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Commentationes Mathematicae Universitatis Carolinae, Vol. 35 (1994), No. 1, 55--62

Persistent URL: <http://dml.cz/dmlcz/118641>

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Some remarks about the p -Dirichlet integral*

MARIANO GIAQUINTA, GIUSEPPE MODICA, JIŘÍ SOUČEK

Abstract. We discuss variational problems for the p -Dirichlet integral, p non integer, for maps between manifolds, illustrating the role played by the geometry of the target manifold in their weak formulation.

Keywords: variational problems, p -Dirichlet integral

Classification: 49Q20

Let \mathcal{X} and \mathcal{Y} be two compact Riemannian manifolds of dimension respectively n and m . Suppose that \mathcal{Y} be without boundary and isometrically embedded in \mathbf{R}^N as a submanifold. For a given domain Ω in \mathcal{X} consider the variational problem

$$(1) \quad \begin{aligned} \mathcal{D}_p(u) &:= \int_{\Omega} |Du|^p dx \rightarrow \min \\ u &: \Omega \rightarrow \mathcal{Y}, u = \varphi \text{ on } \partial\Omega \end{aligned}$$

where $\varphi : \overline{\Omega} \rightarrow \mathcal{Y}$ is a given smooth map and p is a real number with $1 < p < \underline{n} := \min(n, m)$.

It is usual to seek for a minimizer of (1) in the Sobolev class

$$W_{\varphi}^{1,p}(\Omega, \mathcal{Y}) := \{ u \in W^{1,p}(\Omega, \mathbf{R}^N) \mid u(x) \in \mathcal{Y} \text{ for a.e. } x \in \Omega, \\ u = \varphi \text{ on } \partial\Omega \}.$$

However, in the generic situation in which the geometry of \mathcal{Y} is non trivial a *gap phenomenon* appears, i.e. we have

$$(2) \quad \begin{aligned} \inf \{ \mathcal{D}_p(u) \mid u \in W_{\varphi}^{1,p}(\Omega, \mathcal{Y}) \} &< \\ &< \inf \{ \mathcal{D}_p(u) \mid u \in C^1(\Omega, \mathcal{Y}) \cap C_{\varphi}^0(\overline{\Omega}, \mathcal{Y}) \}, \end{aligned}$$

compare [9], [10]. Moreover, in the weak limit process of sequences of smooth maps with equibounded \mathcal{D}_p -energies *concentrations* are produced in such a way that

$$\int_{\Omega} |Du|^p dx$$

*This work has been partially supported by the Ministero dell'Università e della Ricerca Scientifica, by C.N.R. contract n. 91.01343.CT01, by the European Research Project GADGET, and by Grant n.11957 of the Czech Acad. of Sciences.

is not the relaxed energy of \mathcal{D}_p , at least if p is an integer. Those phenomena are primarily due to the fact that maps in $W^{1,p}(\Omega, \mathcal{Y})$ lack the fundamental homological property of having boundaryless graphs in $\Omega \times \mathcal{Y}$ enjoyed instead by smooth maps, compare [6], [5], [8]. In order to overcome those difficulties we proposed in [5], [8] to replace in the generalized approach to (1) the Sobolev classes with the class of *Cartesian currents* and in general with the class of (r, ℓ) -currents, still in the case of integer p 's.

The situation is slightly different if p is not an integer, and this note aims to state a few remarks in this case. If p is not an integer, no concentration is produced in the weak limit procedure of sequences of smooth maps with equibounded \mathcal{D}_p -energies, and the gap is not anymore due to the energy associated to concentrations. Correspondingly, the limit graphs of sequences of smooth graphs do not contain *vertical parts* and they may be identified as a strict subclass of $W^{1,p}(\Omega, \mathcal{Y})$.

Introducing in fact the class

$$(3) \quad RW^{1,p}(\Omega, \mathcal{Y}) := \{u \in W^{1,p}(\Omega, \mathcal{Y}) \mid \partial G_u \llcorner \Omega \times \mathbf{R}^N = 0\}$$

where G_u is the current carried by the graph of u in the sense of (r, ℓ) -currents, and denoting by

$$(4) \quad H^{1,p}(\Omega, \mathcal{Y}) := \text{sequential weak closure of} \\ C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y}) \text{ in } W^{1,p}(\Omega, \mathcal{Y})$$

we shall see that

$$(5) \quad H^{1,p}(\Omega, \mathcal{Y}) \subset RW^{1,p}(\Omega, \mathcal{Y}) \subset W^{1,p}(\Omega, \mathcal{Y}).$$

Moreover,

$$(6) \quad RW^{1,p}(\Omega, \mathcal{Y}) \subsetneq W^{1,p}(\Omega, \mathcal{Y})$$

if \mathcal{Y} has a non trivial homology in a suitable sense.

