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On extended frames

Jorge Picado*

Abstract. Some aspects of extended frames are studied, namely, the behaviour of ideals, covers, admissible systems of covers and uniformities.

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1. Introduction

Recall that A is a *frame* if it is a complete lattice satisfying the frame distributive law $a \wedge \bigvee S = \bigvee\{a \wedge t \mid t \in S\}$, for every $a \in A$ and $S \subseteq A$, and that a function $f : L \longrightarrow M$ between frames is a *frame homomorphism* provided that it preserves finitary meets (including the unit 1) and arbitrary joins (including the zero 0).

For general facts about frames we refer to Johnstone [1] and Vickers [8].

In [7] A. Pultr introduced an algebraic structure extending that of a frame (called *extended frame* or, shortly, *E-frame*) in which the operation $X \cdot a = \bigvee\{x \in X \mid x \wedge a \neq 0\}$, defined, in a frame A, for every cover X of A and every $a \in A$, which has been very useful in the study of uniformities and metrics in frames (cf. e.g. [2], [4], [5] and [6]), becomes an intrinsic operation:

Definition 1.1 [7, Definition 1.1]**.** A pair $(A, \cdot)$ is an *E-frame* if A is a frame and $\cdot$ is a binary operation on A satisfying the following axioms:

(0) $0 \cdot 1 = 0$;
(1) $x \wedge y \leq x \cdot y$;
(2) $(x \wedge y) \cdot 1 = (x \cdot 1) \wedge (y \cdot 1)$;
(3) $(x \wedge y) \cdot 1 = 0 \Rightarrow x \cdot y = 0$;
(4) $(x \cdot y) \cdot 1 = x \cdot (y \cdot 1)$;
(5) $x \cdot (\bigvee Y) = \bigvee_{y \in Y}(x \cdot y)$;
(6) $(\bigvee X) \cdot (y \cdot 1) = (\bigvee_{x \in X}(x \cdot y)) \cdot 1$.

Remarks 1.2. (i) We added axiom (0) to Definition 1.1 of [7] because this condition is necessary to conclude, in the proof of Theorem 3.3 of [7], that $\lambda(0) = 0$ and it is not a consequence of the other axioms. For example, take $(A, \cdot)$ (A with, at least, two elements) where $x \cdot 0 = 0$ and $x \cdot y = 1$ otherwise.

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(ii) Note that, by (1), $1 \cdot 1 = 1$ and $x \leq x \cdot 1$ for every x, and that by (5) $x \cdot 0 = 0$ for all x. Also by (5), $x \leq y$ implies $a \cdot x \leq a \cdot y$, for every a. In particular, $0 \cdot x \leq 0 \cdot 1 = 0$.

By (2), $x \leq y$ implies $x \cdot 1 \leq y \cdot 1$ and, consequently, $\bigvee_{x \in X}(x \cdot 1) \leq (\bigvee X) \cdot 1$.

Example 1.3. Let A be a frame such that $x \wedge y = 0$ only if $x = 0$ or $y = 0$. Each one of the operations

$$x \bullet y = \begin{cases} 0 & \text{if } x \wedge y = 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad x * y = \begin{cases} 0 & \text{if } x \wedge y = 0 \\ x & \text{otherwise} \end{cases}$$

gives A a structure of E-frame.

An *E-frame homomorphism* $f : (A, \cdot) \longrightarrow (B, \cdot)$ is just a frame homomorphism $f : A \longrightarrow B$ such that $f(x) \cdot 1 = f(x \cdot 1)$ and $f(x) \cdot f(y) \leq f(x \cdot y)$.

The category of E-frames and E-frame homomorphisms is denoted by $\mathcal{EF}rm$.

There are functors $E : \mathcal{F}rm \longrightarrow \mathcal{EF}rm$ and $K : \mathcal{EF}rm \longrightarrow \mathcal{F}rm$ such that $E \dashv K$. The extended frame $E(A)$ is defined as follows: consider, in the family of all non-empty subsets of A, the equivalence relation $X \sim Y$ if $X \prec Y$ and $Y \prec X$, where $X \prec Y$ means that for each $x \in X$ there is an $y \in Y$ such that $x \leq y$; then, $E(A)$ is the family of equivalence classes $[X]$ with the operations $[X] \wedge [Y] = [\{x \wedge y \mid x \in X, y \in Y\}]$, $\bigvee_{i \in I}[X_i] = [\bigcup_{i \in I} X_i]$ and $[X] \cdot [Y] = [\{X \cdot y \mid y \in Y\}]$. The E-frame homomorphism $E(f) : E(A) \longrightarrow E(B)$ is defined by $E(f)([X]) = [f(X)]$.

$K(A, \cdot)$ is the frame of *loci* of $(A, \cdot)$, i.e. $K(A, \cdot) = \{a \in A \mid a \cdot 1 = a\}$, in which finitary meet is the same as in A and join $\bigsqcup$ is given by $\bigsqcup S = (\bigvee S) \cdot 1$, $\bigvee$ being join in A. For $f : (A, \cdot) \longrightarrow (B, \cdot)$ in $\mathcal{EF}rm$ the map $K(f)$ is just the restriction of f to $K(A, \cdot)$.

The adjunction is given by the identity transformation $id \overset{\cdot}{\longrightarrow} KE$ and $\lambda : EK \overset{\cdot}{\longrightarrow} id$ defined by $\lambda_{(A,\cdot)}([X]) = \bigvee X$. For more details see [7]. These functors define an equivalence between $\mathcal{F}rm$ and the subcategory $\mathcal{EF}rm_{7,8,9}$ of $\mathcal{EF}rm$ of all E-frames $(A, \cdot)$ satisfying the following additional axioms:

(7) If $x \cdot 1 \leq \bigvee Y$ then there exists an $y \in Y$ such that $x \cdot 1 \leq y$;
(8) For each $x \in A$ there exists $X \subseteq K(A, \cdot)$ such that $x = \bigvee X$;
(9) If $(x \cdot 1) \wedge (y \cdot 1) \neq 0$ then $(x \cdot 1) \cdot (y \cdot 1) = x \cdot 1$.

