A function related to the central limit theorem

Paul Bracken

Abstract. A number of properties of a function which originally appeared in a problem proposed by Ramanujan are presented. Several equivalent representations of the function are derived. These can be used to evaluate the function. A new derivation of an expansion in inverse powers of the argument of the function is obtained, as well as rational expressions for higher order coefficients.

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1. In this article, a number of results which pertain to a function of Ramanujan are presented. The function originally appeared in a problem which was proposed by Ramanujan ([1], [2]) and has received periodic attention in the literature ([3]). In particular, this function $\vartheta(n)$ has a connection with the Poisson distribution and the central limit theorem ([4]). The sum $1 + \cdots + n^{n-1}/(n-1)!$ multiplied by e^{-n} represents the probability that a random variable with a Poisson distribution of mean n be less than n. By the Gaussian approximation of Poisson laws, this probability is close to $1/2$, and so Ramanujan's observations give refinements of this probabilistic consideration. The main results will be briefly summarized. A number of equivalent representations are presented which allow for the evaluation of the function easily for integer values, as well as for arbitrary positive values of the argument. An expansion of the function in inverse powers of the argument is derived. This has appeared before [3], but a new derivation is presented. In addition to this expansion, several other properties of $\vartheta(n)$ are derived and presented in a concise way. It is also shown that $\vartheta(n)$ decreases monotonically as the variable increases. The main advantage of the approach is that several new proofs are given and the techniques which are employed are quite elementary in nature.

2. Define the following series which are functions of the integer n

$$R(n) = \sum_{k=0}^{\infty} \frac{n!\, n^k}{(n+k)!} \tag{1}$$

$$Q(n) = \sum_{k=1}^{\infty} \frac{n!}{(n-k)!\, n^k}\,. \tag{2}$$

By reindexing these sums, these functions can be written in the form

$$R(n) = \frac{n!}{n^n}\Big(e^n - \sum_{k=0}^{n-1} \frac{n^k}{k!}\Big),$$

$$Q(n) = \frac{n!}{n^n}\Big(e^n - \sum_{k=n}^{\infty} \frac{n^k}{k!}\Big) = \frac{n!}{n^n}\sum_{k=0}^{n-1} \frac{n^k}{k!}.$$

By forming the difference of these two series, the following expression is obtained

$$R(n) - Q(n) = \frac{n!}{n^n}\Big(e^n - 2\sum_{k=0}^{n-1} \frac{n^k}{k!}\Big). \tag{3}$$

Define the sequence $\vartheta(n)$ by means of the equation

$$\sum_{k=0}^{n-1} \frac{n^k}{k!} + \frac{n^n}{n!}\vartheta(n) = \frac{1}{2}e^n, \tag{4}$$

Solving for the sum on the left,

$$\sum_{k=0}^{n-1} \frac{n^k}{k!} = \frac{e^n}{2} - \frac{n^n}{n!}\vartheta(n). \tag{5}$$

Substituting this into the right hand side of $R(n)-Q(n)$, one obtains the following simple relationship between the series defined in equations (1) and (2), and $\vartheta(n)$

$$2\,\vartheta(n) = R(n) - Q(n). \tag{6}$$

This provides a convenient way of calculating the function $\vartheta(n)$ for integer values of n. When the function is generalized, other methods for calculating it will be obtained.

By solving the defining equation for $\vartheta(n)$, the following expression for $\vartheta(n)$ can be written down

$$\vartheta(n) = 1 + \frac{1}{2}e^n \frac{n!}{n^n} - \frac{n!}{n^n}(1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}). \tag{7}$$

This function can be defined in terms of integrals and in so doing, it can be extended to a much larger domain, so that n need not be an integer. Also, it could be taken as a more general definition of the function to start with. After presenting the new definition, the following theorem will show that the definition in terms of an integral agrees with that in (7) given above.

