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Daniela Jarušková

Abstract. Recently Hušková (1998) has studied the least squares estimator of a change-point in gradually changing sequence supposing that the sequence increases (or decreases) linearly after the change-point. The present paper shows that the limit behavior of the change-point estimator for more complicated gradual changes is similar. The limit variance of the estimator can be easily calculated from the covariance function of a limit process.

Keywords: gradual type of change, polynomial regression, estimator, limit distribution

Classification: 62G20, 62E20, 60F17

1. Introduction

In applications we often observe a sequence of variables that at some unknown time point starts gradually changing its behaviour. Such series we encounter in engineering problems as well as in ecology. Sometimes, we even know from our experience how the series behaves before and after change point. The inference about broken line regression considered by Hinkley (1969) or Siegmund and Zhang (1994) belongs to such problems. Recently Hušková (1998) has studied the model where at an unknown time point a linear trend appears in the mean of an observed time series $\{Y_i, i = 1, \ldots, n\}$:

$$Y_i = \mu + \delta_n \Big(\frac{i - k^\star}{n}\Big)_+ + e_i,$$

where $a_+ = \max\{a, 0\}$, μ, δ_n, $k^\star$ are unknown parameters and $\{e_i, i = 1, \ldots, n\}$ are random errors. We show that her approach can be generalized to a case of a gradual change in polynomial regression. We consider a sequence $\{Y_i, i = 1, \ldots, n\}$ satisfying

$$Y_i = \tilde{\alpha_0} + \tilde{\alpha_1}\Big(\frac{i}{n}\Big) + \ldots + \tilde{\alpha_p}\Big(\frac{i}{n}\Big)^p + \beta\Big(\Big(\frac{i - k^\star}{n}\Big)_+\Big)^m + e_i, \quad i = 1, \ldots, n,$$

$p = 0, 1, \ldots,$ $m \geq 1$ are known integers, $\tilde{\alpha_0}, \ldots, \tilde{\alpha_p}$, $\beta \in R_1$ and $k^\star \in N$ are unknown parameters. The errors $\{e_i\}$ are i.i.d. satisfying $Ee_i = 0$, $Ee_i^2 = \sigma^2$, $E|e_i|^{2+\delta} < \infty$ for some $\delta > 0$. The parameter σ^2 is supposed to be known and

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we can assume without loss of generality that it is equal to 1. The aim of this paper is to estimate the change point $k^\star$ and to find the limit distribution of this estimator. Therefore, we suppose that β may depend on the sample size n and may go to zero as n tends to infinity. Similarly as in Hušková (1998) we use the least squares method.

To present the least squares estimator $\widehat{k^\star}$ in more apparent way it is convenient to express the variables $\{Y_i, i = 1, \dots, n\}$ with the help of orthogonal vectors $\phi_0^n = (\phi_0^n(1), \dots, \phi_0^n(n))', \dots, \phi_p^n = (\phi_p^n(1), \dots, \phi_p^n(n))'$. More precisely

$$(1.1)\qquad Y_i = \alpha_0\phi_0^n(i) + \dots + \alpha_p\phi_p^n(i) + \beta_n\Big(\Big(\frac{i-k^\star}{n}\Big)_+\Big)^m + e_i, \quad i = 1, \dots, n,$$

$p = 0, 1, \dots$ and $m \geq 1$ are some known integers, $\alpha_0, \dots, \alpha_p$, $\beta_n \in R_1$ and $k^\star \in N$ are unknown parameters. The errors $\{e_i\}$ have the same properties as given above. The orthogonal vectors may be chosen so that

$$\phi_0^n(i) = 1, \quad i = 1, \dots, n,$$

and for $j = 1, \dots, p$

$$\phi_j^n(i) = \Big(\frac{i}{n}\Big)^j + C_{j-1}(j,n)\Big(\frac{i}{n}\Big)^{j-1} + \dots + C_0(j,n), \quad i = 1, \dots, n,$$

so that for every $j \neq j'$

$$\sum_{i=1}^n \phi_j^n(i)\phi_{j'}^n(i) = 0,$$

see Anderson (1971). The first few vectors are

$$\begin{aligned}
\phi_0^n(i) &= 1,\\
\phi_1^n(i) &= \Big(\frac{i}{n}\Big) - \frac{1}{2}\Big(1 + \frac{1}{n}\Big),\\
\phi_2^n(i) &= \Big(\frac{i}{n}\Big)^2 - \Big(\frac{i}{n}\Big)\Big(1 + \frac{1}{n}\Big) + \frac{1}{6}\Big(1 + \frac{1}{n}\Big)\Big(1 + \frac{2}{n}\Big),\\
&\ \vdots
\end{aligned}$$

