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The area formula for $W^{1,n}$ -mappings

JAN MALÝ

Abstract. Let f be a mapping in the Sobolev space $W^{1,n}(\Omega, \mathbf{R}^n)$. Then the change of variables, or area formula holds for f provided removing from counting into the multiplicity function the set where f is not approximately Hölder continuous. This exceptional set has Hausdorff dimension zero.

Keywords: Sobolev spaces, change of variables, area formula, Hölder continuity

Classification: 28A75, 26B15

1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be an open set. Let $f: \Omega \rightarrow \mathbf{R}^n$ be a mapping and $S \subset \Omega$. We define the multiplicity function (Banach indicatrix) \mathcal{N} by

$$\mathcal{N}(y, f, S) = \#\{x \in S: f(x) = y\}.$$

If f is a Lipschitz mapping on Ω , then f is differentiable a.e. (Rademacher theorem) and the area formula

$$(1.1) \quad \int_S |\det \nabla f(x)| dx = \int_{\mathbf{R}^n} \mathcal{N}(y, f, S) dy$$

holds for any measurable set $S \subset \Omega$ (see [2]). The same is true if f is a continuous representative of a mapping in $W^{1,p}(\Omega, \mathbf{R}^n)$ with $p > n$ (as the Lusin (N)-property holds, cf. Proposition 1.1 and [1]). There are continuous mappings in $W^{1,p}(\Omega, \mathbf{R}^n)$ with $p \leq n$ for which the area formula does not hold (see [13], [9] and references therein). The problem of the area formula for Sobolev mappings is continuously stimulating. For interesting recent results we refer to [10]. One approach consists in looking for “partial area formulae”: a set S_0 of full measure is found such that

$$\int_S |\det \nabla f(x)| dx = \int_{\mathbf{R}^n} \mathcal{N}(y, f, S \cap S_0) dy$$

for all measurable $S \subset \Omega$.

As shown by Federer [3], for any f which has partial derivatives almost everywhere there are sequences f_j of Lipschitz mappings and M_j of disjoint measurable sets such that $f_j = f$ on M_j and $\Omega \setminus \bigcup_j M_j$ has zero measure. The following proposition is then an easy and well known consequence (cf. [10], [5]).

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1.1 Proposition. *Let $S \subset \Omega$ be a measurable set and $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbf{R}^n)$. Then the following assertions are equivalent:*

- (i) *“the (N)-property holds for f on S ”: $|f(E)| = 0$ for each $E \subset S$ with $|E| = 0$,*
- (ii) *“the area formula holds for f on S ”:*

$$\int_{S'} |\det \nabla f(x)| \, dx = \int_{\mathbf{R}^n} \mathcal{N}(y, f, S') \, dy$$

for each measurable set $S' \subset S$,

- (iii) *“the change of variables formula holds for f on S ”:*

$$\int_S u(f(x)) |\det \nabla f(x)| \, dx = \int_{\mathbf{R}^n} u(y) \mathcal{N}(y, f, S) \, dy$$

for each nonnegative Borel measurable function u on \mathbf{R}^n .

Let f be a function in L_{loc}^1 defined a.e. in Ω . Then the function

$$\tilde{f}(x) := \lim_{r \rightarrow 0^+} \int_{B(x,r)} f.$$

is defined in Ω except a set of measure zero. The function \tilde{f} is called the *Lebesgue representative* of f and we say that f is *Lebesgue precise* if $f = \tilde{f}$. If, in addition, $f \in W^{1,p}(\Omega)$ with $p > 1$, then \tilde{f} is defined up to a set of p -capacity zero and p -finely continuous except for a set of p -capacity zero (see [14, Section 3.3]), which means that it is p -quasi-continuous ([6, Theorem 8]). These references are also recommended for definitions of p -capacity, p -quasi-continuity and p -fine topology (by the p -capacity cap_p we understand the Bessel capacity denoted by $B_{1,p}$ in [14]).

The following theorem gives a good choice of a set of canonical nature for which the area formula holds ([5], cf. also [4]).

1.2 Proposition. *Let $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbf{R}^n)$ be Lebesgue precise and S be the set of all points of Ω at which f is approximately differentiable. Then the area formula holds for f on S .*

The aim of this paper is to use a slight refinement of methods from [9] to show that for $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ the set to be removed for validity of the area formula can be found even smaller. The following theorem will be proved in the next section.

