delta M K_x +$$

$$+ \frac{1}{\varepsilon}\, c K_x (\delta M + Q)\, s^{\delta - 1} .$$

The term on the right hand side of this inequality is integrable, and thus (18) holds for $g = 0$ as well. So we may compute J_2 integrating by parts, obtaining

$$J_2 = -c^\delta\, S(c)\, [\varepsilon A(\varepsilon^{-1} t_1, x)] + \int_0^c \frac{\mathrm{d}}{\mathrm{d}s} (s^\delta\, S(s))\, [\varepsilon A(\varepsilon^{-1}(t_2 - s), x)]\, \mathrm{d}s .$$

Using now (17) together with the fact that the function under the integral sign is bounded by const. $s^{\delta - 1}$ (the constant independent of ε!) we have the desired equality $\lim_{\varepsilon \to 0+} J_2 = 0$. Q.E.D.

Remark. If we treat the equation (10) with the operator A_α independent of α and generating a holomorphic semigroup we can obviously weaken the assumption (8) in the way just mentioned, i.e. we may assume only

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} [a_\alpha(t, x) - a_0(t, x)]\, \mathrm{d}t = 0 , \quad x \in H , \quad 0 \leqq t_1 \leqq t_2 \leqq t_1 + \Delta_0 .$$

If A_α depends on α we must realize that the constant Q_α in the estimate $\|d/dt\, S_\alpha(t)\| \leqq \leqq Q_\alpha t^{-1}$, $t \in (0, 1]$, depends on the domain of analyticity of $S_\alpha(\cdot)$. In order to formulate an appropriate analogue of Lemma 3 we must assume in addition that there exists $\gamma > 0$ independent of α and such that all $S_\alpha(\cdot)$ have a holomorphic extension in $\{\xi \in C,\ \mathrm{Re}\, \xi > 0,\ |\arg \xi| \leqq \gamma < \pi/2\}$.

Up to now we have dealt with the Wiener process with a nuclear covariance operator. However, as we have already mentioned in the introduction, the situation for the cylindrical Wiener process does not differ substantially. Let us adopt the following assumptions.

(IIc) Let A: $D(A) \to H$ be a generator of a (C_0)-semigroup $S(t)$ on H such that $\int_0^T \|S(t)\|_{HS}^2\, dt < \infty$.

(Vc) Let there exist $\Delta_0 > 0$ such that for all $x \in H$ and $t_1, t_2 \in [0, T]$ such that $t_2 - t_1 \leqq \Delta_0$ we have

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} S(t_2 - s) [a_\alpha(s, x) - a_0(s, x)]\, ds = 0\,,$$

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} \|b_\alpha(s, x) - b_0(s, x)\|^p\, ds = 0\,.$$

Theorem 5. *Suppose that the assumptions* (I), (IIc), (III), (IV), (Vc) *are fulfilled. Denote by* $x_\alpha(t)$ *the mild solutions of the equations*

$$dx_\alpha(t) = (A\, x_\alpha(t) + a_\alpha(t, x_\alpha(t)))\, dt + b_\alpha(t, x_\alpha(t))\, dB(t)\,,$$
$$x_\alpha(0) \ = \varphi_\alpha\,.$$

If $\lim_{\alpha \to 0+} \|\varphi_\alpha - \varphi_0\|_p = 0$ *then*

$$\lim_{\alpha \to 0+} \sup_{t \in [0,T]} \|x_\alpha(t) - x_0(t)\|_p = 0\,.$$

Proof. We can repeat the proof of Theorem 3 almost step by step, so we point out only the differences. We adopt the same notation as in that proof; now the constants C_i may also depend on $\|S(\cdot)\|_{HS}$.

Again we may assume that $\varphi_\alpha = \varphi_0$, $\alpha > 0$, relying on Lemma 2 instead of Gronwall's inequality. We choose a partition $\{\tau_i\}_{i=1}^N$ fine enough to ensure $\int_0^h \|S(t)\|_{HS}^2\, dt < \eta^2$, where $h = \max\{\tau_i - \tau_{i-1},\ i = 1, \ldots, N\}$.