It is easy to see that each one of the axioms (7), (8) and (9) is independent from the others.

Remarks 1.4. (i) In an extended frame $(A, \cdot)$ satisfying (7), $1 = \bigvee X$ only if $1 \in X$. In particular, A is compact. Further, the relation $x \prec y$ ("x is rather below y"), meaning there exists z such that $x \wedge z = 0$ and $y \vee z = 1$, is not interesting because, in this case, $x = 0$ or $y = 1$ necessarily.

(ii) Observe that, by (3) and (0) in Definition 1.1, it follows that, for every $x, y \in A$, $x \cdot y = 0$ whenever $x \wedge y = 0$. Since $x \cdot 1 \in K(A, \cdot)$ for every $x \in A$, (9) is equivalent to

$$x \wedge y \neq 0 \Rightarrow x \cdot y = x, \quad \text{for every } x, y \in K(A, \cdot).$$

So $(A, \cdot)$ satisfies axiom (9) if and only if the restriction of $\cdot$ to $K(A, \cdot)$ is the operation $*$. Since the operation $\cdot$ defined in $E(A)$ is the unique extension of $*$ to $E(A)$ satisfying (5), (6) and the equality $[X] \cdot 1 = [\{\bigvee X\}]$, we conclude that axiom (9) ensures us that the functor E recovers the operation $\cdot$ from its restriction to $K(A, \cdot)$.

In this note we study some aspects of E-frames. Namely, motivated by Remark 4.4 of [7], we study the role of the axioms (7), (8) and (9) in the behaviour of ideals (Section 2), covers and admissibility of systems of covers (Section 3) and uniformities (Section 4) of an E-frame.

2. Ideals

Definition 2.1 [7, Remark 4.1]**.** An element a of an E-frame $(A, \cdot)$ is called an *ideal* if it satisfies the implication

$$x \cdot 1 \vee y \cdot 1 \leq a \;\Rightarrow\; (x \cdot 1 \vee y \cdot 1) \cdot 1 \leq a, \text{ for every } x, y \in A.$$

Trivially any locus is an ideal of $(A, \cdot)$.

The system of ideals is obviously closed under general meets and, hence, it is a complete lattice. We denote it by $\mathcal{I}(A, \cdot)$. Similarly, we denote the frame of all ideals of A by $\mathcal{I}(A)$.

Remarks 2.2. (i) Since $(x \cdot 1 \vee y \cdot 1) \cdot 1 = (x \vee y) \cdot 1$, it is easy to see that a is an ideal if and only if $x \vee y \leq a$ implies that $(x \vee y) \cdot 1 \leq a$, for every $x, y \in K(A, \cdot)$. Therefore, if $x \vee y \in K(A, \cdot)$ whenever $x, y \in K(A, \cdot)$, then $\mathcal{I}(A, \cdot) = A$.

(ii) Let A be a frame and consider the corresponding E-frame $E(A)$. A subset I of A is an ideal of A if and only if the class $[I]$ is an ideal of $E(A)$ and I is decreasing. So, $[X]$ is an ideal of $E(A)$ if and only if $[X] = [I]$ for some (necessarily unique) ideal I of A. The map given by $I \longmapsto [I]$ defines an isomorphism between $\mathcal{I}(A)$ and $\mathcal{I}(E(A))$.

Lemma 2.3. *Let $(A, \cdot)$ be an E-frame satisfying axiom (7). Then, for every ideal I of $K(A, \cdot)$, $\lambda_{(A,\cdot)}([I])$ is an ideal of $(A, \cdot)$.*

Proof: In fact, for every $x, y \in K(A, \cdot)$, $x \vee y \leq \bigvee I$ implies that $(x \vee y) \cdot 1 \leq (a_1 \vee a_2) \cdot 1$, for some $a_1, a_2 \in I$ such that $x \leq a_1$ and $y \leq a_2$, by (7). But $(a_1 \vee a_2) \cdot 1 \in I$ by hypothesis, hence $(x \vee y) \cdot 1 \leq \bigvee I$. □

In summary, we have a map

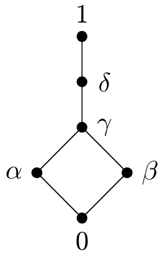
$$\begin{aligned} \overline{\lambda} : \mathcal{I}(K(A, \cdot)) &\longrightarrow \mathcal{I}(A, \cdot) \\ I &\longmapsto \lambda([I]) = \bigvee I \end{aligned}$$

which is injective by (7). Indeed, if $\lambda([I]) = \lambda([J])$, then we have, for each $a \in I$, that $a = a \cdot 1 \leq \bigvee J$ and thus, that there is a $b \in J$ such that $a \leq b$. Therefore $I \subseteq J$. Similarly, $J \subseteq I$. Hence $I = J$ and $\overline{\lambda}$ is one-one.

Now, let us consider the following property:

(8′) For every $a \in \mathcal{I}(A, \cdot)$, there exists $X \subseteq K(A, \cdot)$ such that $a = \bigvee X$.

Remark 2.4. This property is, even in the presence of (7) and (9), weaker than (8). For example, the E-frame



$\cdot$	0	α	β	γ	δ	1
0	0	0	0	0	0	0
α	0	α	0	α	α	α
β	0	0	β	β	β	β
γ	0	α	β	γ	γ	1
δ	0	α	β	γ	δ	1
1	0	1	1	1	1	1

satisfies (7), (9) and (8′), since $\mathcal{I}(A, \cdot) = K(A, \cdot) = \{0, \alpha, \beta, 1\}$, but it does not satisfy (8).