As $n \to \infty$, the functions $\{\phi_j^n([nt]), t \in [0,1]\}\ j = 0, \dots, p$ converge on [0,1] to functions $\{\phi_j(t)\}\ j = 0, \dots, p$ forming an orthogonal system, i.e., for $j \neq j'$

$$\int_0^1 \phi_j(t)\phi_{j'}(t)\,dt = 0$$

and

$$\frac{1}{n}\sum_{i=1}^n (\phi_j^n(i))^2 \to \int_0^1 (\phi_j(t))^2\,dt.$$

The first few functions are

$$\phi_0(t) = 1,$$
$$\phi_1(t) = t - \frac{1}{2},$$
$$\phi_2(t) = t^2 - t + \frac{1}{6},$$
$$\vdots$$

Denote X the design matrix, i.e., $X = (\phi_0^n, \phi_1^n, \dots, \phi_p^n)$. It holds

$$(X'X)^{-1} = \begin{pmatrix} 1/\sum_i(\phi_0^n(i))^2 & 0 & \dots & 0 \\ 0 & 1/\sum_i(\phi_1^n(i))^2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1/\sum_i(\phi_p^n(i))^2 \end{pmatrix}.$$

If we denote

$$c_k = \Big(0, \dots, 0, \Big(\frac{1}{n}\Big)^m, \dots, \Big(\frac{n-k}{n}\Big)^m\Big)',$$

then the least squares estimator $\widehat{k^\star}$ of the change point $k^\star$ can be expressed

$$\widehat{k^\star} = \min\Big\{\tilde{k}, \frac{(c_{\tilde{k}}'MY)^2}{c_{\tilde{k}}'Mc_{\tilde{k}}} = \max_{p \le k < n-p} \frac{(c_k'MY)^2}{c_k'Mc_k}\Big\} \tag{1.2}$$

where the matrix $M = I - X(X'X)^{-1}X'$. It is clear that if the variables $\{e_i\}$ are distributed according to a normal distribution, then the estimator $\widehat{k^\star}$ is the maximum likelihood estimator of $k^\star$ in model (1.1) where all parameters $\alpha_0, \dots \alpha_p, \beta$ are unknown.

The paper is organized as follows. Section 2 contains our main result. To prove the assertion of the main theorem several auxiliary lemmas have to be stated and proved. Section 3 presents three examples how to apply the main theorem to obtain the limit distribution of $k^\star$ in different situations.

2. Main result

The aim of this section is to derive asymptotic distribution of $k^\star$. First, for $n \in N$ we introduce variables $\{S^n(k), k = p, \dots, n-p-1\}$ by

$$\begin{aligned} S^n(k) &= \frac{1}{\sqrt{n}} c_k' Me = \frac{1}{\sqrt{n}} \sum_{i=k}^n \Big(\frac{i-k}{n}\Big)^m e_i \\ &- \sum_{j=0}^p \Big\{ \Big(\frac{1}{n}\sum_{i=k}^n \Big(\frac{i-k}{n}\Big)^m \phi_j^n(i)\Big)\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n \phi_j^n(i)e_i\Big) \Big/ \Big(\frac{1}{n}\sum_{i=1}^n (\phi_j^n(i))^2\Big)\Big\}. \end{aligned} \tag{2.1}$$