The bounds for the terms R_1, R_2 remain valid, I_2 being the only term to be estimated in a different manner:

$$\|I_2\|_p \leqq \int_0^t \|S(t - s) [a_\alpha(s, x_\alpha(s)) - a_\alpha(s, x_0(s))]\|_p\, ds \leqq$$
$$\leqq MK \int_0^t \|x_\alpha(s) - x_0(s)\|_p\, ds \leqq C_{14} \left(\int_0^t \|x_\alpha(s) - x_0(s)\|_p^2\, ds \right)^{1/2}\,.$$

We obtain a constant $\alpha_7 > 0$ such that for $\alpha \in (0, \alpha_7]$

$$\|R_1\|_p + \|R_2\|_p \leqq C_{15} \left(\eta + \left(\int_0^t \|x_\alpha(s) - x_0(s)\|_p^2 \mathrm{d}s\right)^{1/2}\right).$$

Further, we split $R_3 = K_1 + K_2 + K_3$ as before and estimate

$$\|K_1\|_p \leqq C(p)\left(\int_{\sigma(t)}^t |\,\|S(t-s)[b_\alpha(s, x_\alpha(s)) - b_0(s, x_0(s))]\|_{HS}^2|_{p/2}\,\mathrm{d}s\right)^{1/2} \leqq$$

$$\leqq C_{16}(1 + \|\varphi_0\|_p)\left(\int_0^{t-\sigma(t)} \|S(s)\|_{HS}^2\,\mathrm{d}s\right)^{1/2} \leqq C_{17}\eta\,;$$

$$\|K_2\|_p \leqq C(p)\left(\int_0^{\sigma(t)} |\,\|S(t-s)[b_\alpha(s,x_\alpha(s)) - b_\alpha(s,x_0(s))]\|_{HS}^2|_{p/2}\,\mathrm{d}s\right)^{1/2} \leqq$$

$$\leqq KC(p)\left(\int_0^t \|S(t-s)\|_{HS}^2\,\|x_\alpha(s) - x_0(s)\|_p^2\mathrm{d}s\right)^{1/2}.$$

For the term K_3 we have

$$\|K_3\|_p \leqq C(p)\left(\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i} |\,\|S(t-s)[b_\alpha(s, x_0(s)) - b_0(s, x_0(s))]\|_{HS}^2|_{p/2}\,\mathrm{d}s\right)^{1/2} \leqq \sqrt{3}\,C(p)\,.$$

$$.\left(\sum_{i=1}^{\tau(i)}\int_{\tau_{i-1}}^{\tau_i} |\,\|S(t-s)[b_\alpha(s, x_0(s)) - b_\alpha(s, x_0(\tau_{i-1}))]\|_{HS}^2|_{p/2}\,\mathrm{d}s +\right.$$

$$+\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i} |\,\|S(t-s)[b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1}))]\|_{HS}^2|_{p/2}\,\mathrm{d}s +$$

$$\left.+\sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i} |\,\|S(t-s)[b_0(s, x_0(\tau_{i-1})) - b_0(s,x_0(s))]\|_{HS}^2|_{p/2}\,\mathrm{d}s\right)^{1/2} \equiv$$

$$\equiv \sqrt{3}\,C(p)\,(J_1 + J_2 + J_3)^{1/2}\,.$$

It can be easily shown that $J_1 + J_3 \leqq C_{18}\eta^2$. Moreover, if $\{e_i\}_{i=1}^\infty$ is an orthonormal basis of H, then

$$J_2 = \sum_{i=1}^{\tau(t)}\int_{\tau_{i-1}}^{\tau_i} |\sum_{j=1}^{\infty}\|[b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1})))]^*.\, S(t-s)^* e_j\|^2|_{p/2}\,\mathrm{d}s \leqq$$

$$\leqq \sum_{i=1}^{N}\int_{\tau_{i-1}}^{\tau_i}\sum_{j=1}^{\infty} |\,\|[b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1}))]^*\,.\, S(t-s)^*\, e_j\|^2|_{p/2}\,\mathrm{d}s\,.$$