Proposition 2.5. *Let $(A, \cdot)$ be an E-frame satisfying properties* (7) *and* (8′). *Then $\overline{\lambda}$ is a frame isomorphism.*

Proof: First, let us prove the surjectivity. For $a \in \mathcal{I}(A, \cdot)$ we can write $a = \bigvee X_a$, for some $X_a \subseteq K(A, \cdot)$. Let $\downarrow X_a$ denote the set $\{y \in K(A, \cdot) \mid y \leq x$ for some $x \in X_a\}$. Then $\downarrow X_a$ is an ideal of $K(A, \cdot)$. In fact, if $y, z \in \downarrow X_a$ then $y \vee z \leq x_1 \vee x_2$ for some $x_1, x_2 \in X_a$. So $y \vee z \leq a$ and then $y \sqcup z \leq a$, which implies that $y \sqcup z \in \downarrow X_a$ by (7). In conclusion, we have that $a = \overline{\lambda}(\downarrow X_a)$, with $\downarrow X_a \in \mathcal{I}(K(A, \cdot))$.

Trivially, $\overline{\lambda}$ preserves finite meets.

Finally, let $(I_\gamma)_{\gamma \in \Gamma}$ be a family of ideals of $K(A, \cdot)$. Easily, one can observe that

$$\overline{\lambda}\left(\bigvee_{\gamma \in \Gamma}^{\mathcal{I}(K(A,\cdot))} I_\gamma\right) \geq \bigvee_{\gamma \in \Gamma}^{\mathcal{I}(A,\cdot)} \overline{\lambda}(I_\gamma).$$

On the other hand, for $a = \bigvee_{\gamma \in \Gamma}^{\mathcal{I}(A,\cdot)} \overline{\lambda}(I_\gamma)$ there exists a (unique) ideal I of $K(A, \cdot)$ such that $a = \overline{\lambda}(I)$. Then $\overline{\lambda}(I_\gamma) \leq \overline{\lambda}(I)$, for each $\gamma \in \Gamma$. Thus $I_\gamma \subseteq I$, for each $\gamma \in \Gamma$, by (7) and, consequently, $\bigvee_{\gamma \in \Gamma}^{\mathcal{I}(K(A,\cdot))} I_\gamma \subseteq I$. So, we have proved that $\overline{\lambda}$ preserves arbitrary joins. $\square$

Corollary 2.6. *If $(A,\cdot)$ satisfies properties (7) and (8′), then $\mathcal{I}(A,\cdot)$ is a compact frame.* □

Further, it is obvious that the join in $\mathcal{I}(A)$ is given by the formula

$$\bigvee_{\gamma\in\Gamma}^{\mathcal{I}(A)} I_\gamma = \left\{x_1 \vee \cdots \vee x_n \mid x_j \in I_{\gamma_j} \text{ for some } \gamma_j \in \Gamma\right\}.$$

Therefore, if $(A,\cdot)$ satisfies (7) and (8′), then the join in $\mathcal{I}(A,\cdot)$ is given by

$$\bigvee_{\gamma\in\Gamma}^{\mathcal{I}(A,\cdot)} a_\gamma = \bigvee \left\{x_1 \vee \cdots \vee x_n \mid x_j \in I_{\gamma_j} \text{ for some } \gamma_j \in \Gamma\right\},$$

where $I_\gamma = \downarrow X_{a_\gamma}$. As we saw above, rule (9) does not have much use in the study of the behaviour of ideals. However, as we shall see, it has a decisive role in the study of admissible systems of covers and uniformities.

Remark 2.7. Consider the E-frames $(A,*)$ of Example 1.3. They do not satisfy (7) but satisfy (8) and (9). Considering A non-compact, then $\mathcal{I}(A,\cdot)$ is not compact since $\mathcal{I}(A,\cdot) = A$.

3. Admissible systems of covers

From now on let $(A,\cdot)$ denote an E-frame.

Lemma 3.1. (i) *For every $X, Y \subseteq K(A,\cdot)$, $\bigvee X \cdot \bigvee Y \leq \bigvee_{y\in Y} \bigsqcup\{x\cdot y \mid x \in X, x\wedge y \neq 0\} \leq \bigvee_{y\in Y} \bigsqcup\{x \mid x \in X, x \wedge y \neq 0\}$.*

(ii) *In* (i) *the left member of the inequality is equal to the right member if and only if $(A,\cdot)$ satisfies* (9).

(iii) *Let $a, b, c, d \in A$. If $d = \bigvee D$ for some $D \subseteq K(A,\cdot)$, then $a\cdot c \leq b \cdot d$ whenever $a \leq b$ and $c \leq d$.*

(iv) *If $(A,\cdot)$ satisfies* (8) *and* (9), *then, for every $a, b, c \in A$, $a\cdot b\wedge c = 0$ if and only if $b \wedge a\cdot c = 0$.*

(v) *If $(A,\cdot)$ satisfies* (8) *and* (9), *then $(c\cdot b)\cdot a = c\cdot(b\cdot(c\cdot a))$, for every $a, b, c \in A$.*

Proof: (i) Cf. Lemma 3.2 of [7].

(ii) Let $x, y \in K(A,\cdot)$ such that $x \wedge y \neq 0$. Just take $X = \{x\}$ and $Y = \{y\}$. The converse is obvious by the proof of ([7, Lemma 3.2]).

(iii) We have $b\cdot d = \bigvee_{d'\in D}(b\cdot d')$ and $b\cdot d' = (a\vee b)\cdot(d'\cdot 1) = (a\cdot d' \vee b\cdot d')\cdot 1 \geq (a\cdot d')\cdot 1 = a\cdot d'$ thus $b\cdot d \geq \bigvee_{d'\in D}(a\cdot d') = a\cdot d$. By (5), $a\cdot d \geq a\cdot c$.