The covariance function of the sequence $\{S^n(k), k = p, \dots, n-p-1\}$ satisfies for $l \le k$

$$(2.2)\qquad R^n(k,l) = \frac{1}{n} c_k{}' M c_l = \frac{1}{n}\sum_{i=k}^{n}\Big(\frac{i-k}{n}\Big)^m\Big(\frac{i-l}{n}\Big)^m$$
$$-\sum_{j=0}^{p}\Big\{\Big(\frac{1}{n}\sum_{i=k}^{n}\Big(\frac{i-k}{n}\Big)^m\phi_j^n(i)\Big)\Big(\frac{1}{n}\sum_{i=l}^{n}\Big(\frac{i-l}{n}\Big)^m\phi_j^n(i)\Big)\Big/\Big(\frac{1}{n}\sum_{i=1}^{n}(\phi_j^n(i))^2\Big)\Big\}.$$

Further, for $n \in N$ we define the processes $\{Y^n(t)\}$ on $[0,1]$ by

$$Y^n(t) = S^n([nt]), \quad t \in [0,1].$$

Using the limit theorem for random processes, see Theorem 15.6 of Billingsley (1968), the processes $\{Y^n(t)\}$ converge in distribution on $D[0,1]$ to a process

$$(2.3)\qquad Y(t) = \int_t^1 (z-t)^m\, dW(z)$$
$$-\sum_{j=0}^{p}\Big\{\Big(\int_t^1 (z-t)^m\phi_j(z)\, dz\Big)\Big(\int_0^1 \phi_j(z)\, dW(z)\Big)\Big/\Big(\int_0^1(\phi_j(z))^2\, dz\Big)\Big\},\ t\in[0,1],$$

where $\{W(t),\, t \ge 0\}$ denotes Wiener process. The covariance function of the process $\{Y(t)\}$ satisfies

$$(2.4)\qquad R(t,s) = \int_t^1 (z-t)^m (z-s)^m\, dz$$
$$-\sum_{j=0}^{p}\Big\{\Big(\int_t^1 (z-t)^m\phi_j(z)\, dz\Big)\Big(\int_s^1 (z-s)^m\phi_j(z)\, dz\Big)\Big/\Big(\int_0^1(\phi_j(z))^2\, dz\Big)\Big\},\ s\le t.$$

Moreover, for $n \in N$ we introduce a sequence $\{\dot{S}^n(k), k = p, \dots, n-p-1\}$ by

$$(2.5)\qquad \dot{S}^n(k) = -\frac{1}{\sqrt{n}}\sum_{i=k}^{n} m\Big(\frac{i-k}{n}\Big)^{m-1} e_i$$
$$+\sum_{j=0}^{p}\Big\{\Big(\frac{1}{n}\sum_{i=k}^{n} m\Big(\frac{i-k}{n}\Big)^{m-1}\phi_j^n(i)\Big)\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_j^n(i)e_i\Big)\Big/\Big(\frac{1}{n}\sum_{i=1}^{n}(\phi_j^n(i))^2\Big)\Big\}$$

and processes $\{\dot{Y}^n(t)\}$ by

$$\dot{Y}^n(t) = \dot{S}^n([nt]), \quad t \in [0,1].$$

The processes $\{\dot{Y}^n(t)\}$ converge in distribution on $D[0,1]$ to the process

$$\dot{Y}(t) = -\int_t^1 m(z-t)^{m-1}\,dW(z) \tag{2.6}$$

$$+\sum_{j=0}^{p}\Big\{\Big(\int_t^1 m(z-t)^{m-1}\phi_j(z)\,dz\Big)\Big(\int_0^1 \phi_j(z)\,dW(z)\Big)\Big/\Big(\int_0^1 (\phi_j(z))^2\,dz\Big)\Big\},$$

that is the derivative of the process $\{Y(t)\}$.

In the main theorem given below it is stated that under certain conditions on the limit behavior of β_n the asymptotic distribution of $k^\star$ is normal with the variance which may be computed from the covariance function of the process $\{Y(t)\}$.

Theorem. *Let random variables $\{Y_i, i=1,\dots,n\}$ satisfy the properties of model (1.1) with $k^\star = [n\theta^\star]$ for some $\theta^\star \in (0,1)$. Let*

$$\beta_n = O(1) \quad \text{and} \quad \frac{\beta_n^2 n}{(\ln\ln n)^2} \to \infty \quad \text{as} \quad n\to\infty. \tag{2.7}$$

Then, as $n\to\infty$,

$$\beta_n \frac{\widehat{k^\star} - k^\star}{\sqrt{n}}\sqrt{A(\theta^\star)} \xrightarrow{\mathcal{D}} N(0,1), \tag{2.8}$$

where $A(\theta^\star) = \Big(\frac{\partial^2 R(t,s)}{\partial t\partial s}\Big)_{t=s=\theta^\star} - \Big(\frac{\partial R(t,s)}{\partial t}\Big)^2_{t=s=\theta^\star}\Big/R(\theta^\star,\theta^\star)$.

