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Properties of operators occurring in the Penrose transform

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Abstract. It is shown that operators occurring in the classical Penrose transform are differential. These operators are identified depending on line bundles over the twistor space.

Keywords: Penrose transform, conformally invariant operators

Classification: 53C28, 32L25

1. Introduction

A scheme for a study of the generalized Penrose transform on flag manifolds was described in the book [B-E]. The geometrical setting of the transform is given by the double fibration of homogeneous spaces of a form G/P, where G is a (complex) simple Lie group and P its parabolic subgroup. The classical case is that of the double fibration

$$\begin{tikzcd} & F_{1,2}(\mathbb{C}^4) \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ P_3(\mathbb{C}) & & G_2(\mathbb{C}^4) \end{tikzcd}$$

where the Grassmann manifold $G_2(\mathbb{C}^4)$ is the conformal compactification of the complex Minkowski space. The Penrose transform in this case is described in the paper [Ea], where the cohomology groups on subsets of the twistor space $P^3(\mathbb{C})$ with values in holomorphic vector bundles are interpreted by means of invariant operators on $G_2(\mathbb{C}^4)$ acting between sections of appropriate vector bundles. It is a sort of a common knowledge that all these operators are differential operators and that their explicit form can be found from the known classification of conformally invariant differential operators. The aim of the article is to supply a full proof of this fact for the classical Penrose transform. In Section 2 we review shortly the main classification result for conformally invariant differential operators (for details, see [Sl]) and we introduce at the same time the necessary notation. Section 3 contains the proof of the statement that invariant local and continuous operators are differential operators. In Section 4 we are proving that operators appearing in the Penrose transform are continuous, hence by previous result necessarily differential operators.

2. Notation

We will denote the group $SL(4,\mathbb{C})$ by G or by its Dynkin diagram $\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet$. Its parabolic subgroups will be denoted by the Dynkin diagram with corresponding roots crossed. We will use the same symbol for the corresponding homogeneous space. So for example $\bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$ will denote the homogeneous space $\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet \,/\, \bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$, too. The irreducible representations of parabolic groups are classified by their lowest weight λ. Let ρ be the sum of fundamental weights, $\rho=\omega_1+\omega_2+\omega_3$. An irreducible representation will be denoted by $-\lambda+\rho$ writing coefficients λ_i+1 over corresponding nodes, where $-\lambda=\sum\lambda_i\omega_i$. So, the integers over the uncrossed nodes must be positive, over the crossed nodes just integer. For example all irreducible representations of $\bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$ are classified by $\overset{a}{\bullet}\!-\!\!-\!\overset{b}{\times}\!-\!\!-\!\overset{c}{\bullet}$ where b is an integer and a,c are positive integers. We will use the same notation for the corresponding associated bundle.

Let us consider the following double fibration of homogeneous spaces with natural projections.

$$\begin{tikzcd}
 & \times\!-\!\!-\!\times\!-\!\!-\!\bullet \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\
\times\!-\!\!-\!\bullet\!-\!\!-\!\bullet & & \bullet\!-\!\!-\!\times\!-\!\!-\!\bullet
\end{tikzcd}$$

The homogeneous space $\bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$ is in fact the Grassmann manifold $G_2(\mathbb{C}^4)$, the space $\times\!-\!\!-\!\bullet\!-\!\!-\!\bullet$ is the projective space $P_3(\mathbb{C})$ and the space $\times\!-\!\!-\!\times\!-\!\!-\!\bullet$ is the flag manifold $F_{2,1}(\mathbb{C}^4)$.

We can fix a basis of $\mathbb{C}^4$ and define a dense open submanifold $\mathbb{M}^I$ of $\bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$ as a set of all 2-dimensional subspaces of $\mathbb{C}^4$ generated by two vectors of the form $(1,0,z_{00},z_{10})$, $(0,1,z_{01},z_{11})$. In this way we get coordinates $\{z_{00},z_{10},z_{01},z_{11}\}$ on $\mathbb{M}^I$.