(iv) If a, b, c, belong to $K(A,\cdot)$, then $a\cdot b\wedge c = 0$ if and only if $a\wedge b = 0$ or $a \wedge c = 0$, by (ii). Also $b \wedge a\cdot c = 0$ if and only if $a\wedge c = 0$ or $a\wedge b = 0$, thus $a\cdot b\wedge c = 0$ if and only if $b\wedge a\cdot c = 0$. Now let a, b, c, belong to A. Then we can write $a=\bigvee X$, $b=\bigvee Y$, $c=\bigvee Z$ for some families X,Y and Z of loci of $(A,\cdot)$. We have

$$\begin{aligned}
a\cdot b\wedge c=0 &\Leftrightarrow \bigvee_{y\in Y}\left(\left(\bigvee_{x\in X}(x\cdot y)\right)\cdot 1\right)\wedge\bigvee Z=0\\
&\Leftrightarrow \bigvee_{z\in Z}\bigvee_{y\in Y}\left(\left(\bigvee_{x\in X}(x\cdot y)\right)\cdot 1\wedge z\right)=0\\
&\Leftrightarrow \forall z\in Z\ \ \forall y\in Y\quad \bigsqcup_{x\in X}(x\cdot y\wedge z)=0\\
&\Leftrightarrow \forall z\in Z\ \ \forall y\in Y\ \ \forall x\in X\quad x\cdot y\wedge z=0.
\end{aligned}$$

Similarly, $b\wedge a\cdot c=0$ if and only if, for any $z\in Z$, $y\in Y$ and $x\in X$, $y\wedge x\cdot z=0$. Hence $a\cdot b\wedge c=0$ if and only if $b\wedge a\cdot c=0$.

(v) By axiom (5) it suffices to prove (v) for $a\in K(A,\cdot)$. In this case

$$(c\cdot b)\cdot a=(c\cdot\bigvee Y)\cdot(a\cdot 1)=\bigsqcup_{y\in Y}((c\cdot y)\cdot a)=\bigsqcup\{c\cdot y\mid y\in Y, c\cdot y\wedge a\neq 0\}.$$

Using (iv) we obtain

$$\begin{aligned}
(c\cdot b)\cdot a&=\bigsqcup\{c\cdot y\mid y\in Y, y\wedge c\cdot a\neq 0\}=c\cdot\bigsqcup\{y\in Y\mid y\wedge c\cdot a\neq 0\}\\
&=c\cdot\bigsqcup_{y\in Y}(y\cdot(c\cdot a))=c\cdot\Big((\bigsqcup Y)\cdot(c\cdot a)\Big)=c\cdot(b\cdot(c\cdot a)).
\end{aligned}$$

□

Remarks 3.2. (i) The four-element chain $0<\alpha<\beta<1$ with the operation

$\cdot$	0	α	β	1
0	0	0	0	0
α	0	α	α	1
β	0	α	1	1
1	0	α	β	1

is an E-frame which satisfies (7) and (9) but not (8′). However, it does not satisfy the properties (iii) and (v) of the Lemma.

(ii) Instead, if we consider the operation

$\cdot$	0	α	β	1
0	0	0	0	0
α	0	α	α	α
β	0	α	β	β
1	0	β	β	1

we get an E-frame which satisfies (7) and (8) but not (9) and it also fails property (v) of the Lemma.

Definition 3.3 [7, Remark 4.1]**.** An element a of A is a *cover* of $(A,\cdot)$ if $a\cdot 1 = 1$.

The covers in $(A,\cdot)$ form, obviously, a filter in A that we denote by $cov(A,\cdot)$.

Lemma 3.4. *Assume that* $(A,\cdot)$ *satisfies axiom* (8)*. Then:*

(i) $x \le c\cdot x$ *for every* $x \in A$ *and every* $c \in cov(A,\cdot)$.
(ii) $c\cdot(b\cdot a) \le (c\cdot b)\cdot a$, *for every* $a,b \in A$ *and* $c \in cov(A,\cdot)$, *provided* $(A,\cdot)$ *also satisfies axiom* (9).

PROOF: (i) Let $x = \bigvee X$, where each element of X is a locus. For any $y \in X$, we have $y = 1 \wedge y = (c\cdot 1)\wedge(y\cdot 1) = (c\wedge y)\cdot 1 \le (c\cdot y)\cdot 1 = c\cdot y$. Consequently, $x \le c\cdot x$.

(ii) By (i) we have that $a \le c\cdot a$. Therefore $c\cdot(b\cdot a) \le c\cdot(b\cdot(c\cdot a))$. The conclusion follows from Lemma 3.1 (v). $\square$

Remarks 3.5. (i) Consider the E-frame of Remark 2.4, which does not satisfy (8) but satisfies (7), (8$'$) and (9). The element γ is a cover but $\gamma\cdot\delta < \delta$. So, from now on, axiom (8) will be decisive in all the results that we present.

(ii) The examples of Remarks 3.2 also fail property (ii) of the above Lemma.

For any $\mathcal{C},\mathcal{D} \subseteq cov(A,\cdot)$, we say that $\mathcal{C}$ *majorizes* $\mathcal{D}$, and write $\mathcal{C}\ maj\ \mathcal{D}$, if for every $c \in \mathcal{C}$ there is a $d \in \mathcal{D}$ such that $d \le c$. In particular, $\mathcal{C} \subseteq \mathcal{D}$ implies that $\mathcal{C}\ maj\ \mathcal{D}$.

For any $x,y \in A$, $x\overset{\mathcal{C}}{\lhd}y$ means that $c\cdot x \le y$ for some $c \in \mathcal{C}$. By Lemma 3.1 (iii) we have that

$$\mathcal{C}\ maj\ \mathcal{D} \Rightarrow (x\overset{\mathcal{C}}{\lhd}y \Rightarrow x\overset{\mathcal{D}}{\lhd}y)$$

in case $(A,\cdot)$ satisfies axiom (8). This result is not true for a general $(A,\cdot)$: take, for instance, in the E-frame of Remark 3.2 (i), $\mathcal{C} = \{1\}$, $\mathcal{D} = \{\beta\}$ and $x = y = \beta$.