By a result of Bethuel [1] we also know, still when p is not an integer, that $H^{1,p}(\Omega, \mathcal{Y})$ agrees with the strong closure of $C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y})$ in $W^{1,p}(\Omega, \mathcal{Y})$, and under quite restrictive assumptions on \mathcal{Y} that $RW^{1,p}(\Omega, \mathcal{Y}) = H^{1,p}(\Omega, \mathcal{Y})$, see [2], though in general

$$(7) \quad H^{1,p}(\Omega, \mathcal{Y}) \subsetneq RW^{1,p}(\Omega, \mathcal{Y}).$$

In conclusion we may summarize the situation, and hopefully make it clearer by comparison with the much simpler case of scalar maps, as follows, see for more details [4]. Denote by $\mathcal{A}^p(\Omega, \mathbf{R})$ the class of functions $u \in L^p(\Omega, \mathbf{R})$, $1 \leq p < \infty$,

which are almost everywhere approximately differentiable in Ω with approximate differential $\text{ap } Du$ in L^p . Then, in terms of distributional derivatives, the condition

$$\partial G_u \llcorner \Omega \times \mathbf{R} = 0$$

for $u \in \mathcal{A}^p(\Omega)$ is equivalent to $u \in W^{1,p}(\Omega, \mathbf{R})$. If $p > 1$, $W^{1,p}(\Omega, \mathbf{R})$ agrees with the sequential weak closure of $C^1 \cap W^{1,p}(\Omega, \mathbf{R})$ in $W^{1,p}$, if $p = 1$, the sequential weak closure of smooth functions with equibounded \mathcal{D}_1 -energies is instead $BV(\Omega, \mathbf{R})$. In the vector valued case $W^{1,p}(\Omega, \mathcal{Y})$ plays an analogous role as $\mathcal{A}^p(\Omega, \mathbf{R})$ in the scalar case; the space $W^{1,p}(\Omega, \mathbf{R})$ has a natural substitute in the class of Cartesian currents or (r, ℓ) -currents (according to which we are in the rectifiable case $p = \underline{n}$ or in the non-rectifiable case $p < \underline{n}$) if p is an integer or in the class $RW^{1,p}(\Omega, \mathcal{Y})$ if p is not an integer. Notice that in the vector valued case, if $p \in \mathbf{N}$ and the geometry of \mathcal{Y} is not trivial, with respect to the analogy to the scalar case we are close to the BV -situation more than to $W^{1,p}$ -situation. Apart from some specific and particular cases, compare [7], [2], and [11], the problem of characterizing “strong” and “sequential weak” closure is still largely open.

As in the sequel the structure of manifold for \mathcal{X} is irrelevant, from now on we shall assume that Ω is a bounded open set in \mathbf{R}^n . The relevant geometry of \mathcal{Y} will be expressed in terms of the cohomology groups of \mathcal{Y} . We assume that for values ℓ to be specified later the De Rham cohomology group of order ℓ , $H_{DR}^\ell(\mathcal{Y}, \mathbf{R})$ be non-zero and denote by $[\sigma_1], \dots, [\sigma_{\bar{s}}]$ a basis of $H_{DR}^\ell(\mathcal{Y}, \mathbf{R})$, where $\sigma_1, \dots, \sigma_{\bar{s}}$ are ℓ -forms regarded as ℓ -forms in \mathbf{R}^N , or better in a neighborhood of \mathcal{Y} in \mathbf{R}^N . Coordinates in \mathbf{R}^n and \mathbf{R}^N with respect to the standard bases (e_1, \dots, e_n) , $(\varepsilon_1, \dots, \varepsilon_N)$ are denoted by (x^1, \dots, x^n) and (y^1, \dots, y^N) .