Define the function $\vartheta(n)$ in the following way

$$\vartheta(n) = 1 + \frac{1}{2}n\,e^n\Big(\int_0^1 t^n e^{-nt}\,dt - \int_1^\infty t^n e^{-nt}\,dt\Big). \tag{8}$$

Theorem 1. *This definition agrees with equation* (7) *when* n *is a positive integer.*

PROOF: Beginning with the integral,

$$\int_1^\infty t^n e^{-nt}\, dt = \frac{1}{n^{n+1}} \int_n^\infty u^n e^{-u}\, du = \frac{e^{-n}}{n^{n+1}} \int_0^\infty (s+n)^n\, e^{-s}\, ds,$$

the bracket $(s+n)^n$ is expanded to obtain the following series of equations

$$\int_1^\infty t^n e^{-nt}\, dt = \frac{e^{-n}}{n^{n+1}} \sum_{k=0}^n n^{n-k} \binom{n}{k} \int_0^\infty s^k e^{-s}\, ds = \frac{e^{-n} n!}{n^{n+1}} \sum_{k=0}^n \frac{n^{n-k}}{(n-k)!}$$
$$= \frac{e^{-n} n!}{n^{n+1}} (1 + \frac{n}{1!} + \cdots + \frac{n^n}{n!}),$$

and

$$\frac{n!}{n^{n+1}} = \int_0^\infty (te^{-t})^n\, dt.$$

Comparing these results with the expression for $\vartheta(n)$, the proof is complete. $\square$

The results which follow hold if n is positive.

Theorem 2.

$$\vartheta(n) = 1 + \frac{e^n}{2n^n}\Gamma(n+1) - \int_0^\infty e^{-s}\, (1 + \frac{s}{n})^n\, ds.$$

PROOF: Consider the expression

$$\frac{n}{2} \int_0^1 t^n e^{n(1-t)}\, dt - \frac{n}{2} \int_1^\infty t^n e^{n(1-t)}\, dt.$$

It is useful to introduce a particular name for the variable t to the left of 1, and another name to represent t to the right of 1. This is motivated by looking at the graph of xe^{-x} as a function of x. For $x > 0$, there is a maximum at $x = 1$ where the function equals e^{-1}. Introduce the variable $\upsilon = t$ in the first integral for $0 \le t < 1$ and the variable $\omega = t$ for $1 < t < \infty$ in the second integral to obtain

$$\frac{n}{2} \int_0^1 \upsilon^n e^{n(1-\upsilon)}\, d\upsilon - \frac{n}{2} \int_1^\infty \omega^n e^{n(1-\omega)}\, d\omega$$
$$= \frac{n}{2} \int_0^\infty \upsilon^n e^{n(1-\upsilon)}\, d\upsilon - n \int_1^\infty \omega^n e^{n(1-\omega)}\, d\omega.$$

Set $x = n\,v$ in the first integral and $s = (\omega - 1)\,n$ in the second integral,

$$\frac{e^n}{2n^n}\int_0^\infty x^n e^{-x}\,dx - n\int_0^\infty (1+\frac{s}{n})^n e^{-s}\frac{ds}{n}$$
$$= \frac{e^n}{2n^n}\Gamma(n+1) - \int_0^\infty e^{-s}(1+\frac{s}{n})^n\,ds.$$

Let us define the function v or ω implicitly as a function of some parameter t by means of the general equation

$$e^{-t} = ue^{1-u},$$

where u might either represent v or ω. By taking the logarithm of this equation, one obtains

$$t = u - 1 - \log(u).$$

Consequently, it is clear that, on the lower branch, as $u \to 0$, one has $t \to \infty$, and as $u \to 1^-$, one can see that $t \to 0$. Finally, on the upper branch, as $u \to \infty$, $t \to \infty$, and as $u \to 1^+$, it is seen that $t \to 0$. Consequently, as an integral over the new variable t, $\vartheta(n)$ can be written