$\mathcal{C}$ is said to be *meet-closed* if for every $c_1,c_2 \in \mathcal{C}$ there is a $c \in \mathcal{C}$ such that $c \le c_1 \wedge c_2$.

The following properties of $\lhd$ are easily proved:

Proposition 3.6. (a) *If* $x\overset{\mathcal{C}}{\lhd}y$, $x' \le x$ *and* $y \le y'$ *then* $x'\overset{\mathcal{C}}{\lhd}y'$.

(b) *If* $(A,\cdot)$ *satisfies* (8) *then:*

(i) $x\overset{\mathcal{C}}{\lhd}y \Rightarrow x \le y$;
(ii) $x_i\overset{\mathcal{C}}{\lhd}y_i$ $(i = 1,2)$ *imply that* $x_1\wedge x_2\overset{\mathcal{C}}{\lhd}y_1\wedge y_2$ *and* $x_1\vee x_2\overset{\mathcal{C}}{\lhd}y_1\vee y_2$ *whenever* $\mathcal{C}$ *is meet-closed. Moreover, if* $(A,\cdot)$ *satisfies* (9) *then* $x_1\cdot x_2\overset{\mathcal{C}}{\lhd}y_1\cdot y_2$.

$\square$

Let us denote by $\prec_K$ the relation "rather below" in the frame $K(A,\cdot)$.

Let A_1 be a subset of A and denote by $\mathcal{D}((A,\cdot),A_1)$ the system of all covers c of $(A,\cdot)$ for which $c = a\vee b$, for some loci a,b in A_1.

Lemma 3.7. *Let $\mathcal{C}$ be a system of covers such that $\mathcal{D}((A,\cdot),A)$ maj $\mathcal{C}$. Then, for every $x, y \in K(A,\cdot)$, $x \prec_K y$ implies $x \overset{\mathcal{C}}{\lhd} y$.*

PROOF: Consider $a \in K(A,\cdot)$ such that $a \wedge x = 0$ and $a \sqcup y = 1$. Since $a \vee y \in \mathcal{D}((A,\cdot),A)$, there is a $c \in \mathcal{C}$ with $c \leq a \vee y$. Then $c \cdot x \leq (a \vee y) \cdot x = (a \cdot x \vee y \cdot x) \cdot 1$. But $a \cdot x = 0$, thus $c \cdot x \leq y \cdot x \leq y$. □

In the presence of rule (9) we have:

Lemma 3.8. *Assume $(A,\cdot)$ satisfies (8) and (9) and let $\mathcal{C}$ be a system of covers. If $x \overset{\mathcal{C}}{\lhd} y$ then $x \cdot 1 \prec_K y \cdot 1$.*

PROOF: Let $c \cdot x \leq y$ with $c \in \mathcal{C}$ and consider the pseudocomplement x^* of x. It follows immediately that $(x \cdot 1) \wedge (x^* \cdot 1) = 0$. On the other hand, if $c = \bigvee C$ and $x = \bigvee X$, where $C, X \subseteq K(A,\cdot)$, we have

$$\begin{aligned} y \vee x^* \geq c \cdot x \vee x^* &= \bigvee_{x' \in X} \bigsqcup \{c' \in C \mid c' \wedge x' \neq 0\} \vee x^* \\ &\geq \bigvee_{x' \in X} \bigvee \{c' \in C \mid c' \wedge x' \neq 0\} \vee x^* \geq c. \end{aligned}$$

Hence $y \cdot 1 \sqcup x^* \cdot 1 = 1$. □

Therefore we can conclude that if $(A,\cdot)$ satisfies (8) and (9) and $\mathcal{D}((A,\cdot),A)$ *maj* $\mathcal{C}$, the restriction of $\overset{\mathcal{C}}{\lhd}$ to $K(A,\cdot)$ is exactly the relation $\prec_K$.

Now, for any $\mathcal{C} \subseteq cov(A,\cdot)$, put $(A,\cdot)_{\mathcal{C}} = \{a \in A \mid a = \bigvee\{b \in A \mid b \overset{\mathcal{C}}{\lhd} a\}\}$. The set $\mathcal{C}$ is called *admissible* if $(A,\cdot)_{\mathcal{C}} = A$.

The following lemma is easy to prove:

Lemma 3.9. *Assume $(A,\cdot)$ satisfies (8). Then:*

(i) *$\mathcal{C}$ maj $\mathcal{D} \Rightarrow (A,\cdot)_{\mathcal{C}} \subseteq (A,\cdot)_{\mathcal{D}}$;*

(ii) *$(A,\cdot)_{\mathcal{C}}$ is an upper sub-semilattice of A. Moreover, if $\mathcal{C}$ is meet-closed, then $(A,\cdot)_{\mathcal{C}}$ is a subframe of A.* □

Proposition 3.10. *Assume $(A,\cdot)$ satisfies (8) and (9) and let $\mathcal{C}$ be a system of covers such that $\mathcal{D}((A,\cdot),A)$ maj $\mathcal{C}$. Then the following assertions are equivalent:*

(i) *$\mathcal{C}$ is admissible;*

(ii) *$K(A,\cdot)$ is regular and, for every $a \in K(A,\cdot)$, $\bigvee\{b \in K(A,\cdot) \mid b \overset{\mathcal{C}}{\lhd} a\} \in K(A,\cdot)$;*

(iii) *For every $a \in K(A,\cdot)$, $a = \bigvee\{b \in K(A,\cdot) \mid b \prec_K a\}$.*

