Equations with discontinuous nonlinear semimonotone operators

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Abstract. The aim of this paper is to present an existence theorem for the operator equation of Hammerstein type $x + KF(x) = 0$ with the discontinuous semimonotone operator F. Then the result is used to prove the existence of solution of the equations of Urysohn type. Some examples in the theory of nonlinear equations in $L_p(\Omega)$ are given for illustration.

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1. Introduction

Let X be a real Banach space and X^* be its dual which are uniformly convex. For the sake of simplicity, the norms of X and X^* will be denoted by one symbol $\|\cdot\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x \in X^*$ and $x \in X$. Let $F : X \to X^*$ be a bounded, discontinuous and semimonotone operator and $K : X^* \to X$ a bounded (i.e. image of any bounded subset is bounded), linear and nonnegative operator.

Consider the nonlinear operator equation of Hammerstein type

$$x + KF(x) = 0. \tag{1.1}$$

Integral equations of Hammerstein type with a nonlinear smooth operator F are studied in [1]–[3], [6], [17]. When F is discontinuous, they are investigated in [5], [7], [15], [16] by introducing a new concept of solution. But, throughout this paper, the word 'solution' is meant in the classical sense. We shall prove an existence theorem for solution for discontinuous F. Using this result, we get a new result regarding the solvability of a class of nonlinear equations of Urysohn type

$$x + \sum_{j=1}^{m} K_j F_j(x) = 0, \tag{1.2}$$

where each K_j and F_j has the properties as K and F, respectively. Then, these theoretical results are applied to study the nonlinear integral equations in the spaces of type $L_p(\Omega)$. It should be mentioned that quasilinear elliptic equations with nonlinear discontinuous part are usually used to describe the state of the systems with variable structure (see [10]). These equations are studied recently (see [12]–[14]) and can be transformed to equations of Hammerstein type (see [12]).

Below, the symbols $\rightarrow$ and $\rightharpoonup$ denote convergence in norm and weak convergence, respectively.

2. Main result

Definition 1 (see [13]). A point $x \in X$ is called a point of h-continuity of the operator $G : X \to X^*$ if

$$\forall\, l \in X \;\; \lim_{t \to 0_+} \langle G(x + tl), l\rangle = \langle G(x), l\rangle.$$

A point $x \in X$ is called a point of discontinuity if x does not satisfy the condition in Definition 1.

Definition 2. A point of discontinuity x of G is called regular if

$$\exists\, l \in X : \lim_{t \to 0_+} \langle G(x + tl), l\rangle < 0.$$

Theorem 2.1. *Assume that all the above conditions hold, all the points of discontinuity of F are regular and that there exists a positive constant r such that*

$$\langle F(x), x\rangle > 0 \;\; \text{if} \;\; \|x\| > r.$$

Then equation (1.1) *has a solution x.*

Proof: As in [6], consider the regularized equation

$$x + B_n F(x) = 0, \quad B_n = B + \alpha_n V, \tag{2.1}$$

where V is the standard dual mapping of X^*, i.e. $V : X^* \to X$,

$$\langle V(x^*), x^*\rangle = \|V(x^*)\|\,\|x^*\| = \|x^*\|^2, \quad \forall\, x^* \in X^*,$$

and α_n is a sequence of positive real numbers such that $\alpha_n \to 0$ as $n \to +\infty$. Then $R(B_n) = X$, $B_n^{-1}(0) = 0$, B_n^{-1} is an one-to-one mapping and B_n^{-1} is continuous (see [4]). Therefore, all the points of discontinuity of F are points of discontinuity of $\widetilde{B}_n + F$ and, conversely, all points of discontinuity of $\widetilde{B}_n + F$ are points of discontinuity of F, where $\widetilde{B}_n(x) = -B_n^{-1}(-x)$. Obviously, we can rewrite equation (2.1) in the form