(9)
$$\vartheta(n) = 1 + \frac{n}{2}\int_\infty^0 e^{-nt}\frac{dv}{dt}\,dt - \frac{n}{2}\int_0^\infty e^{-nt}\frac{d\omega}{dt}\,dt$$
$$= 1 - \frac{n}{2}\int_0^\infty e^{-nt}\Big(\frac{dv}{dt} + \frac{d\omega}{dt}\Big)\,dt.$$

Differentiating the defining equation $u\,e^{1-u} = e^{-t}$ with respect to t, one obtains

$$e^{1-u}\,(1-u)\frac{du}{dt} = -e^{-t},$$

and solving for the derivative, one obtains

$$\frac{du}{dt} = \frac{u}{u-1}\,.$$

Consequently, on the lower branch, as $t \to \infty$, it can be seen that $v \to 0$, and therefore

$$\lim_{t\to\infty}\frac{dv}{dt} = 0,$$

and if one is on the upper branch, then $\omega \to \infty$ as $t \to \infty$, so the derivative has the limit

$$\lim_{t\to\infty}\frac{d\omega}{dt} = 1.$$

It is especially useful to investigate how the first derivative behaves as $t \to 0$, which means that v and ω go to 1 from above and below, respectively. This can be seen by plotting $t = u - 1 - \log(u)$ as a function of u, then plotting u as a function of t.

3. In order to determine the nature of the solutions v and ω, and in particular, their expansions about $t = 0$, consider the equation $t = u - 1 - \log(u)$ in the following form

$$\frac{\xi^2}{2} = w - \log(1+w),$$

where $t = \xi^2/2$ and $u - 1$ has been replaced by the variable w. We first expand about $w = 0$. This equation defines w as a function of the variable ξ, which could be complex ([5]).

Now ξ, regarded as a function of w is two valued in the neighborhood of $t = 0$, with its two branches given by

$$\xi = \pm w\left(1 - \frac{2}{3}w + \frac{2}{4}w^2 - \cdots\right)^{1/2}.$$

Each branch is an analytic function of w, which is regular when $|w| < 1$, with a simple zero at $w = 0$.

Since each branch is an analytic function of w, regular when $|w| < 1$ with a simple zero at $w = 0$, it follows from this that the equation

$$\xi = w\left(1 - \frac{2}{3}w + \frac{2}{4}w^2 - \cdots\right)^{1/2}$$

possesses a unique solution

$$w = \xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5 + a_6\xi^6 + \cdots$$

regular in a neighborhood $|\xi| < \rho$ of the origin. To obtain the a_i, substitute the series expansion for the function w into the equation

$$w - \log(1 + w) - \frac{1}{2}\xi^2 = 0.$$

Expanding this out and equating coefficients to zero, one finds the first six nonzero a_i which are given below.

$$\begin{aligned} a_2 &= \frac{1}{3}, & a_3 &= \frac{1}{36}, & a_4 &= -\frac{1}{270}, \\ a_5 &= \frac{1}{4320}, & a_6 &= \frac{1}{17010}, & a_7 &= -\frac{139}{5443200}. \end{aligned}$$

If this solution is called $w_1(\xi)$, the solution of the other branch is $w_2(\xi) = w_1(-\xi)$ which is regular in some neighborhood $|\xi| < \rho$. Setting $w = u - 1$ and $\xi^2 = 2t$, one obtains the following expansions for the functions υ and ω respectively

$$\begin{aligned} \upsilon &= 1 - \xi + a_2\xi^2 - a_3\xi^3 + a_4\xi^4 - a_5\xi^5 + \cdots \\ &= 1 - \sqrt{2}t^{1/2} + 2a_2t - 2\sqrt{2}a_3t^{3/2} + 4a_4t^2 - 4\sqrt{2}a_5t^{5/2} + \cdots, \end{aligned}$$

and

$$\begin{aligned} \omega &= 1 + \xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5 + \cdots \\ &= 1 + \sqrt{2}t^{1/2} + 2a_2t + 2\sqrt{2}a_3t^{3/2} + 4a_4t^2 + 4\sqrt{2}a_5t^{5/2} + \cdots. \end{aligned}$$