PROOF: (i) ⇒ (ii) Consider $a \in K(A,\cdot)$. Note that $\bigvee\{b \in K(A,\cdot) \mid b \overset{\mathcal{C}}{\lhd} a\}$ is equal to $\bigvee\{b \in A \mid b \overset{\mathcal{C}}{\lhd} a\}$ so if $\mathcal{C}$ is admissible then $\bigvee\{b \in K(A,\cdot) \mid b \overset{\mathcal{C}}{\lhd} a\} = a$, that is,

$\bigvee\{b \in K(A,\cdot) \mid b \overset{\mathcal{C}}{\lhd} a\}$ is a locus. Also $a = (\bigvee\{b \in A \mid b \overset{\mathcal{C}}{\lhd} a\}) \cdot 1$, thus it follows that $a \leq \bigsqcup\{b \in K(A,\cdot) \mid b \prec_K a\} \leq a$ by Lemma 3.8. Hence $a = \bigsqcup_{b \prec_K a} b$.

(ii) $\Rightarrow$ (iii) Let $a \in K(A,\cdot)$. Then

$$a = \bigsqcup\{b \in K(A,\cdot) \mid b \prec_K a\} = \bigvee\{b \in K(A,\cdot) \mid b \prec_K a\}$$

since $b \prec_K a$ if and only if $b \overset{\mathcal{C}}{\lhd} a$.

(iii) $\Rightarrow$ (i) Since $K(A,\cdot) \subseteq (A,\cdot)_{\mathcal{C}}$, the conclusion follows from the fact that $(A,\cdot)_{\mathcal{C}}$ is an upper sub-semilattice of A. $\square$

Similarly one can show that, in case $\mathcal{C}$ is finite, $\mathcal{C}$ is admissible if and only if $K(A,\cdot)$ is an atomic Boolean algebra and, for every $a \in K(A,\cdot)$, $\bigvee\{b \in K(A,\cdot) \mid b \overset{\mathcal{C}}{\lhd} a\} \in K(A,\cdot)$.

On the other hand, in the presence of (7), we have an alternative characterization:

Proposition 3.11. *If $(A,\cdot)$ satisfies (7) and (8) then a system of covers $\mathcal{C}$ is admissible if and only if $a \overset{\mathcal{C}}{\lhd} a$ for every locus a.*

Proof: Consider $a \in K(A,\cdot)$. By hypothesis, $a = \bigvee\{b \in A \mid b \overset{\mathcal{C}}{\lhd} a\}$, so, using (7), we conclude that, for some b, $a \leq b \overset{\mathcal{C}}{\lhd} a$. Hence $a \overset{\mathcal{C}}{\lhd} a$.

The converse is trivial since $(A,\cdot)_{\mathcal{C}}$ is an upper sub-semilattice. $\square$

By Lemmas 3.7 and 3.8, if $(A,\cdot)$ satisfies (8) and (9) and $\mathcal{D}((A,\cdot),A)$ *maj* $\mathcal{C}$, then the condition $a \overset{\mathcal{C}}{\lhd} a$ for every $a \in K(A,\cdot)$ means that $K(A,\cdot)$ is a Boolean algebra. Thus, Proposition 3.11 gives us the following corollary:

Corollary 3.12. *If $(A,\cdot) \in \mathcal{EF}rm_{7,8,9}$, then $(A,\cdot)$ has an admissible system of covers if and only if $K(A,\cdot)$ is a Boolean algebra.* $\square$

4. Uniformities

We say that a subset $\mathcal{U}$ of $cov(A,\cdot)$ is a *u-basis* on $(A,\cdot)$ if for each $a \in \mathcal{U}$ there is a $b \in \mathcal{U}$ such that $b \cdot b \leq a$.

In this case, when $(A,\cdot)$ satisfies (8) and (9), the relation $\overset{\mathcal{U}}{\lhd}$ interpolates. In fact, if $x \overset{\mathcal{U}}{\lhd} y$, i.e. $a \cdot x \leq y$ ($a \in \mathcal{U}$), then, for $z = b \cdot x$ (where $b \in \mathcal{U}$ is such that $b \cdot b \leq a$), $x \overset{\mathcal{U}}{\lhd} z$ and, by Lemma 3.4 (ii), $z \overset{\mathcal{U}}{\lhd} y$. Then one easily obtains a similar characterization to Proposition 3.10:

Proposition 4.1. *If $(A,\cdot)$ satisfies (8) and (9) and $\mathcal{U}$ is a u-basis such that $\mathcal{D}((A,\cdot),A)$ maj $\mathcal{U}$, then $\mathcal{U}$ is admissible if and only if $K(A,\cdot)$ is a completely regular frame and $\bigvee\{b \in K(A,\cdot) \mid b \overset{\mathcal{U}}{\lhd} a\} \in K(A,\cdot)$, for every $a \in K(A,\cdot)$.* $\square$

Based on the usual notion of uniformity for frames, it is natural to define a uniformity of an E-frame as follows:

Definition 4.2. Let $\mathcal{U} \subseteq cov(A,\cdot)$. Then $\mathcal{U}$ is a *uniformity* on $(A,\cdot)$ provided that:

(α) $\mathcal{U}$ is a filter with respect to $\leq$;

(β) $\mathcal{U}$ is a u-basis;

(γ) $\mathcal{U}$ is admissible.

An E-frame $(A,\cdot)$ with a uniformity $\mathcal{U}$ is called a *uniform E-frame.* A *uniform E-frame homomorphism* $f : ((A,\cdot),\mathcal{U}) \longrightarrow ((B,\cdot),\mathcal{V})$ is just an E-frame morphism $f : (A,\cdot) \longrightarrow (B,\cdot)$ such that $f(\mathcal{U}) \subseteq \mathcal{V}$. This way we have the category $\mathcal{UEF}rm$ of uniform E-frames and uniform E-frame homomorphisms.