$$\widetilde{B}_n(x) + F(x) = 0. \tag{2.2}$$

By virtue of [17], equation (2.2) has a unique solution, henceforth denoted by x_n. Moreover, $\|x_n\| \leq r$, $\forall\, n$. As F is bounded, the sequence $\{F(x_n)\}$ is bounded, too. Without loss of generality, assume that

$$x_n \rightharpoonup x_0 \quad\text{and}\quad F(x_n) \rightharpoonup y_0^*.$$

From (2.1) it follows that

$$x_0 + By_0^* = 0. \tag{2.3}$$

Now, we have to prove that $y_0^* = F(x_0)$. Since F is semimonotone, we have $F = T + C$, with a monotone operator T and a compact operator C. Therefore,

$$\langle F(x) - C(x) - (F(x_n) - C(x_n)), x - x_n\rangle > 0, \quad \forall\, x \in X.$$

Hence,

$$\begin{aligned}\langle F(x) - C(x), x - x_n\rangle - \langle F(x_n) - C(x_n), x\rangle \geq \langle F(x_n), BF(x_n)\rangle \\ - \langle C(x_n), x_n\rangle + \alpha_n\langle F(x_n), VF(x_n)\rangle.\end{aligned}$$

By passing $n \to +\infty$ in the last equality, because of

$$\liminf_{n\to\infty}\langle F(x_n), BF(x_n)\rangle \geq \langle y_0^*, By_0^*\rangle,$$
$$\lim_{n\to+\infty} \alpha_n\langle F(x_n), VF(x_n)\rangle = 0,$$
$$\lim_{n\to+\infty}\langle C(x_n), x_n\rangle = \langle C(x_0), x_0\rangle,$$

and (2.3) we obtain

$$\langle F(x) - C(x), x - x_0\rangle - \langle y_0^* - C(x_0), x\rangle \geq \langle y_0^*, By_0^*\rangle - \langle C(x_0), x_0\rangle.$$

Thus,

$$\langle T(x) - (y_0^* - C(x_0)), x - x_0\rangle \geq 0. \tag{2.4}$$

Replacing x by $x_0 + tl$ for any $l \in X$ and $t > 0$ in (2.4) we see that

$$\langle F(x_0 + tl) - (y_0^* + C(x_0)), l\rangle \geq 0, \quad \forall\, l \in X.$$

Hence, x_0 is a point of h-continuity of T. Consequently, from (2.4) and Minty's lemma (see [17]) $T(x_0) = y_0^* - C(x_0)$, i.e. $y_0^* = F(x_0)$. □

Now, consider equation (2.1). Let the following conditions hold:

- $K_j : X^* \to X$ are linear and bounded operators satisfying the condition: $\sum_{j=1}^m\langle K_j x_j^*, x^*\rangle \geq 0$, $x^* = \sum_{i=1}^m x_i^*$, $x_i^* \in X^*$,
- $F_j : X \to X^*$ are bounded, discontinuous and semimonotone, and
- $\langle F_j(x), x\rangle \geq a_j\|x\|^2 - b_j\|x\| - c_j$, $\;a_j, b_j, c_j > 0$ (see [8]).

Operator equation (1.2) is investigated in [8]–[9], [11] with some smoothness property of F_j. Here, applying Theorem 2.1, we can prove the following result.

Theorem 2.2. *Under the above conditions on K_j and F_j, equation* (1.2) *has a solution in X.*

PROOF: Denote $Z = X \times \cdots \times X$ (m times). For $z = (x_1, \ldots, x_m) \in Z$, let

$$\|z\| = \Big(\sum_{j=1}^{m} \|x_j\|^2\Big)^{1/2}.$$

Then, Z is uniformly convex Banach space with respect to this norm with dual $Z^* = X^* \times \cdots \times X^*$. $(x_1, \ldots, x_m)$ means the column vector $(x_1, \ldots, x_m)^T$. Let $K : Z^* \to Z$ and $F : Z \to Z^*$ be defined as follows

$$K = \begin{bmatrix} K_1 & K_2 & \ldots & K_m \\ K_1 & K_2 & \ldots & K_m \\ \vdots & \vdots & \ddots & \vdots \\ K_1 & K_2 & \ldots & K_m \end{bmatrix}, \qquad F = \begin{bmatrix} F_1 & 0 & \ldots & 0 \\ 0 & F_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & F_m \end{bmatrix}. \tag{2.5}$$