Proof: Similarly as in Hušková (1998), the estimator $\widehat{k^\star}$ can be defined as the solution of the following maximization problem

$$\max_{p\le k<n-p} D_k = \max_{p\le k<n-p}\Big\{\frac{(c_k{}'MY)^2}{c_k{}'Mc_k} - \frac{(c_{k^\star}{}'MY)^2}{c_{k^\star}{}'Mc_{k^\star}}\Big\}. \tag{2.9}$$

Under the assumptions that the variables $\{Y_i, i=1,\dots,n\}$ satisfy model (1.1), the variables D_k can be expressed as follows:

$$D_k = \beta_n^2 C_k + 2\beta_n B_k + A_k, \tag{2.10}$$

where

$$C_k = n\Big(\frac{(R^n(k,k^\star))^2}{R^n(k,k)} - R^n(k^\star,k^\star)\Big),$$

$$B_k = \sqrt{n}\Big(\frac{R^n(k,k^\star)-R^n(k,k)}{R^n(k,k)}\Big)S^n(k) + S^n(k) - S^n(k^\star),$$

$$A_k = \frac{(S^n(k))^2}{R^n(k,k)} - \frac{(S^n(k^\star))^2}{R^n(k^\star,k^\star)}.$$

We consider a sequence $\{r_n\}$ such that

$$r_n \to \infty, \quad \frac{|\beta_n|\sqrt{n}}{r_n \ln\ln n} \to \infty.$$

To prove the main result, we have to verify the validity of two assertions

$$P\Big(\max_{p\le k<n-p} D_k = \max_{|\beta_n|\frac{|k-k^\star|}{\sqrt{n}}\le r_n} D_k\Big) \to 1 \quad \text{as} \quad n\to\infty \tag{2.11}$$

and uniformly for $|\beta_n(k-k^\star)|/\sqrt{n} \le r_n$

(2.12)

$$C_k = n\Big(Q\Big(\frac{k}{n},\frac{k^\star}{n}\Big) - Q\Big(\frac{k^\star}{n},\frac{k^\star}{n}\Big)\Big) + o\Big(\frac{|k-k^\star|}{\sqrt{n}}\Big),$$

(2.13)

$$B_k = \sqrt{n}\Big(Z\Big(\frac{k}{n},\frac{k^\star}{n}\Big) - 1\Big)S^n(k^\star) + \dot{S}^n(k^\star)\frac{k-k^\star}{\sqrt{n}} + o_P\Big(\frac{|k-k^\star|}{\sqrt{n}}\Big),$$

(2.14)

$$A_k = o_P\Big(\frac{|k-k^\star|}{\sqrt{n}}\Big),$$

where $Q(t,s) = \big(R(t,s)\big)^2/R(t,t)$ and $Z(t,s) = R(t,s)/R(t,t)$.

The proof of (2.12, 2.13, 2.14) follows from the following lemmas.

Lemma 1. *It holds uniformly in $|\beta_n(k-k^\star)|/\sqrt{n} \le r_n$*

(2.15)

$$\Big(Q\Big(\frac{k}{n},\frac{k^\star}{n}\Big) - Q\Big(\frac{k^\star}{n},\frac{k^\star}{n}\Big)\Big) - \Big(\frac{(R^n(k,k^\star))^2}{R^n(k,k)} - R^n(k^\star,k^\star)\Big) = o\Big(\frac{|k-k^\star|}{n^{3/2}}\Big),$$

$$\Big(Z\Big(\frac{k}{n},\frac{k^\star}{n}\Big) - 1\Big) - \Big(\frac{R^n(k,k^\star)}{R^n(k,k)} - 1\Big) = o\Big(\frac{|k-k^\star|}{n^{3/2}}\Big). \tag{2.16}$$