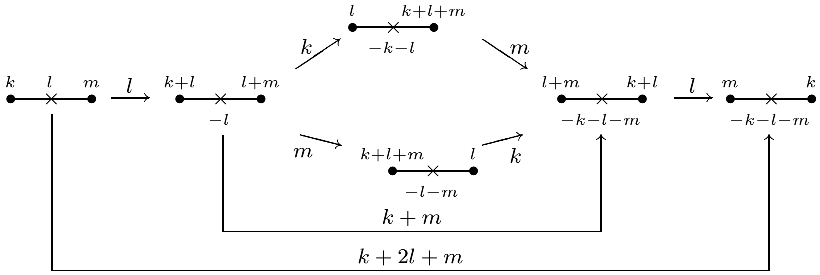
3. Invariant operators on $\bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$

It is important, that the invariant differential operators over $\bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$ are classified. The proof of the following lemma can be found for example in [Sl, 8.13, 8.14.]

Lemma 1. *Let E_λ and F_ρ be two irreducible homogeneous vector bundles over $\bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$. We consider the vector space D^{E_λ,F_ρ} of all G-invariant differential operators from E_λ to F_ρ.*

The dimension of D^{E_λ,F_ρ} is 0 (there is only a zero operator) or 1. The dimension is 1 if and only if there exist three positive integers k,l,m so that E_λ and F_ρ

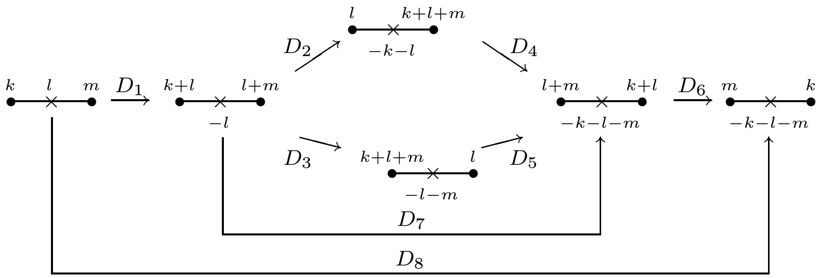
occur in the following diagram and they are joined by some arrow:



In this case D^{E_λ,F_ρ} is generated by one differential operator d^{E_λ,F_ρ}. The order of this operator is given by the number written over the arrow in the diagram.

Because we will be interested in Ker and Coker of these operators, we will choose one of them, depending on coefficients k, l, m and on its position in diagram as follows:

(DIG)



We omit the sheaves and write just bundles. So for example for $k = 2$, $l = 3$, $m = 4$ we have

$$D_3^{2,3,4} : \mathcal{O}(\overset{5}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{7}{\bullet}) \to \mathcal{O}\,(\overset{9}{\bullet}\!-\!\overset{-7}{\times}\!-\!\overset{3}{\bullet})$$

and the order of this operator is 4. We say that operator $D_k^{a,b,c}$ is an operator of type k. Operators from diagram (DIG) are G-invariant and if we disregard bottom arrows, we get an exact sequence.

Lemma 2. *Let E_λ, F_ρ be two homogeneous irreducible vector bundles over the space $\bullet\!-\!\times\!-\!\bullet$ corresponding to representations λ, ρ. Then any G-invariant operator $L : \mathcal{O}\,(E_\lambda) \to \mathcal{O}\,(F_\rho)$, which is continuous in the topology of local uniform convergence for all derivations, is differential. So these operators are classified in the above way.*

PROOF: Let L be such an operator and let us consider the action of element

$$o = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e^{-1} & 0 \\ 0 & 0 & 0 & e^{-1} \end{pmatrix} \in \bullet\!-\!\times\!-\!\bullet\,.$$

The element o acts by multiplication by e^{-2} on coordinates z and so by multiplication by e^{2k} on homogeneous polynomials of degree k in z. Because o is in the centre of an L-reductive factor of $\bullet\!\!-\!\!\times\!\!-\!\!\bullet$, it must also act by multiplication by numbers a_λ, a_ρ on $\mathbb{E}_\lambda$, $\mathbb{F}_\rho$.

Let $p_k = \sum_{|I|=k} p_I.z^I$ be a homogeneous polynomial in z of degree k with values in $\mathbb{E}_\lambda$ ($p_I \in \mathbb{E}_\lambda$). Its image $L(p_k)$ can be written as a series $\sum_{k=0,|I|=k}^{\infty} f_I^{p_k}.z^I$, $f_I^{p_k} \in \mathbb{F}_\rho$. By invariance with respect to action of o we get

$$e^{2k}.a_\lambda.\sum_{|I|=0}^{\infty} f_I^{p_k}.z^I = L(o(p_k)) = o(L(p_k)) = a_\rho.\sum_{|I|=0}^{\infty} e^{2|I|}.f_I^{p_k}.z^I$$

which implies that

$$e^{2k}.a_\lambda.f_I^{p_k} = a_\rho.e^{2|I|}.f_I^{p_k} \text{ for any } k, I.$$

So either $f_I^{p_k} = 0$ or $|I| = k - \log(a_\lambda/a_\rho)/2$. Let us denote $l = \log(a_\lambda/a_\rho)/2$. If l is integer then, for any k, $L(p_k)$ is a homogeneous polynomial of degree $k-l$, otherwise $L(p_k) = 0$.