The category of uniform frames (cf. [4]) is denoted by $\mathcal{UF}rm$.

Proposition 4.3. *If $(A,\cdot)$ satisfies (8) and (9) and $\mathcal{C}$ is a meet-closed system of covers, then $a \overset{\mathcal{C}}{\lhd} a$ for every locus a if and only if $K(A,\cdot)$ is a Boolean algebra and $\mathcal{D}((A,\cdot),A)$ maj $\mathcal{C}$.*

PROOF: The fact that $K(A,\cdot)$ is a Boolean algebra follows from Lemma 3.8.

Let $c = c_1 \vee c_2$ be a cover of $\mathcal{D}((A,\cdot),A)$. By hypothesis, there are $d_1, d_2 \in \mathcal{C}$ such that $d_1 \cdot c_1 \leq c_1$ and $d_2 \cdot c_2 \leq c_2$. Consider $d \in \mathcal{C}$ with $d \leq d_1 \wedge d_2$. We claim that $d \leq c$:

Let us write $d = \bigvee D$, $D \subseteq K(A,\cdot) \setminus \{0\}$. For every $d' \in D$, $d' \wedge c \cdot 1 \neq 0$, i.e. $c \cdot d' \neq 0$. But $c \cdot d' \leq \bigsqcup\{c_j \mid j \in \{1,2\}, c_j \wedge d' \neq 0\}$, so for every $d' \in D$ there is a $j \in \{1,2\}$ such that $c_j \wedge d' \neq 0$. Hence

$$d' \leq \bigsqcup\{d'' \in D \mid d'' \wedge c_1 \neq 0\} \vee \bigsqcup\{d'' \in D \mid d'' \wedge c_2 \neq 0\}. \tag{1}$$

On the other hand, $d'' \wedge c_j \neq 0$ $(d'' \in D, j \in \{1,2\})$ implies that $d'' \leq d \cdot c_j$ since $d'' \geq d'' \wedge d \cdot c_j = d'' \wedge \bigsqcup_{d' \in D}(d' \cdot c_j) = \bigsqcup_{d' \in D}(d'' \wedge (d' \cdot c_j)) \geq d'' \wedge (d'' \cdot c_j) = d'' \wedge d'' = d''$. Therefore, from (1) it follows that $d' \leq d \cdot c_1 \vee d \cdot c_2 \leq d_1 \cdot c_1 \vee d_2 \cdot c_2 \leq c_1 \vee c_2 = c$. Hence $d \leq c$.

The converse is an immediate consequence of Lemma 3.7. $\square$

If q is a system of covers of a frame A, then $\mathcal{U}_q = \{[C] : C \in q\}$ is a system of covers of the E-frame $E(A)$. Since $\mathcal{D}(A,A)$ *maj* q (cf. [2]) if and only if $\mathcal{D}(E(A),E(A))$ *maj* $\mathcal{U}_q$ and, moreover, $KE = id$, the following corollary is a particular case of the previous proposition.

Corollary 4.4. *If A is a frame and q is a system of covers of A such that for every $C_1, C_2 \in q$ there is a refinement $C \in q$ of $C_1 \wedge C_2$, then $a \overset{q}{\lhd} a$ for every $a \in A$ if and only if A is a Boolean algebra and $\mathcal{D}(A,A)$ maj q.* $\square$

We call a uniform frame (A,q) *Boolean* if it satisfies the equivalent conditions of Corollary 4.4. By $\mathcal{BUF}rm$ we denote the (full) subcategory of $\mathcal{UF}rm$ of all Boolean uniform frames.

Proposition 4.5. *Let $(A,q) \in \mathcal{UF}rm$. Then $(E(A),\mathcal{U}_q)$ is a uniform E-frame if and only if (A,q) is Boolean.*

Proof: If $(E(A),\mathcal{U}_q)$ is a uniform E-frame, then $[\{a\}]\overset{\mathcal{U}_q}{\lhd}[\{a\}]$, for every $a \in A$, by Proposition 3.11, that is, $a\overset{q}{\lhd}a$ for every $a \in A$.

Conversely, conditions (α) and (β) are trivially satisfied. Condition (γ) is a consequence of Proposition 3.11 since $KE(A) = A$ is a Boolean algebra and $\mathcal{D}(A,A)$ *maj* q. $\square$

Furthermore, for any $f : (A,q_1) \longrightarrow (B,q_2)$ in $\mathcal{UF}rm$, the E-frame map $E(f)$ is always uniform. Immediately, the functor

$$\begin{aligned} E : \mathcal{BUF}rm &\longrightarrow \mathcal{UEF}rm \\ (A,q) &\longmapsto (E(A),\mathcal{U}_q) \\ f &\longmapsto E(f) \end{aligned}$$

is a full embedding.

In order to have a functor $K : \mathcal{UEF}rm \longrightarrow \mathcal{BUF}rm$ and an adjunction $E \dashv K$ we need the conditions (7), (8) and (9). For $((A,\cdot),\mathcal{U}) \in \mathcal{UEF}rm_{7,8,9}$ and $a \in \mathcal{U}$ let $(X_a^i)_{i\in I_a}$ be the family of all sets X_a^i of loci such that $a = \bigvee X_a^i$. By (8) each I_a is non-empty.

Proposition 4.6. *The family $q_{\mathcal{U}} = (X_a^i)_{a\in\mathcal{U},i\in I_a}$ is a uniformity on $K(A,\cdot)$.*

Proof: Each X_a^i is a cover of K since $\bigsqcup X_a^i = (\bigvee X_a^i)\cdot 1 = a \cdot 1 = 1$.

Let $X_a^i \in q_{\mathcal{U}}$ and let C be a cover of $K(A,\cdot)$ such that $X_a^i \prec C$. Obviously $a = \bigvee X_a^i \le \bigvee C$. Thus $\bigvee C \in \mathcal{U}$ and, therefore, $C \in q_{\mathcal{U}}$.