Sufficiently close to $t = 0$, these series for υ and ω can be added together to obtain the expansion for the function $\upsilon + \omega$, in terms of t,

$$\upsilon + \omega = 2 + 4a_2\, t + 8a_4 t^2 + 16a_6 t^3 + 32a_8 t^4 + \cdots ,$$

or in a more compact form,

$$\upsilon + \omega = 2 + \sum_{n=1}^{\infty} 2^n a_{2n} t^n .$$

It now becomes clear that one can obtain an expression for $\vartheta(n)$ in powers of n^{-1} by integrating the integral in the expression for $\vartheta(n)$ by parts, and then evaluating the large t limit from the derivatives in terms of υ or ω and the limit for small t by using the small t expansion for $\upsilon + \omega$.

The following theorem will make this clear and provide an additional way of calculating $\vartheta(n)$, such as evaluating the integrals numerically in Theorem 2, or evaluating the series (6).

Theorem 3 (Ramanujan [2])**.**

$$\vartheta(n) = \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{1}{2n^2}\int_0^{\infty} e^{-nt}\, h^{(4)}(t)\, dt, \tag{10}$$

where

$$h(t) = \upsilon(t) + \omega(t).$$

PROOF: Starting from the first derivative,

$$u^{(1)} = \frac{du}{dt} = \frac{u}{u-1}$$

where we write the m-th derivative of u as $u^{(m)}$ in what follows, it is found that the first four derivatives can be written down in the following form

$$u^{(2)} = -\frac{u}{(u-1)^3}, \quad u^{(3)} = \frac{2u^2+u}{(u-1)^5}, \quad u^{(4)} = \frac{-6u^3-8u^2-u}{(u-1)^7}.$$

Inductively, and from the expression for $u^{(1)}$, it is clear that there is at least a linear factor of u in the numerator of $u^{(m)}$ and the highest power of u in the denominator, which goes like $(u-1)$ to a power, is at least one greater than in the numerator. Hence, the derivatives are differentiable, rational functions of u such that $\upsilon^{(m)} \to 0$ as $t \to \infty$ on the lower branch and $\omega^{(m)} \to 0$ as $t \to \infty$ on the upper branch. The derivatives as t approaches zero can be found from the series expansions. Moreover, this implies that all derivatives are bounded and well defined on the interval of integration, and independent of the variable n.

Now, to obtain the required expansion, start with the integral expression (9)

$$\vartheta(n) = 1 - \frac{n}{2}\int_0^\infty e^{-nt}\, h^{(1)}(t)\, dt,$$

and integrating this by parts, one obtains

$$\vartheta(n) = \frac{1}{3} - \frac{1}{2}\int_0^\infty e^{-nt}\, h^{(2)}(t)\, dt.$$

Since the limits on the derivatives can be evaluated in a straightforward way using the previous remarks, this simple procedure can be continued arbitrarily many times, in particular, to obtain (10). That is, since all of the derivatives of $h(t)$ are bounded and can be evaluated at the endpoints, this simple procedure can be continued to arbitrarily high orders in n^{-1}. An expansion of the following form will result

$$\vartheta(n) = \sum_{k=0}^{N} b_k n^{-k} + O(n^{-N-1}).$$

The first ten coefficients in this expansion are presented in Table 1. By continuing the calculation to order twenty or more, the resulting expansion gives extremely good numerical results for values of n greater than one. □

Theorem 4. *The function $\vartheta(n)$ decreases monotonically from $1/2$ to $1/3$ as the argument increases.*