1.3 Theorem. *Suppose that f is an n -quasi-continuous representative of a mapping in $W_{\text{loc}}^{1,n}(\Omega, \mathbf{R}^n)$ and S is the set of all points of Ω at which f is approximately Hölder continuous. Then the area formula holds for f on S .*

The size of the exceptional set is estimated in the following result, proved in Section 3.

1.4 Theorem. *Suppose that f is an n -quasi-continuous representative of a mapping in $W^{1,n}_{loc}(\Omega, \mathbf{R}^n)$ and S is the set of all points of Ω at which f is approximately Hölder continuous. Then the set $\Omega \setminus S$ has Hausdorff dimension zero.*

2. Points of approximate Hölder continuity

Let f be a measurable function on Ω . We say that f is *approximately Hölder continuous* at $x \in \Omega$ if there is $\alpha \in (0, 1]$ and a set M such that

$$\limsup_{y \rightarrow x, y \in M} \frac{|f(y) - f(x)|}{|y - x|^\alpha} < \infty$$

and the Lebesgue density of M at x is one.

We need the following version of the Gehring oscillation lemma.

2.1 Lemma. *Let f be a quasi-continuous representative of a mapping in $W^{1,n}(B(x, r), \mathbf{R}^m)$. Then for almost all $t \in (0, r)$ the restriction of f to $\partial B(x, t)$ is a continuous representative of an element of $W^{1,n}(\partial B(x, t), \mathbf{R}^m)$ and the inequality*

$$(2.1) \quad \left(\text{diam } f(\partial B(x, t))\right)^n \leq ct \int_{\partial B(x, t)} |\nabla f|^n dS.$$

holds.

PROOF: The estimate follows from the Sobolev inequality and a similarity argument if f is C^1 . In the general case there are C^1 mappings f_j such that

$$\sum_j \|f_j - f\|_{1,p}^p < \infty.$$

Using integration over radii it follows that there is $N_1 \subset (0, r)$ with $|N_1| = 0$ such that

$$\sum_j \int_{\partial B(x, t)} (|f_j - f|^p + |\nabla f_j - \nabla f|^p) dS < \infty.$$

Since f is n -quasi-continuous and $f_j \rightarrow f$ in $W^{1,n}$, we know (after selecting a subsequence) that $f_j \rightarrow f$ except a set E of n -capacity zero ([12, Theorem 5.4]). By well known relations between capacity and Hausdorff measure ([12]) it follows that the linear measure of E is zero, so that there is $N_2 \subset (0, r)$ with $|N_2| = 0$ such that $f_j \rightarrow f$ everywhere on $\partial B(x, t)$ for each $t \in (0, r) \setminus N_2$. If $t \in (0, r) \setminus (N_1 \cup N_2)$, then the Sobolev inequality implies uniform convergence $f_j \rightarrow f$ on $\partial B(x, t)$ and a routine passage to limit yields (2.1). □

The following tool is essentially Lemma 4.5 of [9].

2.2 Lemma. *Let B be a ball $B(x, r)$ and $\tau \in (0, 1)$. Suppose $f \in W^{1,n}(B, \mathbf{R}^n)$. Then there is a measurable set $A \subset B$ such that*

$$|B \setminus A| \leq \tau|B|$$

and

$$(\text{diam } f(A))^n \leq c \int_A (1 + |\nabla f|^n) dy,$$

c depends only on n and τ .

PROOF: Although the proof in fact follows the idea in [9], we present it here, as in the original the mapping f is assumed to be continuous. Set

$$\omega_0 = \inf\{\text{diam } f(\partial B(x, t)): t \in [r/2, r]\}.$$

Find $t_0 \in [r/2, r]$ such that $\text{diam } f(\partial B(x, t_0)) < \omega_0 + r$ and choose $z_0 \in \partial B(x, t_0)$. Denote

$$\begin{aligned} u(y) &= |f(y) - f(z_0)|, \\ \lambda_i &= i(r + \omega_0), \\ E_i &= \{y \in B(x, r): |f(y) - f(z_0)| < \lambda_i\}, \\ v_{i,j} &= \max(\min(u, \lambda_j), \lambda_i). \end{aligned}$$