Proof: For every integer $q\ge 0$, $j\ge 0$ and $k<k^\star$ it holds:

$$\int_{k/n}^1 \Big(z-\frac{k}{n}\Big)^q \phi_j(z)\,dz = \frac{1}{n}\sum_{i=k}^n \Big(\frac{i-k}{n}\Big)^q \phi_j^n(i) + O\Big(\frac{1}{n}\Big),$$

$$\int_{k/n}^{k^\star/n} \Big(z-\frac{k}{n}\Big)^q \phi_j(z)\,dz = \frac{1}{n}\sum_{i=k}^{k^\star-1} \Big(\frac{i-k}{n}\Big)^q \phi_j^n(i) + O\Big(\frac{(k^\star-k)}{n}\Big).$$

Clearly for arbitrary integer k and $k^\star$

$$R\Big(\frac{k}{n},\frac{k}{n}\Big) - R^n(k,k) = O\Big(\frac{1}{n}\Big) \quad \text{and} \quad R\Big(\frac{k}{n},\frac{k^\star}{n}\Big) - R^n(k,k^\star) = O\Big(\frac{1}{n}\Big).$$

Moreover

$$R\Big(\frac{k}{n},\frac{k^\star}{n}\Big)-R\Big(\frac{k}{n},\frac{k}{n}\Big)=O\Big(\frac{|k-k^\star|}{n}\Big),\qquad R^n(k,k^\star)-R^n(k,k)=O\Big(\frac{|k-k^\star|}{n}\Big).$$

Using the equality

$$\Big(z-\frac{k}{n}\Big)^m-\Big(z-\frac{k^\star}{n}\Big)^m=-\bigg(m\,\Big(z-\frac{k}{n}\Big)^{m-1}\Big(\frac{k-k^\star}{n}\Big)+\ldots+\Big(\frac{k-k^\star}{n}\Big)^m\bigg),$$

it can be proved that

$$\bigg(R\Big(\frac{k}{n},\frac{k^\star}{n}\Big)-R\Big(\frac{k}{n},\frac{k}{n}\Big)\bigg)-\bigg(R^n(k,k^\star)-R^n(k,k)\bigg)=O\Big(\frac{|k-k^\star|}{n^2}\Big).$$

Notice that all the equalities above hold uniformly for $k<k^\star$. The assertion of Lemma 1 is their consequence. □

Lemma 2. *For $k^\star<k$, uniformly in $|\beta_n(k-k^\star)|/\sqrt{n}\le r_n$, we have*

$$\frac{1}{\sqrt{n}}\sum_{i=k^\star}^{k}\Big(\frac{i-k}{n}\Big)^m e_i=o_P\Big(\frac{k-k^\star}{n}\Big), \tag{2.17}$$

$$\sum_{j=0}^{p}\bigg\{\bigg(\frac{1}{n}\sum_{i=k^\star}^{k}\Big(\frac{i-k}{n}\Big)^m\phi_j^n(i)\bigg)\bigg(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_j^n(i)e_i\bigg)\Big/\bigg(\frac{1}{n}\sum_{i=1}^{n}(\phi_j^n(i))^2\bigg)\bigg\}=o_P\big(\frac{k-k^\star}{n}\big). \tag{2.18}$$

The analogous result holds for $k<k^\star$.

Proof: Assertion (2.17) follows from the law of iterated logarithm, see Theorem 3, Chapter 7, par. 3 of Petrov (1971). For $k^\star<k$, uniformly in $|\beta_n(k-k^\star)|/\sqrt{n}\le r_n$:

$$\frac{1}{\sqrt{n}}\sum_{i=k^\star}^{k}\Big(\frac{i-k}{n}\Big)^m e_i=\bigg(\frac{\frac{1}{\sqrt{n}}\sum_{i=k^\star}^{k}\big(\frac{i-k}{n}\big)^m e_i}{\big(\frac{k-k^\star}{n}\big)^{\frac{2m+1}{2}}}\bigg)\Big(\frac{k-k^\star}{n}\Big)^{m+1/2}$$

$$=O_P(\sqrt{\ln\ln n})\Big(\frac{k-k^\star}{n}\Big)^{m+1/2}\le\Big(\frac{k-k^\star}{n}\Big)\frac{r_n^{1/2}O_p(\sqrt{\ln\ln n})}{\beta_n^{1/2}n^{1/4}}=o_P\Big(\frac{k-k^\star}{n}\Big).$$