Proof: Begin with the expression for $\vartheta(n)$ in terms of the integrals

$$\vartheta(n) = 1 + \frac{1}{2}ne^n\Big\{\int_0^1 t^n e^{-nt}\, dt - \int_1^\infty t^n e^{-nt}\Big\}\, dt.$$

Let the function y be defined implicitly by the equation $ye^{-y} = te^{-t}$. Using this equation, the second integral from 1 to ∞ can be written as an integral from 0 to 1

$$\int_1^\infty t^n e^{-nt}\, dt = \int_1^0 (ye^{-y})^n \frac{dt}{dy}\, dy$$

where

$$\frac{dt}{dy} = \frac{(1-y)}{y}\frac{t}{(1-t)},$$

and so $\vartheta(n)$ can be written in terms of a single integral

$$\vartheta(n) = 1 + \frac{1}{2}ne^n\int_0^1 (ye^{-y})^n \frac{1-y}{y} g(y)\, dy,$$

where

$$g(y) = \frac{y}{1-y}\Big(1 + \frac{dt}{dy}\Big).$$

Since
$$\frac{d}{dy}(y\,e^{-y})^n = n(y\,e^{-y})^n \frac{1-y}{y}\,,$$
it is easy to integrate $\vartheta(n)$ in this form by parts to obtain
$$\vartheta(n) = 1 + \frac{1}{2}g(1) - \frac{e^n}{2}\int_0^1 (ye^{-y})^n g^{(1)}(y)\,dy.$$

If it can be shown that $g^{(1)}(y) < 0$ on the domain of integration then since the inequality $(ye^{1-y})^{n+1} < (ye^{1-y})^n$ holds, one obtains the inequality
$$-\frac{e^{n+1}}{2}\int_0^1 (ye^{-y})^{n+1} g^{(1)}(y)\,dy < -\frac{e^n}{2}\int_0^1 (ye^{-y})^n g^{(1)}(y)\,dy,$$
which implies that $\vartheta(n{+}1) < \vartheta(n)$. This means that $\vartheta(n)$ decreases monotonically as $n \to \infty$. Differentiating $g(y)$ with respect to y, one finds that
$$g^{(1)}(y) = \frac{1-y}{y}\Big(\frac{y}{(1-y)^3} + \frac{t}{(1-t)^3}\Big).$$

To investigate the behavior of this function, introduce the variable, or function of y, called $s(y)$ such that $t = ys(y)$. Substituting this into the equation $te^{-t} = ye^{-y}$, one has $se^{-ys} = e^{-y}$. Solving for y as a function of s, one has
$$y = \frac{\log(s)}{s-1}\,,$$
and s is restricted to the interval $1 \le s < \infty$. From $g^{(1)}(y)$ it must be shown that
$$\frac{y}{(1-y)^3} \le \frac{t}{(t-1)^3}\,.$$

In terms of s,
$$\frac{1}{(s-1-\log(s))^3} \le \frac{s}{(s\log(s)-s+1)^3}\,.$$

Since $s-1-\log(s)$ and $s\log(s)-s+1$ are both positive for $s \in (1,\infty)$, this inequality can be written as
$$\frac{\log(s)}{s-1} \le \frac{1+s^{1/3}}{s+s^{1/3}}\,.$$

This in fact can be shown to hold for s in the given interval. It remains to calculate the limiting values of $\vartheta(n)$.

To calculate the limiting values of $\vartheta(n)$, expand y in powers of $(s-1)$ near $s=1$ in the form

$$\frac{\log(s)}{s-1} = 1 - \frac{1}{2}(s-1) + \frac{1}{3}(s-1)^2 - \frac{1}{4}(s-1)^3 + \cdots .$$

Then, it is easy to show that

$$\lim_{y\to 1} g(y) = -\frac{4}{3},$$

and so the integral in terms of y is given by

$$\vartheta(n) = \frac{1}{3} - \frac{e^n}{2}\int_0^1 (ye^{-y})^n g^{(1)}(y)\,dy.$$