First we will derive the estimate

$$(2.2) \quad \omega_0^n \leq c \int_{B(x,r) \cap E_3} |\nabla f|^n dy.$$

If there exists $t_1 \in [r/2, r]$ such that $\partial B(x, t_1) \cap E_2 = \emptyset$, we observe that $v_{1,2} = \lambda_1$ on $\partial B(x, t_0)$ and $v_{1,2} = \lambda_2$ on $\partial B(x, t_1)$. It follows

$$(r + \omega_0)^n = (\lambda_2 - \lambda_1)^n \leq c \int_{B(x,r)} |\nabla v_{1,2}|^n dy \leq c \int_{E_3 \cap B(x,r)} |\nabla f|^n dy,$$

which proves (2.2) in this case. Now we may assume that $\partial B(x, t)$ intersects E_2 for all $t \in [r/2, r]$. We write $F = \{t \in [r/2, r] : \partial B(x, t) \subset E_3\}$. Using Lemma 2.1 to f and $v_{2,3}$, for almost all $t \in [r/2, r]$ we get

$$(2.3) \quad t^{-1} \omega_0^n \leq c \int_{E_3 \cap \partial B(x,t)} |\nabla f|^n dS.$$

Indeed, if $t \in F$, we estimate

$$\begin{aligned} \omega_0^n &\leq (\text{diam } f(\partial B(x, t)))^n \leq ct \int_{\partial B(x,t)} |\nabla f|^n dS \\ &= ct \int_{E_3 \cap \partial B(x,t)} |\nabla f|^n dS, \end{aligned}$$

while for $t \in [r/2, r] \setminus F$ we have

$$\begin{aligned} \omega_0^n &\leq (\lambda_3 - \lambda_2)^n \leq (\text{diam } v_{2,3}(\partial B(x, t)))^n \leq ct \int_{\partial B(x, t)} |\nabla v_{2,3}|^n dS \\ &\leq ct \int_{E_3 \cap \partial B(x, t)} |\nabla f|^n dS. \end{aligned}$$

Integrating (2.3) over $t \in [r/2, r]$ we obtain (2.2). Now, if $|B \setminus E_3| \leq \tau|B|$, we are done. Indeed, setting $A = B \cap E_3$, we obtain

$$(\text{diam } f(A))^n \leq c(\omega_0 + r)^n \leq c \int_A |\nabla f|^n dy + cr^n \leq c \int_A (1 + |\nabla f|^n) dy.$$

Otherwise we find $k \geq 3$ such that

$$|B \setminus E_{k+1}| \leq \tau|B| < |B \setminus E_k|$$

and set $A = B \cap E_{k+1}$. Since $v_{1,k} - \lambda_1 = 0$ on $\partial B(x, t_0)$, a Poincaré-type inequality ([14, Section 4.5]) yields

$$\begin{aligned} (\text{diam } f(A))^n &\leq (2\lambda_{k+1})^n \leq c\tau^{-1}r^{-n} \int_B (v_{1,k} - \lambda_1)^n dy \leq c\tau^{-1} \int_B |\nabla v_{1,k}|^n dy \\ &\leq c\tau^{-1} \int_A |\nabla f|^n dy, \end{aligned}$$

which concludes the proof. □

PROOF OF THEOREM 1.3: We verify the condition (i) of Proposition 1.1. Recall that S is the set of all points where f is approximately Hölder continuous. Let $E \subset S$ be a set of zero measure. Decomposing E if necessary into a countable union, we may assume that f is $1/m$ -Hölder continuous at all $x \in E$ for $m \in \mathbf{N}$ fixed. Choose an open set $G \subset \Omega$ containing E . Fix $x \in E$. There are $K > 0$ and a set $M \subset \Omega$, such that the Lebesgue density of M at x is one and

$$|f(y) - f(x)| \leq K|y - x|^{1/m}$$

for all $y \in M$. Find $r_0 > 0$ such that $B(x, r_0) \subset G$ and

$$|B(x, r) \setminus M| \leq \frac{1}{4}|B(x, r)|$$

for all $r \in (0, r_0)$. For $k = 0, 1, \dots$ we denote

$$\begin{aligned} r_k &= r_0 2^{-k}, \\ B_k &= B(x, r_k) \end{aligned}$$

and using Lemma 2.2 we find a measurable set $A_k \subset B_k$ such that

$$|B_k \setminus A_k| \leq 2^{-n-2}|B_k|$$

and

$$(2.4) \quad (\text{diam } f(A_k))^n \leq c \int_{A_k} (1 + |\nabla f|^n) dy.$$

Then (for $k \geq 1$)

$$\begin{aligned} |B_k \setminus A_{k-1}| &\leq 2^{-n-2}|B_{k-1}| = \frac{1}{4}|B_k|, \\ |B_k \setminus A_k| &\leq 2^{-n-2}|B_k|, \\ |B_k \setminus M| &\leq \frac{1}{4}|B_k|, \end{aligned}$$