Now for an arbitrary $J \in \mathbb{N}$

$$\int_{\tau_{i-1}}^{\tau_i} \sum_{j=1}^{J} | \|[b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1}))]^* S(t-s)^* e_j\|^2|_{p/2} \, ds \leqq$$

$$\leqq JM^2 \int_{\tau_{i-1}}^{\tau_i} \|b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1}))\|_p^2 \, ds \leqq$$

$$\leqq JM^2 T^{1-2/p} \mathsf{E} \int_{\tau_{i-1}}^{\tau_i} \|b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1}))\|^p \, ds)^{2/p}$$

and the Lebesgue dominated convergence theorem together with the assumption (Vc) yields that this term tends to 0 as $\alpha \to 0+$. Further,

$$\int_{\tau_{i-1}}^{\tau_i} \sum_{j=J+1}^{\infty} | \|[b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1}))]^* S(t-s)^* e_i\|^2|_{p/2} \, ds \leqq$$

$$\leqq \int_{\tau_{i-1}}^{\tau_i} \sum_{j=J+1}^{\infty} \|S(t-s)^* e_i\|^2 \|b_\alpha(s, x_0(\tau_{i-1})) - b_0(s, x_0(\tau_{i-1}))\|_p^2 \, ds \leqq$$

$$\leqq C_{18} \int_{\tau_{i-1}}^{\tau_i} \sum_{j=J+1}^{\infty} \|S(t-s)^* e_i\|^2 \, ds$$

and this term is arbitrarily small for J sufficiently large; that is, we can find $\alpha_8 > 0$ such that $J_2 \leqq \eta^2$ for $\alpha \in (0, \alpha_8]$. Combining all the estimates we obtain

$$\|R_3\|_p \leqq C_{19}\left(\eta + \left(\int_0^t \|S(t-s)\|_{\mathrm{HS}}^2 \|x_\alpha(s) - x_0(s)\|_p^2 \, ds\right)^{1/2}\right),$$

hence

$$\|x_\alpha(t) - x_0(t)\|_p^2 \leqq C_{20}\left(\eta^2 + \int_0^t f(t-s)\| x_\alpha(s) - x_0(s)\|_p^2 \, ds\right),$$

where $f \in \mathrm{L}^1([0, T])$ and is positive, thus using Lemma 2 we complete the proof.

Q.E.D.

The "cylindrical version" of Theorem 4 is a consequence of Theorem 5 just in the same way as Theorem 4 follows from Theorem 3.

As the last topic we will briefly consider an interesting and important modification of the theory of semilinear equations driven by a cylindrical Wiener process suggested by G. DaPrato and J. Zabczyk ([2], see also [14], [17], [18]). They investigated the problem

(19) $$dx(t) = (A\,x(t) + f(t, x(t)))\,dt + dB(t),$$
$$x(0) = \varphi,$$

where the state space V, on which the nonlinear coefficient f is defined, is not identical with the Hilbert space H on which the cylindrical process $B(t)$ is defined. To be more correct, let us formulate this problem in the following precise form. First, let us introduce some assumptions:

(Z1) Let H be a real separable Hilbert space, V a Banach space embedded continuously and as a Borel subset into H. Let $A: D(A) \to V$, $D(A) \subseteq V$, define a (C_0)-semigroup $S(t)$ on V, extendable to a (C_0)-semigroup $S_0(t)$ on H with the infinitesimal generator A_0. Let $B(t)$ be an $(\mathscr{F}_t)$-adapted cylindrical Wiener process in H, defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathsf{P})$.

(Z2) Let $A_0: D(A_0) \to H$ be self-adjoint and negative definite. Suppose that the operator A_0^{-1} is nuclear and that the Gaussian measure $N(0, -\frac{1}{2}A_0^{-1})$ (with the mean 0 and the covariance operator $-\frac{1}{2}A_0^{-1}$) on H is supported by V.