In particular, one has

$$\vartheta(0) = \frac{1}{3} - \frac{1}{2}\int_0^1 g^{(1)}(y)\,dy = \frac{1}{3} + \frac{1}{2}\left(g(0) - g(1)\right).$$

As $w \to 0$, one has $t \to \infty$, hence $g(0^+) = -1$, and so

$$\vartheta(0) = \frac{1}{2}.$$

Clearly,

$$\vartheta(\infty) = \frac{1}{3},$$

and so the function $\vartheta(n)$ decreases monotonically from $1/2$ to $1/3$. □

Theorem 5. *Define the function $\alpha(n)$ in terms of the function $\vartheta(n)$ as follows*

$$\vartheta(n) = \frac{1}{3} + \frac{4}{135\,(n+\alpha(n))}, \tag{11}$$

and the function $\beta(n)$ also in terms of $\vartheta(n)$ through the equation

$$\vartheta(n) = \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835(n^2+n\beta(n))}. \tag{12}$$

Then both functions $\alpha(n)$ and $\beta(n)$ are continuous over the entire interval $[0,\infty)$, and have expansions in powers of n^{-1}. The first few terms in each of these expansions are given as follows

$$\alpha(n) = \frac{2}{21} + \frac{32}{441n} - \frac{50752}{4584195n^2} - \frac{27070592}{1251485235n^3} + \frac{243682048}{26281189935n^4} + \cdots \tag{13}$$

$$\beta(n) = -\frac{2}{3} + \frac{3104}{4455n} - \frac{4544}{11583n^2} + \frac{28248448}{258011325n^3} - \frac{4265809664}{41868201375n^4} + \cdots . \tag{14}$$

These equations (13) *and* (14) *imply that the following limits hold*

$$\lim_{n\to\infty} \alpha(n) = \frac{2}{21}\,, \qquad \lim_{n\to\infty} \beta(n) = -\frac{2}{3}\,.$$

PROOF: Consider $\alpha(n)$ first. Solving the defining equation (11) for $\alpha(n)$, one obtains

$$\alpha(n) = \frac{4}{135\,(\vartheta(n) - \frac{1}{3})} - n.$$

This implies that $\alpha(n)$ is continuous on $[0,\infty)$ since $\vartheta(n)$ is continuous on $[0,\infty)$, and with $\vartheta(0) = 1/2$ from Theorem 4, one has $\alpha(0) = 8/45$. Substituting the asymptotic expansion for $\vartheta(n)$ with the b_k given in Table 1 into the equation for $\alpha(n)$, one can calculate the expansion (13) given above.

Similarly, by calculating $\beta(n)$ as a function of n and $\vartheta(n)$ from (12), one has

$$\beta(n) = -\frac{2835n^2\,\vartheta(n) - 945n^2 - 84n + 8}{2835n\,\vartheta(n) - 945n - 84}\,.$$

This implies that $\beta(0) = 2/21$ when $n = 0$ is substituted into this equation. Substituting the expansion for $\vartheta(n)$ into this equation for $\beta(n)$ and expanding, one obtains the expansion (14) for $\beta(n)$. These expansions (13) and (14) imply the stated limits as $n \to \infty$. □

Table 1: Coefficients for the Expansion of $\vartheta(n)$ in Powers of n^{-1}.

i	b_i
0	$\frac{1}{3}$
1	$\frac{4}{135}$
2	$-\frac{8}{2835}$
3	$-\frac{16}{8505}$
4	$\frac{8992}{12629925}$
5	$\frac{334144}{492567075}$
6	$-\frac{698752}{1477701225}$
7	$-\frac{23349012224}{39565450299375}$
8	$\frac{1357305243136}{2255230667064375}$
9	$\frac{6319924923392}{6765692001193125}$

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Centre de Recherches Mathématiques, Université de Montréal, Montréal, Québec H3C 3J7, Canada

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