and thus there is $x_k \in A_k \cap A_{k-1} \cap M$, $x_k \neq x$. We have

$$(2.5) \quad |f(x_k) - f(x)| \leq c|x_k - x|^{1/m} \leq c2^{-k/m}.$$

Choosing b with $(1 + 1/b)^{-1} > 2^{-1/m}$ we claim that the set

$$I(x) = \{k \in \mathbf{N} : |f(x_{k+1}) - f(x)| \leq b|f(x_k) - f(x_{k+1})|\}$$

is infinite. Indeed, assuming that $\max I(x) = k_0$, we get

$$|f(x_k) - f(x)| \leq |f(x_{k+1}) - f(x)| + |f(x_k) - f(x_{k+1})| \leq (1 + 1/b)|f(x_{k+1}) - f(x)|$$

for each $k > k_0$, which leads to a contradiction with (2.5), as an iteration yields

$$|f(x_k) - f(x)| \geq c(1 + 1/b)^{-k}.$$

Denote

$$R_k(x) = \text{diam } f(A_k) + |f(x_{k+1}) - f(x)|.$$

Since $x_k, x_{k+1} \in A_k$, we have

$$f(A_k) \subset B(f(x), R_k)$$

and (using (2.4))

$$\begin{aligned} R_k(x) &\leq \text{diam } f(A_k) + b|f(x_{k+1}) - f(x_k)| \leq (1 + b) \text{diam } f(A_k) \\ &\leq c(1 + b) \left(\int_{A_k} (1 + |\nabla f|^n) dy \right)^{1/n} \end{aligned}$$

whenever $k \in I(x)$. For the next step of the proof we write this estimate in the form

$$(2.6) \quad (R_k(x))^n \leq c \int_{f^{-1}(B(f(x), R_k(x)))} (1 + |\nabla f|^n) dy, \quad k \in I(x).$$

The balls $B(f(x), R_k)$, $x \in E$, $k \in I(x)$, form a Vitali cover of $f(E)$. By the Vitali covering theorem there is a disjoint subcover $B(f(x_\tau), R_\tau)$ such that the set $N := E \setminus \bigcup_\tau B(f(x_\tau), R_\tau)$ has zero Lebesgue measure. Using (2.6) it follows that

$$\begin{aligned} |f(E)| &\leq |N| + 2^n \sum_\tau R_\tau^n \\ &\leq c \sum_\tau \int_{f^{-1}(B(f(x_\tau), R_\tau^n))} (1 + |\nabla f|^n) dy \leq c \int_G (1 + |\nabla f|^n) dy. \end{aligned}$$

Varying the set G we get the required conclusion $|f(E)| = 0$. □

3. Size of the exceptional set

3.1 Lemma. *Let $p > 1$. Then the p -fine closure of any open set $A \subset \mathbf{R}^n$ has the same p -capacity as A .*

PROOF: See [11, Proposition 3.2]. □

The following tool is an immediate consequence of Theorem 7 of [8].

3.2 Proposition. *Let f be an n -quasi-continuous representative of a mapping in $W^{1,n}(\Omega, \mathbf{R}^n)$, $\varepsilon > 0$ and $p < n$. Then there is an open set $G \subset \mathbf{R}^n$ such that $\text{cap}_p G < \varepsilon$ and the restriction of f to $\Omega \setminus G$ is locally Hölder continuous.*

PROOF OF THEOREM 1.4: Using Proposition 3.2 we find open sets $G_{i,j}$ and $p_i \nearrow n$ such that $\text{cap}_{p_i}(G_{i,j}) < 1/j$ and f is locally Hölder continuous on $\Omega \setminus G_{i,j}$. Denote by $S_{i,j}$ the p_i -fine interior of $\mathbf{R}^n \setminus G_{i,j}$ and $S = \Omega \cap \bigcup_{i,j} S_{i,j}$. By Lemma 3.1, $\text{cap}_{p_i}(\mathbf{R}^n \setminus S_{i,j}) < 1/j$, and thus the Hausdorff dimension of E is zero (see e.g. [7, Theorem 2.26]). Consider a point $x \in S$. Then there are i, j such that $x \in \Omega \cap S_{i,j}$. Since the set $S_{i,j}$ is p_i -finely open and f is locally Hölder continuous on $S_{i,j}$, it follows that the Lebesgue density of S at x is one ([14, Section 3.3]) and f is approximately Hölder continuous at x . □

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