Consider the Hammerstein equation

$$z + KF(z) = 0, \quad z \in Z \tag{2.6}$$

with K and F from (2.5). It is easy to see that K is a linear, bounded and nonnegative operator on Z^* and F is a semicontinuous operator on Z. Moreover,

$$\begin{aligned} \langle F(z), z\rangle &= \sum_{j=1}^{m} \langle F_j(x_j)\rangle \geq \sum_{j=1}^{m} \Big(a_j\|x_j\|^2 - b_j\|x_j\| - c_j\Big) \\ &\geq a\|z\|^2 - b\|z\| - c, \end{aligned}$$

where $a = \min a_j$, $b = \sqrt{m} \max b_j$ and $c = \max c_j$. Therefore, there exists a positive constant R such that $\langle F(z), z\rangle > 0$, if $\|z\| > R$. By virtue of Theorem 2.1, equation (2.6) has a solution $z_* = (x_{1*}, \ldots, x_{m*})$. Consequently, equation (1.2) has a solution $x = x_{1*}$ $(= x_{2*} = \cdots = x_{m*})$. □

3. Application

a. Consider the nonlinear integral equation of second kind

$$x(s) + \int_\Omega k(s,t)F(x(t))\,dt = 0, \tag{3.1}$$

where the kernel function $k(s,t)$ is such that the operator K defined by

$$(Kx)(s) = \int_\Omega k(s,t)x(t)\,dt$$

is bounded, nonnegative and K acts from $L_q(\Omega)$ into $L_p(\Omega)$ with $\Omega \subset \mathbb{R}^n$ measurable and $p^{-1} + q^{-1} = 1$. The nonlinear function $f(t)$ satisfies the following conditions:

(a) $f(t)t \geq a_0|t|^p + b_0|t|^\gamma + c_0$, $a_0 > 0$, $b_0 < 0$, $c_0 < 0$, $\gamma < p$ (see [14]),
(b) $f(t)$ is nondecreasing, rightcontinuous and at any point of discontinuity t_0 $f(t_0 - 0) < 0$, $f(t_0) < 0$,
(c) $|F(t)| \leq a_1 + b_1|t|^{p-1}$, $\forall\, t \in R^1$, $a_1 + b_1 > 0$, $a_1 \geq 0$, $b_1 \geq 0$.

By virtue of (c) we can define the operator $F : X = L_p(\Omega) \to X^* = L_q(\Omega)$ as

$$F(x)(t) = F(x(t)), \quad \forall\, x(t) \in L_p(\Omega).$$

Then equation (3.1) can be rewritten in the form (1.1), where the defined operator F possesses all the properties from Section 1. Indeed, condition (a) guarantees the existence of r in Theorem 2.1, the monotone property and the regularity of all points of discontinuity of F follows from (b) (see [13]) and the remaining properties are verified on the base of (c). Therefore, equation (3.1) has a solution, and this solution is unique if one of the operators K, F is strictly monotone.

b. Consider the nonlinear integral equation

$$x(t) + \sum_{j=1}^{m} \int_\Omega k_j(t,s) f_j(x(s))\, ds = 0. \tag{3.3}$$

If the operators K_j and F_j defined by

$$(K_j x)(t) = \int_\Omega k_j(t,s) x(s)\, ds,$$
$$(F_j x)(t) = f_j(x(t)),$$

have the same properties as K and F in **a.**, where only instead of the nonnegetiveness of K we assume that

$$\sum_{i=1}^{m} \int_\Omega x_i(t) \int_\Omega \sum_{j=1}^{m} k_j(t,s) x_j(s)\, ds\, dt \geq 0,$$

then (3.3) can be rewritten in the form (1.2). Therefore, equation (3.3) is solvable by Theorem 2.2.

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