For $r \leq \min(n, N)$ and $\ell \leq r$ we denote by $\mathcal{D}^{r,\ell}(\Omega \times \mathbf{R}^N)$ the space of smooth and compactly supported r -forms in $\Omega \times \mathbf{R}^N$ with at most ℓ differentials in the variables y . In coordinates any $\omega \in \mathcal{D}^{r,\ell}(\Omega \times \mathbf{R}^N)$ can be written as

$$(8) \quad \omega = \sum_{\substack{|\alpha|+|\beta|=r \\ |\beta| \leq \ell}} \omega_{\alpha\beta}(x, y) dx^\alpha \wedge dy^\beta.$$

The dual space of $\mathcal{D}^{r,\ell}(\Omega \times \mathbf{R}^N)$ will be referred as the space of (r, ℓ) -currents and denoted by $\mathcal{D}_{r,\ell}(\Omega \times \mathbf{R}^N)$, see [8]. Given a map $u \in W^{1,p}(\Omega, \mathbf{R}^N)$, $p < n$, the (n, ℓ) -graph or simply the graph of u is defined as the (n, ℓ) -current, ℓ being the integer part of p , $\ell := [p]$, given by

$$G_u(\omega) := \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta| \leq \ell}} \sigma(\alpha, \bar{\alpha}) \int_{\Omega} \omega_{\alpha\beta}(x, u(x)) M_{\bar{\alpha}}^\beta(Du(x)) dx$$

where ω is given by (8) and $M_{\bar{\alpha}}^\beta(Du(x))$ denotes the $(\beta, \bar{\alpha})$ minor of the approximate differential matrix $Du(x)$, $\bar{\alpha}$ is the complement of α in $\{1, 2, \dots, n\}$ and

$\sigma(\alpha, \bar{\alpha})$ denotes the sign of the permutation which reorders in increasing way the multiindex $(\alpha, \bar{\alpha})$.

Similarly we denote by $\mathcal{D}^{r,\ell}(\Omega \times \mathcal{Y})$, $\ell \leq r$, $r \leq \min(n, m)$, the space of r -forms in $\Omega \times \mathcal{Y}$ with at most ℓ differentials in \mathcal{Y} . The immersion $i : \mathcal{Y} \rightarrow \mathbf{R}^N$ induces a map

$$(\text{id} \boxtimes i)^\# : \mathcal{D}^{r,\ell}(\Omega \times \mathbf{R}^N) \rightarrow \mathcal{D}^{r,\ell}(\Omega \times \mathcal{Y})$$

which is onto. The space of (r, ℓ) -currents in $\Omega \times \mathcal{Y}$ is then defined as the subspace of (r, ℓ) -currents in $\mathcal{D}_{r,\ell}(\Omega, \times \mathbf{R}^N)$ which vanish on $\ker(\text{id} \boxtimes i)^\#$. It is easily checked that, if $u \in W^{1,p}(\Omega, \mathcal{Y})$, and $\ell = [p]$, then the (n, ℓ) -graph of u is an (n, ℓ) -current in $\Omega \times \mathcal{Y}$, $G_u \in \mathcal{D}_{n,\ell}(\Omega \times \mathcal{Y})$.

With the previous notations we now set

Definition 1. The reduced Sobolev class $RW^{1,p}(\Omega, \mathcal{Y})$ is given by

$$RW^{1,p}(\Omega, \mathcal{Y}) := \{ u \in W^{1,p}(\Omega, \mathcal{Y}) \mid G_u(\pi^\# d\varrho \wedge \hat{\pi}^\# \sigma_s) = 0 \\ \text{for all } s \text{ and for all } \varrho \in \mathcal{D}^{n-\ell-1}(\Omega) \}.$$

Here π and $\hat{\pi}$ denote the orthogonal projections of $\Omega \times \mathbf{R}^N$ into Ω and \mathbf{R}^N respectively.

Of course, if

$$d\varrho = \sum_{|\alpha|=n-\ell} \varrho_\alpha(x) dx^\alpha, \quad \sigma^s = \sum_{|\beta|=\ell} \psi_\beta^{(s)}(y) dy^\beta,$$

we have

$$(9) \quad G_u(\pi^\# d\varrho \wedge \hat{\pi}^\# \sigma_s) = \\ = \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=\ell}} \sigma(\alpha, \bar{\alpha}) \int_{\Omega} \varrho_\alpha(x) \psi_\alpha^{(s)}(u(x)) M_\alpha^\beta(Du(x)) dx.$$