Let $X_a^i, X_b^j \in q_{\mathcal{U}}$. Then, since $a \wedge b \in \mathcal{U}$ and $a \wedge b = \bigvee_{x\in X_a^i}\bigvee_{y\in X_b^j}(x \wedge y)$, $X_a^i \wedge X_b^j \in q_{\mathcal{U}}$.

Let $X_a^i \in q_{\mathcal{U}}$. By hypothesis there is a $b \in \mathcal{U}$ such that $b\cdot b \le a$. We claim that $X_b^j \cdot X_b^j \prec X_a^i$, for every $j \in I_b$. Indeed, for any $y \in X_b^j$, $X_b^j \cdot y = \bigsqcup\{x \in X_b^j \mid x \wedge y \neq 0\}$. On the other hand, we have that $b \cdot b \le a$ if and only if

$$\bigvee_{y\in X_b^j}\Big((\bigvee\{x \mid x \in X_b^j, x\wedge y \neq 0\})\cdot 1\Big) \le \bigvee_{z\in X_a^i} z,$$

by Lemma 3.1 (ii). Thus $(\bigvee\{x \mid x \in X_b^j, x \wedge y \neq 0\})\cdot 1 \le \bigvee_{z\in X_a^i} z$, for every $y \in X_b^j$. Applying (7) we get that for each $y \in X_b^j$ there is a $z \in X_a^i$ such that $(\bigvee\{x \mid x \in X_b^j, x\wedge y \neq 0\})\cdot 1 \le z$, i.e. $X_b^j \cdot y \le z$.

Finally we have to prove that $x = \bigsqcup\{y \mid y\overset{q_{\mathcal{U}}}{\lhd}x\}$, for every locus x. It suffices to show that $x \le \bigsqcup\{y \mid y\overset{q_{\mathcal{U}}}{\lhd}x\}$. By hypothesis, $x = \bigvee\{y \in A \mid y\overset{\mathcal{U}}{\lhd}x\}$. Let $a \in \mathcal{U}$

and $y \in A$ such that $a \cdot y \leq x$. Assuming that $a = \bigvee_{\gamma\in\Gamma} a_\gamma$ and $y = \bigvee_{\delta\in\Delta} y_\delta$ are decompositions of a and y given by (8), we have that $\bigvee_{\delta\in\Delta}((\bigvee\{a_\gamma \mid \gamma \in \Gamma, a_\gamma \wedge y_\delta \neq 0\}) \cdot 1) \leq x$, by Lemma 3.1 (ii). This implies that, for every $\delta \in \Delta$, $((\bigvee\{a_\gamma \mid \gamma \in \Gamma, a_\gamma \wedge y_\delta \neq 0\}) \cdot 1) \leq x$, i.e. $X_a^i \cdot y_\delta \leq x$ for some $X_a^i \in q_{\mathcal{U}}$. Hence

$$\begin{aligned} x = \bigvee_{a\in\mathcal{U}} \bigvee\{y \in A \mid a \cdot y \leq x\} &\leq \bigvee_{X_a^i\in q_{\mathcal{U}}} \bigvee\{y \in K(A,\cdot) \mid X_a^i \cdot y \leq x\} \\ &\leq \bigsqcup_{X_a^i\in q_{\mathcal{U}}} \{y \in K(A,\cdot) \mid X_a^i \cdot y \leq x\} \\ &= \bigsqcup\{y \in K(A,\cdot) \mid y \overset{q_{\mathcal{U}}}{\lhd} x\}. \end{aligned}$$

$\square$

Remark 4.7. Consider again the frame $(A,\cdot)$ of Remark 2.4. The set $\mathcal{U} = \{\delta, 1\}$ is a uniformity but the family $q_{\mathcal{U}} = \{X_1^i, X_\delta^j \mid i \in I_1, j \in I_\delta\} = \{\{1\}\}$ is not a uniformity on $K(A,\cdot)$. In fact $\{1\} \cdot y = 1$ for any $y \neq 0$, so $\bigsqcup\{y \mid y \overset{q_{\mathcal{U}}}{\lhd} x\}$ is not equal to x in case $x = \alpha$ or $x = \beta$.

In case $((A,\cdot),\mathcal{U}) \in \mathcal{UEF}rm_{7,8,9}$, then $\mathcal{D}(K(A,\cdot), K(A,\cdot))$ *maj* $q_{\mathcal{U}}$, since this is equivalent to $\mathcal{D}((A,\cdot), A)$ *maj* $\mathcal{U}$, and $K(A,\cdot)$ is a Boolean algebra. Consequently, $(K(A,\cdot), q_{\mathcal{U}}) \in \mathcal{BUF}rm$ whenever $((A,\cdot),\mathcal{U}) \in \mathcal{UEF}rm_{7,8,9}$.

In conclusion, we have a functor $K : \mathcal{UEF}rm_{7,8,9} \longrightarrow \mathcal{BUF}rm$, and then, easily, the categories $\mathcal{BUF}rm$ and $\mathcal{UEF}rm_{7,8,9}$ are equivalent.

We point out that in order to get a category of E-frames equivalent to $\mathcal{UF}rm$ we should modify condition (γ) in the definition of uniform E-frame as follows:

(γ') For every $a \in K(A,\cdot)$, $\quad a = \bigsqcup\{b \in K(A,\cdot) \mid b \overset{\mathcal{U}}{\lhd} a\}$.

In fact, with this modification, the functor E above may be defined in the category $\mathcal{UF}rm$ and one can conclude similarly that the functors E and K define an equivalence between $\mathcal{UF}rm$ and this category of E-frames.

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Departamento de Matemática, Universidade de Coimbra, Apartado 3008, 3000 Coimbra, Portugal

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