By definition the operator L is differential, if it factorizes through some bundle of jets:

$$\begin{tikzcd} \mathcal{O}\,(E_\lambda) \arrow[r, "L"] \arrow[d, "\pi_l"'] & \mathcal{O}(F_\rho) \arrow[d, "\pi_0"] \\ J^l E_\lambda \arrow[r, "D"'] & F_\rho \end{tikzcd}$$

We will show that L factorizes through $J^l E_\lambda$ in point 0 and so by G-invariance in all points. Let α, β be two germs in 0. So we can write

$$\alpha = \sum_{|I|=0}^{\infty} \alpha_I.z^I = \sum_{i=0}^{\infty} \alpha_i \text{ and } \beta = \sum_{|I|=0}^{\infty} \beta_I.z^I = \sum_{i=0}^{\infty} \beta_i,$$

where α_i, β_i are homogeneous polynomials of degree i. Now if $\pi_l(\alpha) = \pi_l(\beta)$, then $\alpha_l = \beta_l$ by definition. By continuity of L we get $L(\alpha) = \sum_{i=0}^{\infty} L(\alpha_i)$. But $L(\alpha_i)$ is a homogeneous polynomial of degree $i - l$ and so $L(\alpha_i)(0) = 0$ for $i \neq l$. Similarly for β. So we get

$$L(\alpha)(0) = L(\alpha_l)(0) = L(\beta_l)(0) = L(\beta)(0)$$

and L factorizes through $J^l E_\lambda$ in point 0 and by G-invariance in all points. So it is a differential operator of degree l. $\square$

4. Penrose transform

Theorem 1. *Consider the following diagram of homogeneous spaces of $SL(4,\mathbb{C})$ with naturals projections;*

$$\begin{tikzcd} & \times\!-\!\times\!-\!\bullet \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \times\!-\!\bullet\!-\!\bullet & & \bullet\!-\!\times\!-\!\bullet \end{tikzcd}$$

Let C' be an open convex subset of $\mathbb{M}^I \subset \bullet\!-\!\times\!-\!\bullet$, $B' = \tau^{-1}(C')$ and $A' = \eta(B')$. The Penrose transform gives the following isomorphism and exact sequences. Operators are considered on sections of bundles over C'. These identifications are invariant with respect to the infinitesimal action of $sl(4,\mathbb{C})$.

For the zero cohomology on $\times\!-\!\bullet\!-\!\bullet$:

$\boxed{a \geq 1}$

$$H^0(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) \simeq \operatorname{Ker} D_1^{a,b,c},$$

$\boxed{a < 1}$

$$H^0(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) = 0.$$

For the first cohomology on $\times\!-\!\bullet\!-\!\bullet$:

$\boxed{a \geq 1}$

$$H^1(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) \simeq \operatorname{Ker} D_4^{a,b,c},$$

$\boxed{a = 0}$

$$H^1(A', \overset{0}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) \simeq \operatorname{Ker} D_3^{0,b,c} = \operatorname{Ker} D_4^{0,b,c},$$

$\boxed{-b < a < 0}$

$$0 \to \operatorname{Ker} D_4^{-a,a+b,c} \to H^1(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) \to \operatorname{Ker} D_8^{-a,a+b,c} \to 0,$$

$\boxed{a = -b}$

$$H^1(A', \overset{-b}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) \simeq \operatorname{Ker} D_8^{b,0,c} = \operatorname{Ker} D_7^{b,0,c},$$

$\boxed{-b-c < a < -b}$

$$0 \to H^1(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) \to \operatorname{Ker} D_5^{b,c,-a-b-c} \xrightarrow{d_2^{0,1}} \overset{-a-b-c}{\bullet}\!-\!\underset{a}{\times}\!-\!\overset{b}{\bullet} \to 0,$$

$\boxed{a = -b - c}$

$$H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \simeq \operatorname{Ker}\, D_2^{b,c,0} = \operatorname{Ker}\, D_5^{b,c,0},$$

$\boxed{a < -b - c}$

$$H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \simeq \operatorname{Ker}\, D_5^{b,c,-a-b-c}.$$

Proof: These identifications are proved using the Penrose transform, which is described in detail, for example, in [B-E]. We shall quickly describe its principle and then we shall study the problem of identification of operators on $\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet$.