The nuclearity of A_0^{-1} implies $\int_0^T \|S_0(t)\|_{HS}^2 \, dt < \infty$ for any $T > 0$, thus the process $\hat{Z}(t) = \int_0^t S_0(t-s) \, dB(s)$ is defined correctly. Let us further assume:

(Z3) The process $\hat{Z}(t)$ has an $(\mathscr{F}_t)$-adapted modification $Z(t)$ with V-continuous trajectories.

First we mention a simple existence result for the equation (19).

Lemma 4. *Let* (Z1), (Z2), (Z3) *be fulfilled. Let the function* $f: [0, T] \times V \to V$ *satisfy*

$$\sup_{t \in [0,T]} \sup_{x,y \in V} \|x - y\|_V^{-1} \|f(t, x) - f(t, y)\|_V < \infty ,$$

$$\sup_{t \in [0,T]} \|f(t, 0)\|_V < \infty .$$

Let $\varphi \in L^p(\Omega; V)$ *be* $\mathscr{F}_0$*-measurable. Then there exists a unique* V*-valued mild solution of* (19) *in* $\mathscr{C}([0, T]; L^p(\Omega; V))$.

Remark. Much more general existence results (for the autonomous case) can be found in the papers quoted above. The last assertion of our lemma, however, seems not to have appeared explicitly yet.

Proof. The proof is standard and resembles e.g. that of Theorem 1, so we only sketch it. We define an operator

$$\mathfrak{K}x(t) = S(t)\,\varphi + \int_0^t S(t-s) f(s, x(s)) \, ds + Z(t)$$

wanting to prove that this operator maps the subspace of the $(\mathscr{F}_t)$-adapted functions in $\mathscr{C}([0, T]; L^p(\Omega; V))$ into itself and is contractive if that subspace is endowed with an equivalent norm $\|y\| \equiv \sup \{e^{-\alpha t}(\mathsf{E}\|y(t)\|_V^p)^{1/p},\ t \in [0, T]\}$, $\alpha > 0$ sufficiently large. The only difficulty appears when checking that $\mathfrak{K}x(\cdot)$ is continuous, because we need to show that $Z(\cdot) \in \mathscr{C}([0, T]; L^p(\Omega; V))$. To this end, we choose $r \in [0, T]$ and a sequence $\{h_n\}$ of real numbers with the limit 0 arbitrarily and realize that $Z(r + h_n) - Z(r)$ is Gaussian in V. Indeed, in [17], Prop. 3, it is proved that if the assumptions (Z1), (Z2) hold then there exists a sequence $\Pi_n \in \mathscr{L}(H, H)$ of self-adjoint operators with finite-dimensional ranges $\operatorname{Rng} \Pi_n \subseteq V$, such that $\lim_{n\to\infty} \|\Pi_n x - x\|_V = 0$ for arbitrary $x \in V$. $K(r, k) \equiv Z(r + h_k) - Z(r)$ is Gaussian in H, i.e. $\langle K(r, k), h\rangle$ is a real Gaussian random variable for each $h \in H$ ($\langle .,. \rangle$ denotes the scalar product in H). Let $x^* \in V^*$ be arbitrary. Since $\dim \operatorname{Rng} \Pi_n < \infty$, $x^* \in (\operatorname{Rng} \Pi_n, |\cdot|_H)^*$, hence there exists $y_n \in H$ such that $x^*(y) = \langle y, y_n \rangle$ for all $y \in \operatorname{Rng} \Pi_n$, so $x^*(\Pi_n K(r, k)) = \langle \Pi_n K(r, k), y_n \rangle = \langle K(r, k), \Pi_n y_n \rangle$, i.e. $x^*(\Pi_n K(r, k))$ are Gaussian and tend to $x^*(K(r, k))$ a.s. as $n \to \infty$, hence in $L^2(\Omega)$ (see [12], Th.I.1). Consequently, $x^*(K(r, k))$ is Gaussian. By the assumption (Z3)

$$\lim_{k\to\infty} \|K(r, k)\|_V = 0 \quad \text{a.s.} \,. \tag{20}$$

The random variables $K(r, k)$ are Gaussian and centered, which together with (20) yields the existence of $\alpha > 0$ such that

$$\sup_{k\in N} \mathsf{E} \exp\left(\alpha \|K(r, k)\|_V^2\right) < \infty$$

(this strengthening of the theorem of Fernique is proved e.g. in [11], p. 148; we may also apply Prop. 1 in [14]), so the functions $\|K(r, k)\|_V^p$ are uniformly integrable and we obtain $K(r, k) \to 0$, $k \to \infty$, in $L^p(\Omega; V)$. Q.E.D.