Remark 1. For all s one can define the $(n - \ell)$ -current in Ω

$$\mathbf{D}_s(u) := \pi_\#(G_u \llcorner \hat{\pi}^\# \sigma_s)$$

and the $(n - \ell - 1)$ -current in Ω

$$\mathbf{P}_s(u) := \partial \mathbf{D}_s(u).$$

One can see, compare [8, p. 348], that, while $\mathbf{D}_s(u)$ depends on the representative σ_s of $[\sigma_s]$, $\mathbf{P}_s(u)$ depends only on the cohomology class $[\sigma_s]$. Moreover the whole system of conditions

$$\mathbf{P}_s(u) = 0 \quad s = 1, \dots, \bar{s}$$

depends only on the group $H_{DR}^\ell(\mathcal{Y})$ and not on the chosen basis of $H_{DR}^\ell(\mathcal{Y})$, so that

$$RW^{1,p}(\Omega, \mathcal{Y}) = \{ u \in W^{1,p}(\Omega, \mathcal{Y}) \mid \mathbf{P}_s(u) = 0 \forall s \}$$

is a subclass of $W^{1,p}(\Omega, \mathcal{Y})$ which is fixed by the cohomology group $H_{DR}^\ell(\mathcal{Y})$.

Remark 2. We notice that the system of conditions $\mathbf{P}_s(u) = 0 \ \forall s$ reads

$$\int_{\Omega} d\alpha \wedge u^{\#}(\sigma^s) = 0 \quad \forall s = 1 - n, \quad \forall \alpha \in \mathcal{D}^{n-\ell-1}(\Omega).$$

Since the forms σ^s generate all closed forms modulo exact forms and

$$\int_{\Omega} d\alpha \wedge u^{\#}(\beta) = 0$$

for any exact form $\beta \in \mathcal{D}^{\ell}(\mathcal{Y})$, then

$$RW^{1,p}(\Omega, \mathcal{Y}) = \{ u \in W^{1,p}(\Omega, \mathcal{Y}) \mid u^{\#}(\beta) = 0 \\ \text{for any closed form } \beta \in \mathcal{D}^{\ell}(\mathcal{Y}) \}.$$

Compare [2].

Remark 3. One can introduce a notion of boundary of (r, ℓ) -currents, compare [8] and in particular Propositions 2.1 and 3.2. Then we have $\mathbf{P}_s(u) = 0$ for all s if and only if $\partial G_u = 0$ in $\Omega \times \mathcal{Y}$.

Remark 4. In the special case that Ω is the unit ball of \mathbf{R}^3 , $\mathcal{Y} = S^2$, and $\ell = 2$, there is only one generator of $H_{DR}^2(\mathcal{Y}, \mathbf{R}) \simeq \mathbf{R}$ which is represented by the volume form ω_{S^2} of S^2 . In this case

$$\mathbf{D}_1(u)(\alpha) = \int \langle \alpha, D(u) \rangle dx \quad \forall \alpha \in \mathcal{D}^1(B^3)$$

where $D(u)$ is the vector field

$$D(u) := (u \cdot u_{x_2} \times u_{x_3}, u \cdot u_{x_3} \times u_{x_1}, u \cdot u_{x_1} \times u_{x_2}).$$

Moreover, for $2 < p < 3$ we have

$$RW^{1,p}(\Omega, S^2) = \{ u \in W^{1,p}(\Omega, S^2) \mid \operatorname{div} D(u) = 0 \}.$$

Theorem 1. *Suppose that p is not an integer. Then $RW^{1,p}(\Omega, \mathcal{Y})$ is sequentially weakly closed in $W^{1,p}(\Omega, \mathcal{Y})$.*

PROOF: Let $\{u_k\}$ be a weakly converging sequence in $W^{1,p}(\Omega, \mathcal{Y})$. Then $\{Du_k\}$ is equibounded in L^p and $\{M(Du_k)\}$ is equibounded in $L^{p/\ell}$, $p/\ell > 1$. Thus passing to a subsequence

$$M(Du_k) \rightharpoonup M(Du) \quad \text{weakly in } L^{p/\ell} \\ u_k(x) \rightarrow u(x) \quad \text{for a.e. } x.$$

As the $\psi_{\beta}^{(s)}$ are bounded in L^{∞} , we therefore can pass to the limit in

$$G_{u_k}(\pi^{\#} d_{\varrho} \wedge \widehat{\pi}^{\#} \sigma_s) = 0,$$

compare (9), getting also

$$G_u(\pi^{\#} d_{\varrho} \wedge \widehat{\pi}^{\#} \sigma_s) = 0.$$

□

Remark 5. Notice that the proof above shows also that $\mathbf{P}_s(u_k) \rightarrow \mathbf{P}_s(u)$ provided $M(Du_k) \rightarrow M(Du)$ weakly in L^1 .