Assertion (2.18) follows from the obvious fact that, uniformly in $|\beta_n(k-k^\star)|/\sqrt{n}\le r_n$,

$$\frac{1}{n}\sum_{i=k^\star}^{k}\Big(\frac{i-k}{n}\Big)^m\phi_j^n(i)=O\bigg(\Big(\frac{k-k^\star}{n}\Big)^{m+1}\bigg).$$

□

Lemma 3. *It holds uniformly for* $|\beta_n(k-k^\star)|/\sqrt{n} \le r_n$

$$S^n(k) - S^n(k^\star) = \frac{k-k^\star}{n}\dot{S}^n(k^\star) + o_P\Big(\frac{k-k^\star}{n}\Big). \tag{2.19}$$

Proof: For $k > k^\star$ we have

$$S^n(k) - S^n(k^\star) = \frac{1}{\sqrt{n}}\sum_{i=k}^{n}\Big(\frac{i-k}{n}\Big)^m e_i - \frac{1}{\sqrt{n}}\sum_{i=k^\star}^{n}\Big(\frac{i-k^\star}{n}\Big)^m e_i$$

$$-\sum_{j=0}^{p}\frac{\Big(\frac{1}{n}\sum_{i=k}^{n}\big(\frac{i-k}{n}\big)^m\phi_j^n(i) - \frac{1}{n}\sum_{i=k^\star}^{n}\big(\frac{i-k^\star}{n}\big)^m\phi_j^n(i)\Big)\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_j^n(i)e_i\Big)}{\Big(\frac{1}{n}\sum_{i=1}^{n}(\phi_j^n(i))^2\Big)}$$

$$= \frac{1}{\sqrt{n}}\sum_{i=k^\star}^{n}\Big(\Big(\frac{i-k}{n}\Big)^m - \Big(\frac{i-k^\star}{n}\Big)^m\Big)e_i - \frac{1}{\sqrt{n}}\sum_{i=k^\star}^{k}\Big(\frac{i-k}{n}\Big)^m e_i$$

$$-\sum_{j=0}^{p}\frac{\Big(\frac{1}{n}\sum_{i=k^\star}^{n}\big(\big(\frac{i-k}{n}\big)^m - \big(\frac{i-k^\star}{n}\big)^m\big)\phi_j^n(i)\Big)\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_j^n(i)e_i\Big)}{\Big(\frac{1}{n}\sum_{i=1}^{n}(\phi_j^n(i))^2\Big)}$$

$$+\sum_{j=0}^{p}\frac{\Big(\frac{1}{n}\sum_{i=k^\star}^{k}\big(\frac{i-k}{n}\big)^m\phi_j^n(i)\Big)\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_j^n(i)e_i\Big)}{\Big(\frac{1}{n}\sum_{i=1}^{n}(\phi_j^n(i))^2\Big)}.$$

The result follows from Lemma 2 and the expansion

$$\Big(\frac{i-k}{n}\Big)^m - \Big(\frac{i-k^\star}{n}\Big)^m = -m\Big(\frac{i-k^\star}{n}\Big)^{m-1}\Big(\frac{k-k^\star}{n}\Big) + o\Big(\frac{k-k^\star}{n}\Big).$$

□

Assertion (2.12) follows from Lemma 1, assertion (2.13) from Lemma 1 and Lemma 3 and assertion (2.14) from Lemma 2 and Lemma 3. The proof of (2.11) follows the same pattern as in Hušková (1998).