Having checked that the mild solution of the equation (19) lies in $\mathscr{C}([0, T]; L^p(\Omega; V))$ we see that the proof of Theorem 3 can be easily modified to work also for the equations of DaPrato-Zabczyk's type; the modified version of Theorem 3 reads as follows:

Theorem 6. *Assume that the hypotheses* (Z1), (Z2), (Z3) *hold. Let* $f_\alpha: \mathbb{R}_+ \times V \to V$, $\alpha \in [0, 1]$, *be measurable functions and suppose that there exists* $K > 0$ *such that for all* $x, y \in V$, $t \geqq 0$, $\alpha \in [0, 1]$

$$\|f_\alpha(t, x) - f_\alpha(t, y)\|_V \leqq K \|x - y\|_V ,$$
$$\|f_\alpha(t, 0)\|_V \leqq K .$$

Let $\varphi_\alpha \in L^p(\Omega; V)$ *be* $\mathscr{F}_0$*-measurable. Denote by* $x_\alpha(t)$ *the mild solutions of the equations*

$$dx_\alpha(t) = (A\, x_\alpha(t) + f_\alpha(t, x_\alpha(t)))\, dt + dB(t) ,$$
$$x_\alpha(0) = \varphi_\alpha .$$

Suppose that for some $\Delta_0 > 0$ *and each* $x \in H$, $t_1, t_2 \in \mathbb{R}_+$ *such that* $0 \leqq t_1 \leqq t_2 \leqq t_1 + \Delta_0 < \infty$ *we have*

$$\lim_{\alpha\to 0+} \int_{t_1}^{t_2} S(t_2 - s)\,[f_\alpha(s, x) - f_0(s, x)]\, ds = 0 .$$

If $\lim_{\alpha\to 0+} \varphi_\alpha = \varphi_0$ *in* $L^p(\Omega; V)$ *then for all* $T > 0$

$$\lim_{\alpha\to 0+} x_\alpha = x_0 \quad \text{in} \quad \mathscr{C}([0, T]; L^p(\Omega; V)) .$$

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Souhrn

METODA PRŮMĚRŮ PRO STOCHASTICKÉ EVOLUČNÍ ROVNICE

JAN SEIDLER, IVO VRKOČ

Ve stati jsou dokázány věty o integrální spojitosti řešení stochastických parciálních diferenciálních rovnic evolučního typu podle parametru. Tyto rovnice jsou vyšetřovány v rámci semigrupového přístupu jako rovnice v Hilbertově prostoru, přičemž je paralelně uvažován případ rovnic s Wienerovým procesem s nukleárním kovariančním operátorem, rovnic s cylindrickým Wienerovým procesem a rovnic DaPrato-Zabczykova typu. Jako pomocný výsledek je dokázána dosti obecná existenční věta pro rovnice s cylindrickým Wienerovým procesem.

Резюме

МЕТОД УСРЕДНЕНИЯ ДЛЯ СТОХАСТИЧЕСКИХ ЭВОЛЮЦИОННЫХ УРАВНЕНИЙ

JAN SEIDLER, IVO VRKOČ

В статье установлены теоремы об интегральной непрерывности по параметру решений стохастических дифференциальных уравнений в частных производных эволюционного типа. Эти уравнения исследуются методами теории полугрупп как уравнения в гильбертовом пространстве, параллельно изучаются уравнения с процессом Винера с ядерным ковариационным оператором, уравнения с цилиндрическим процессом Винера и уравнения типа ДаПрата-Забчика. Подготовительно доказана довольно общая теорема о существовании решений уравнения с цилиндрическим процессом Винера.

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