The fibres of $\eta : B' \to A'$ are topologically trivial, and so by [Bu]

$$H^r(B', \eta^{-1}\mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})) \simeq H^r(A', \mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})). \tag{1}$$

Due to the fact that A' can be covered by two Stein sets, intersections of which are Stein too, we have $H^r(A', \mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})) = 0$ for $r \geq 2$ ([G-R, 6.D.4]).

On the other hand we have ([B-E, 8.4.1, 8.7], [Ro], [Sl, 8.3]) the BGG resolution of $\eta^{-1}\mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})$

$$0 \to \eta^{-1}\mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \to \Delta^*(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\times}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \tag{2}$$

which is in fact the exact sequence

$$0 \to \eta^{-1}(\mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})) \to \mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\times}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \to \mathcal{O}(\overset{a+b}{\times}\!\!-\!\!\!-\!\!\overset{-b}{\times}\!\!-\!\!\!-\!\!\overset{b+c}{\bullet}) \to \mathcal{O}(\overset{a+b+c}{\times}\!\!-\!\!\!-\!\!\underset{-b-c}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}) \to 0.$$

We construct the double complex by constructing the Dolbeault resolution for every Δ^p. So $K^{p,q} = \mathcal{E}^{0,q}(B', \Delta^p(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\times}\!\!-\!\!\!-\!\!\overset{c}{\bullet}))$, with d' induced by operators in BGG and $d'' = \bar{\partial}$. There is the associated spectral sequence (E, d) such that

$$E_1^{p,q} = H^q(B', \Delta^p(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})) = \Gamma(C', \tau_*^q \Delta^p(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})), \tag{3}$$

the last equation holding because τ is proper and C' is Stein ([W-W, 3.6]). This sequence converges to the global cohomology K^{**} and so by (1)

$$E^{p,q} \Rightarrow H^r(B', \eta^{-1}\mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})) \simeq H^r(A', \mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})). \tag{4}$$

The terms of E_1 can be identified with sections of vector bundles over C' by Bott-Borel-Weyl theorem ([B-E, 5.1]):

$$\tag{5} \tau^0_*\mathcal{O}(\overset{k}{\times}\!-\!\overset{l}{\times}\!-\!\overset{m}{\bullet}) = \begin{cases} \mathcal{O}(\overset{k}{\bullet}\!-\!\overset{l}{\times}\!-\!\overset{m}{\bullet}) & \text{for } k>0 \\ 0 & \text{for } k \le 0 \end{cases},$$

$$\tau^1_*\mathcal{O}(\overset{k}{\times}\!-\!\overset{l}{\times}\!-\!\overset{m}{\bullet}) = \begin{cases} \mathcal{O}(\overset{-k}{\bullet}\!-\!\overset{k+l}{\times}\!-\!\overset{m}{\bullet}) & \text{for } k<0 \\ 0 & \text{for } k \ge 0 \end{cases},$$

$$\tau^i_*\mathcal{O}(\overset{k}{\times}\!-\!\overset{l}{\times}\!-\!\overset{m}{\bullet}) = 0 \text{ for } i \ge 2.$$

Now we can identify the operators d for to obtain the results. Let us write down explicitly the E and identify d for the case $a=-b$, which is the most complicated case. The demonstrations for other cases are essentially the same.

Using the procedure described above, we get

$$E_1^{p,q} = E_2^{p,q} = \left|\begin{array}{ccc} \dots & \dots & \dots \\ 0 & 0 & 0 \\ \mathcal{O}\ (C', \overset{b}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{c}{\bullet}) & 0 & 0 \\ 0 & 0 & \mathcal{O}(C', \overset{c}{\bullet}\!-\!\overset{-b-c}{\times}\!-\!\overset{b}{\bullet}) \\ \hline \end{array}\right.$$

$$d_2^{0,1} : \mathcal{O}\ (C', \overset{b}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{c}{\bullet}) \to \mathcal{O}(C', \overset{c}{\bullet}\!-\!\overset{-b-c}{\times}\!-\!\overset{b}{\bullet}).$$