We shall now prove that $RW^{1,p}(\Omega, \mathcal{Y})$ is a proper subspace of $W^{1,p}(\Omega, \mathcal{Y})$ whenever the homology group $H_\ell(\mathcal{Y}, \mathbf{Z})$ is not trivial in the sense specified below. Denote by $H_{\ell(tf)}(\mathcal{Y}, \mathbf{Z})$ the torsion free part of the singular homology group with integer coefficients $H_\ell(\mathcal{Y}, \mathbf{Z})$, $\ell = [p]$. It is well-known that $H_{\ell(tf)}(\mathcal{Y}, \mathbf{Z})$ is finitely generated and that it can be represented by choosing a finite set of integer rectifiable cycles $\gamma_1, \dots, \gamma_{\bar{s}}, \bar{s}$ being the dimension of $H_{DR}^\ell(\mathcal{Y}, \mathbf{R})$, as

$$H_{\ell(tf)}(\mathcal{Y}, \mathbf{Z}) = \left\{ \sum_{s=1}^{\bar{s}} k_s [\gamma_s]_{\mathbf{Z}} \mid k_s \in \mathbf{Z} \right\}.$$

We now say that a homology class $[\gamma] \in H_\ell(\mathcal{Y}, \mathbf{Z})$ is of the type S^ℓ if $[\gamma]$ contains an S^ℓ -cycle, i.e. there exists a smooth map $\phi : S^\ell \rightarrow \mathcal{Y}$, $\phi \in C^1(S^\ell, \mathcal{Y})$, such that the image by ϕ of the current $[[S^\ell]]$ is the homology class of γ . The subgroup of $H_\ell(\mathcal{Y}, \mathbf{Z})$ of all homology classes $[\gamma]$ of the type S^ℓ will be denoted by $H_\ell^{(sph)}(\mathcal{Y}, \mathbf{Z})$.

Our main assumption on \mathcal{Y} is

(I) *The subgroup*

$$H_{\ell(tf)}^{(sph)}(\mathcal{Y}, \mathbf{Z}) := H_{\ell(tf)}(\mathcal{Y}, \mathbf{Z}) \cap H_\ell^{(sph)}(\mathcal{Y}, \mathbf{Z})$$

of $H_\ell(\mathcal{Y}, \mathbf{Z})$ is not trivial.

This is clearly equivalent to

(I') *There exists a map $\phi \in C^1(S^\ell, \mathcal{Y})$ such that, apart from the zero multiple, no integer multiple of the image of $[[S^\ell]]$ by ϕ is homologous to zero*

or to

(I'') *There exists a map $\phi \in C^1(S^\ell, \mathcal{Y})$ and a closed form $\sigma_1 \in \mathcal{D}^\ell(\mathcal{Y})$ such that $\phi_{\#}[[S^\ell]](\sigma_1) \neq 0$.*

Finally, for the sake of simplicity we shall assume that Ω is bilipschitz homeomorphic to the unit ball of \mathbf{R}^n . We then have

Theorem 2. *Suppose p is not an integer and let $\ell = [p]$. If \mathcal{Y} satisfies (I), then*

$$RW^{1,p}(\Omega, \mathcal{Y}) \subsetneq W^{1,p}(\Omega, \mathcal{Y}).$$

PROOF: It suffices to construct a map $u : B^\ell \times B^{n-\ell} \rightarrow \mathcal{Y}$, $u \in W^{1,p}(\Omega, \mathcal{Y})$, such that $u \notin RW^{1,p}(\Omega, \mathcal{Y})$, B^ℓ being the unit ball in \mathbf{R}^ℓ . We may think of the map $\phi : S^\ell \rightarrow \mathcal{Y}$ in (I'') as a map $\psi : B^\ell \rightarrow \mathcal{Y}$ which is constant, $\psi = c_0 \in \mathcal{Y}$, on ∂B^ℓ . We now extend ψ to be c_0 on $\mathbf{R}^\ell \setminus B^\ell$. Clearly ψ is a Lipschitz map from \mathbf{R}^ℓ into \mathcal{Y} .

Next, for $(w, t) \in \mathbf{R}^\ell \times (-1, 1)$ we define

$$v(w, t) := \begin{cases} \psi\left(\frac{w}{t}\right) & t > 0 \\ c_0 & t \leq 0 \end{cases}.$$

Clearly $v \in \text{Lip}(\mathbf{R}^\ell \times (-1, 1) \setminus \{0, 0\})$ and an easy computation shows that

$$\int_{\mathbf{R}^\ell \times (-1, 1)} |Dv|^{\ell+\delta} dw dt < \infty, \quad 0 < \delta < 1,$$

so that $v \in W^{1,p}(B^\ell(0, 2) \times B^{n-\ell}(0, 2), \mathcal{Y})$.