Now,

$$\begin{aligned}
\Big(\frac{\partial Q(t,s)}{\partial t}\Big)_{t=s=\theta^\star} &= 0,\\
\Big(\frac{\partial^2 Q(t,s)}{\partial t^2}\Big)_{t=s=\theta^\star} &= 2\Big\{\Big(\frac{\partial R(t,s)}{\partial t}\Big)^2_{t=s=\theta^\star}\Big/R(\theta^\star,\theta^\star) - \Big(\frac{\partial^2 R(t,s)}{\partial t\partial s}\Big)_{t=s=\theta^\star}\Big\}\\
&= -2A(\theta^\star),\\
\Big(\frac{\partial Z(t,s)}{\partial t}\Big)_{t=s=\theta^\star} &= \Big(\frac{\partial R(t,s)}{\partial t}\Big)_{t=s=\theta^\star}\Big/R(\theta^\star,\theta^\star).
\end{aligned}$$

Hence, uniformly in $|\beta_n(k-k^\star)|/\sqrt{n} \le r_n$,

$$D_k = \beta_n^2\Big(\frac{k-k^\star}{\sqrt{n}}\Big)^2\frac{1}{2}\Big(\frac{\partial Q(t,s)}{\partial t^2}\Big)_{t=s=\theta^\star}$$
$$+2\beta_n\frac{k-k^\star}{\sqrt{n}}\Big(\Big(\frac{\partial Z(s,t)}{\partial t}\Big)_{t=s=\theta^\star}S^n(k^\star)+\dot{S}^n(k^\star)\Big)+o_P\Big(\frac{|k-k^\star|}{\sqrt{n}}\Big).$$

Similarly as in Hušková (1998), regarding the definition of $\widehat{k^\star}$, the variable $\beta_n\frac{\widehat{k^\star}-k^\star}{\sqrt{n}}A(\theta^\star)$ has the same limit distribution as the variable

$$\Big(\frac{\partial Z(t,s)}{\partial t}\Big)_{t=s=\theta^\star}S^n(k^\star)+\dot{S}^n(k^\star)$$

that is asymptotically normal with the limit variance $A(\theta^\star)$. Thus the asymptotic distribution of $\beta_n\frac{\widehat{k^\star}-k^\star}{\sqrt{n}}\sqrt{A(\theta^\star)}$ is standard normal. □

Remark 1. It is clear that the exact form of $A(\theta^\star)$ depends on the value m and p.

Remark 2. If σ^2 is known but not equal to 1, then

$$\frac{\beta_n}{\sigma}\frac{\widehat{k^\star}-k^\star}{\sqrt{n}}\sqrt{A(\theta^\star)} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad n\to\infty.$$

If σ^2 is unknown it can be replaced by its usual estimator based on residual sum of squares obtained by least squares method.

3. Examples

Example 1 (Hušková)**.** We consider variables $\{Y_i, i=1,\dots,n\}$:

$$Y_i = \mu + \beta_n\Big(\frac{i-k^\star}{n}\Big)_+ + e_i, \quad i=1,\dots,n,$$

where $\mu, \beta_n \in R_1$, $k^\star \in N$ are unknown parameters, $k^\star = [n\theta^\star]$ for some $\theta^\star \in (0,1)$. The errors $\{e_i\}$ are i.i.d. satisfying $Ee_i = 0$, $Ee_i^2 = \sigma^2$, $E|e_i|^{2+\delta} < \infty$ for some $\delta > 0$. As $n \to \infty$, the parameter β_n satisfies condition (2.7).

The limit process $Y(t)$ has the form

$$Y(t) = \int_t^1 (z-t)\,dW(z) - \frac{(1-t)^2}{2}W(1)$$

and its covariance function

$$R(t,s) = \frac{(1-t)^3}{3} + (t-s)\frac{(1-t)^2}{2} - \frac{(1-s)^2(1-t)^2}{4}, \quad s \le t.$$

The limit distribution of $\widehat{k^\star}$ is given by:

$$\frac{\beta_n}{\sigma}\frac{\widehat{k^\star}-k^\star}{\sqrt{n}}\sqrt{\frac{\theta^\star(1-\theta^\star)}{1+3\theta^\star}} \xrightarrow{\mathcal{D}} N(0,1).$$

Example 2. We consider variables $\{Y_i, i = 1, \dots, n\}$:

$$Y_i = \mu + \alpha\Big(\frac{i}{n}\Big) + \beta_n\Big(\frac{i-k^\star}{n}\Big)_+ + e_i, \quad i = 1, \dots, n,$$

where $\mu, \alpha, \beta_n \in R_1$, $k^\star \in N$ are unknown parameters, $k^\star = [n\theta^\star]$ for some $\theta^\star \in (0,1)$. The errors have the same properties as in Example 1. As $n \to \infty$, the parameter β_n satisfies condition (2.7).