Now $d_2^{0,1}$ must be surjective, because Coker $d_2^{0,1} = H^2(A', \mathcal{O}(\overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet})) = 0$. So $d_2^{0,1}$ is non zero operator, and using Lemmas 1 and 2 we can identify $d_2^{0,1} = D_7^{b,0,c} = D_8^{b,0,c}$. So we get the limit of the spectral sequence

$$E_3^{p,q} = E_\infty^{p,q} = \left|\begin{array}{ccc} \dots & \dots & \dots \\ 0 & 0 & 0 \\ \text{Ker } D_8^{b,0,c} & 0 & 0 \\ 0 & 0 & 0 \\ \hline \end{array}\right.$$

and we obtain $H^1(A', \overset{-b}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) \simeq \text{Ker } D_8^{b,0,c}$ as the result of the Penrose transform.

To complete this proof, we must show that the operators acting between sections over C' of bundles occurring in spectral sequences define an invariant operator continuous in the topology of local uniform convergence for all derivations.

The proof of invariance and that L commutes with sheaf restrictions is straightforward.

For the proof of continuity we must first see in more detail the identifications (5), which are in fact

$$
\begin{aligned}
H^0\big(B', \overset{k}{\times}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}\big) &\simeq \Gamma\big(C', \overset{k}{\bullet}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}\big),\\
H^1\big(B', \overset{k}{\times}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}\big) &\simeq \Gamma\big(C', \overset{-k}{\bullet}\!\!-\!\!-\!\!\underset{k+l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}\big).
\end{aligned}
\tag{6}
$$

These identifications are based on Leray spectral sequence and can be made explicit in the following way ([G-H, 3.5]).

The fibres of τ are isomorphic to $\mathbb{P}_1$ and because C' is topologically trivial, we have $B' \simeq C' \times \mathbb{P}_1$ and so we have the local chart $(z, [v])$ on B'. The coordinates for $\mathbb{C} \simeq \{[v_0, v_1] \in \mathbb{P}_1; v_1 \neq 0\} \subset \mathbb{P}_1$ will be denoted by $v = v_0/v_1$. The forms on $\mathbb{P}_1$ are determined by their restrictions to $\mathbb{C}$. To simplify the formulae we will denote $\overset{k}{\times}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}$ by F.

Let us recall that $H^i(B', F)$ is defined as $\operatorname{Ker} \bar\partial_i / \operatorname{Im} \bar\partial_{i-1}$ in the Dolbeault resolution. Identifications (6) are induced by the mappings:

$$P_0 : \operatorname{Ker} \bar\partial_0 \to \mathcal{O}(C', H^0(\mathbb{P}^1, F)),$$

$$P_1 : \operatorname{Ker} \bar\partial_1 \to \mathcal{O}(C', H^1(\mathbb{P}^1, F)),$$

where F is restricted to fibres and $\mathcal{O}(C', H^0(\mathbb{P}^1, F))$, $\mathcal{O}(C', H^1(\mathbb{P}^1, F))$ being identified by BBW theorem.

The maps P_0 and P_1 are obtained from the Leray spectral sequence as follows. Let us define a double complex

$$K^{p,q} := \Big\{\omega = \sum_{\substack{|I|=q\\|J|=p}} f_I(z,v)d\bar v_I \wedge d\bar z_J; f_I \in \mathcal{C}^\infty(B', F)\Big\},$$

the horizontal operator is $\bar\partial_z$ and vertical one is $\bar\partial_v$. Evidently $K^{p,q} = 0$ for $p > 4$ or $q > 1$. The associated complex is exactly the Dolbeault resolution. We have the spectral sequence converging to the total cohomology whose first term is

$$E_1^{p,q} = \operatorname{Ker} \bar\partial_v / \operatorname{Im} \bar\partial_v = \{\omega = \sum_{|J|=p} G_J d\bar z_J; G_J \in C^\infty(C', H^q(\mathbb{P}^1, F)\}.$$

The space $H^q(\mathbb{P}^1, \mathcal{O}(F))$ is a finite dimensional vector space. In such a way,

$$E_1^{p,q} = \mathcal{E}^{0,p}(C', H^q(\mathbb{P}^1, F))$$

and $d_1 = \bar{\partial}_z$. It follows, that

$$E_2^{p,q} = H^p(C', \mathcal{O}(H^q(\mathbb{P}^1, F))) = \begin{cases} \mathcal{O}(C', H^q(\mathbb{P}^1, F)) & \text{for } p = 0 \\ 0 & \text{for } p \geq 1 \end{cases}.$$