Finally, set $\Omega = B^\ell(0, 2) \times B^{n-\ell}(0, 2)$ and consider the map $u : \Omega \rightarrow \mathcal{Y}$ defined by

$$u(w, z) := v(w, |z| - 1).$$

Clearly, $u \in W^{1,p}(\Omega, \mathcal{Y})$ and $u \in \text{Lip}(\Omega \setminus \Sigma, \mathcal{Y})$ where

$$\Sigma := \{(w, z) \mid w = 0, |z| = 1\} = \{0\} \times S^{n-\ell-1}.$$

We shall now prove that $u \notin RW^{1,p}(\Omega, \mathcal{Y})$.

As we can assume by (I'') that $\psi_\# \llbracket B^\ell \rrbracket(\sigma_1) = \int_{B^\ell} \psi^\# \sigma_1 \neq 0$, it suffices to show that there exists r such that

$$\mathbf{P}_1(u) = r\delta_0 \times \llbracket S^{n-\ell-1} \rrbracket$$

and, moreover,

$$r = u_\#(\llbracket S_{x,\varepsilon} \rrbracket)(\sigma_1)$$

where $S_{x,\varepsilon}$ is a small ℓ -sphere centered at a point $x \in \Sigma$ in the $(\ell + 1)$ -plane orthogonal to Σ at x . Choosing $x_0 = (w_0, z_0)$, $z_0 := (1, 0, \dots, 0)$ we have

$$S_{x_0,\varepsilon} = \{(w, z) \mid |w|^2 + (z_1 - 1)^2 = \varepsilon^2, z_2 = \dots = z_{n-\ell} = 0\},$$

therefore by the definition of u

$$u_\# \llbracket S_{x_0,\varepsilon} \rrbracket = u_\# \llbracket \mathbf{R}^\ell \times \{(2, 0, \dots, 0)\} \rrbracket = \psi_\# \llbracket \mathbf{R}^\ell \rrbracket = \phi_\# \llbracket B^\ell \rrbracket.$$

Notice that by a homotopy argument we have

$$u_\# \llbracket S_{x,\varepsilon} \rrbracket \in [\phi_\# \llbracket S^\ell \rrbracket]$$

for any $x \in \Sigma_1$ and $\varepsilon < 1$. Therefore we infer

$$r = u_\# \llbracket S_{x,\varepsilon} \rrbracket(\sigma_1) = \phi_\# \llbracket B^\ell \rrbracket(\sigma_1) \neq 0,$$

i.e. $\mathbf{P}_1(u) \neq 0$. □

As we have already mentioned, Bethuel in [1] showed that the sequential weak closure $H^{1,p}(\Omega, \mathcal{Y})$ of $C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y})$ in $W^{1,p}(\Omega, \mathcal{Y})$ agrees with the strong closure of $C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y})$ in $W^{1,p}(\Omega, \mathcal{Y})$, provided p is not an integer. In particular we see that

$$H^{1,p}(\Omega, \mathcal{Y}) \subset RW^{1,p}(\Omega, \mathcal{Y})$$

as trivially $C^1(\Omega, \mathcal{Y}) \cap W^{1,p}(\Omega, \mathcal{Y}) \subset RW^{1,p}(\Omega, \mathcal{Y})$ by the Stokes theorem. Under the quite restrictive assumption that \mathcal{Y} is $([p]-1)$ -connected, that is, all homotopy groups of order $\leq [p]-1$ of \mathcal{Y} are trivial, it has been proved in [2] that the strong closure of smooth maps in $W^{1,p}$ agrees with $RW^{1,p}(\Omega, \mathcal{Y})$, so that

$$RW^{1,p}(\Omega, \mathcal{Y}) = H^{1,p}(\Omega, \mathcal{Y}).$$

However, the general case seems to be largely open.

Finally, we notice that if p is an integer and \mathcal{Y} has a non trivial geometry as in Theorem 2, then $RW^{1,p}(\Omega, \mathcal{Y})$ is *not* sequentially weakly closed. In order to see that, it suffices to approximate by smooth maps the map u in the proof of Theorem 2 as in [9], [3].

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(Received May 4, 1993)