The limit process $Y(t)$ has the form

$$Y(t) = \int_t^1 (z-t)\, dW(z) - \frac{(1-t)^2}{2} W(1) - (1-t)^2(1+2t)\int_0^1 \Big(z - \frac{1}{2}\Big)\, dW(z)$$

and its covariance function is

$$R(t,s) = \frac{(1-t)^3}{3} + (t-s)\frac{(1-t)^2}{2}$$
$$- \frac{(1-s)^2(1-t)^2}{4} - \frac{(1-t)^2(1-s)^2(1+2t)(1+2s)}{12}, \quad s \le t.$$

The limit distribution of $\widehat{k^\star}$ is given by:

$$\frac{\beta_n}{\sigma}\frac{\widehat{k^\star} - k^\star}{\sqrt{n}}\sqrt{\frac{\theta^\star(1-\theta^\star)}{4}} \xrightarrow{\mathcal{D}} N(0,1).$$

Example 3. We consider variables $\{Y_i, i = 1, \dots, n\}$:

$$Y_i = \mu + \alpha\Big(\frac{i}{n}\Big) + \beta_n\Big(\Big(\frac{i-k^\star}{n}\Big)_+\Big)^2 + e_i, \quad i = 1, \dots, n,$$

where $\mu, \alpha, \beta_n \in R_1$, $k^\star \in N$ are unknown parameters, $k^\star = [n\theta^\star]$ for some $\theta^\star \in (0,1)$. The errors have the same properties as in Example 1. As $n \to \infty$, the parameter β_n satisfies condition (2.7).

The limit process $Y(t)$ has the form

$$Y(t) = \int_t^1 (z-t)^2\, dW(z) - \frac{(1-t)^3}{3} W(1)$$
$$- 12\Big(\int_t^1 (z-t)^2\big(z - \frac{1}{2}\big)\, dz\Big)\Big(\int_0^1 \big(z-\frac{1}{2}\big)\, dW(z)\Big)$$

and its covariance function is

$$R(t,s) = \frac{(1-t)^5}{5} + (t-s)\frac{(1-t)^4}{2} + (t-s)^2\frac{(1-t)^3}{3} - \frac{(1-s)^3(1-t)^3}{9}$$
$$- \Big((1-t)^4 + \Big(t - \frac{1}{2}\Big)\frac{(1-t)^3}{3}\Big)\Big(\frac{(1-s)^4}{4} + \Big(s - \frac{1}{2}\Big)\frac{(1-s)^3}{3}\Big), \quad s \le t.$$

The limit distribution of $\widehat{k^\star}$ is given by:

$$\frac{\beta_n}{\sigma}\frac{\widehat{k^\star} - k^\star}{\sqrt{n}}\sqrt{\frac{(\theta^\star)^3(1-\theta^\star)^3(4+5\theta^\star)}{3 + 15\theta^\star + 45(\theta^\star)^2 + 45(\theta^\star)^3}} \xrightarrow{\mathcal{D}} N(0,1).$$

References

[1] Anderson T.W., *The Statistical Analysis of Time Series*, John Wiley, New York, 1971.
[2] Billingsley P., *Convergence of Probability Measures*, John Wiley, New York, 1968.
[3] Davies R.B., *Hypothesis testing when a nuisance parameter is present only under the alternative*, Biometrika **74** (1987), 33–43.
[4] Hinkley D., *Inference about the intersection in two-phase regression*, Biometrika **56** (1969), 495–504.
[5] Hušková M., *Estimation in location model with gradual changes*, Comment. Math. Univ. Carolinae **39** (1998), 147–157.
[6] Knowles M., Siegmund D., Zhang H., *Confidence regions in semilinear regression*, Biometrika **78** (1991), 15–31.
[7] Petrov V.V., *Predel'nyje teoremy dlja sum nezavisimych slucajnych velicin*, Nauka, Moskva, 1987.
[8] Siegmund D., Zhang H., *Confidence region in broken line regression*, Change-point problems, IMS Lecture Notes - Monograph Series, vol. 23, 1994, pp. 292–316.

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