Evidently $E_\infty^{p,q} = E_2^{p,q}$ and because $E^{p,q}$ converges to the cohomology of K^n we get $H^p(C', \mathcal{O}(H^q(\mathbb{P}^1, F))) \Rightarrow H^{p+q}(B', F)$ and we obtain identities:

$$H^q(B', F) \simeq \Gamma(C', H^q(\mathbb{P}^1, F)) \text{ for } q = 0, 1.$$

In fact, $\mathcal{E}^{0,0}(B', F) = K^{0,0}$, $\mathcal{E}^{0,1}(B', F) = K^{1,0} \oplus K^{0,1}$ and mappings P_0 and P_1 are obtained by double quotient in this spectral sequence. From this we see that these mappings are continuous and P_0 is a bijection.

We define a subspace $\mathcal{H}_{B'}$ of Ker $\bar{\partial}^1$, formed by forms $\omega = f(z, v)d\bar{v}$, where f holomorphic in z and $f(z, v)d\bar{v} \in \mathcal{H}^1(\mathbb{P}^1, F)$ for any z fixed. $\mathcal{H}^1(\mathbb{P}^1, F)$ denotes 1-forms on $\mathbb{P}^1$ with coefficients in F harmonic with respect to any metric. In such a way, we get forms representing $H^1(\mathbb{P}^1, F)$. The P_1 restricted to $\mathcal{H}_{B'}$ is a bijection.

We have to identify the operators d_1 and d_2 with some differential operators. Let us see the proof of continuity for d_2, which is more complicated and contains the case of d_1.

$$\begin{array}{ccccc}
 & & \uparrow \bar{\partial} & & \\
\mathcal{E}^{0,1}(B', F_0) & \xrightarrow{D_1} & \mathcal{E}^{0,1}(B', F_1) & & \\
 & & \uparrow \bar{\partial} & & \\
 & & \mathcal{E}^{0,0}(B', F_1) & \xrightarrow[D_2]{} & \mathcal{E}^{0,0}(B', F_2)
\end{array}$$

The operator d_2 goes from a double quotient of $\mathcal{E}^{0,1}(B', F_0)$ to a double quotient of $\mathcal{E}^{0,0}(B', F_2)$. We claim that by the identification of these double quotients by P_0 and P_1, we get an operator L continuous in the topology of locally uniform convergence for all derivations. This convergence will be denoted by an arrow.

Let $s_n \in \Gamma(C', \mathcal{H}^1(\mathbb{P}^1, F_0)), s_n \to 0$. Let $\{h_j\}$ be a basis for $\mathcal{H}^1(\mathbb{P}^1, F_0)$ and

$$s_n = \sum_1^m s_n^j . h_j,$$

where s_n^j are holomorphic functions on C'. We define (0,1)-forms on B'

$$\alpha_n(z, v) := \sum_1^m s_n^j(z) . h_j(v).$$

Evidently $P_1(\alpha_n) = s_n$ and $\alpha_n \to 0$ on B'. Let us denote $\gamma_n = D_1(\alpha_n)$. Because D_1 is differential, we have $\gamma_n \to 0$ and by the properties of a spectral sequence we have $\gamma_n \in \mathrm{Im}\ \bar\partial$. Spaces $\mathcal{E}^{0,0}(B', F_1)$ and $\mathcal{E}^{0,1}(B', F_1)$ are Fréchet spaces in the topology of locally uniform convergence for all derivations. In addition $\mathrm{Im}\ \bar\partial = \mathrm{Ker}\ \bar\partial$, because $\mathrm{Ker}\ \bar\partial / \mathrm{Im}\ \bar\partial = H^1(B', F_1) = 0$ by BBW theorem. So $\mathrm{Im}\ \bar\partial$ is a Fréchet space too, and

$$\bar\partial : \mathcal{E}^{0,0}(B', F_1) \to \mathrm{Im}\ \bar\partial$$

is an open mapping by the Banach theorem. So there exist $\beta_n \in \mathcal{E}^{0,0}(B', F_1)$ such that $\bar\partial(\beta_n) = \gamma_n$ and $\beta_n \to 0$.

But by definition $L(s_n) = P_0(D_2(\beta_n))$ and evidently $P_0(D_2(\beta_n)) \to 0$ and so L is continuous. □

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