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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 42 (2001), No. 4, 711--719

Persistent URL: <http://dml.cz/dmlcz/119286>

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## On exit laws for semigroups in weak duality

IMED BACHAR

*Abstract.* Let  $\mathbb{P} := (P_t)_{t>0}$  be a measurable semigroup and  $m$  a  $\sigma$ -finite positive measure on a Lusin space  $X$ . An  $m$ -exit law for  $\mathbb{P}$  is a family  $(f_t)_{t>0}$  of nonnegative measurable functions on  $X$  which are finite  $m$ -a.e. and satisfy for each  $s, t > 0$   $P_s f_t = f_{s+t}$   $m$ -a.e. An excessive function  $u$  is said to be in  $\mathcal{R}$  if there exists an  $m$ -exit law  $(f_t)_{t>0}$  for  $\mathbb{P}$  such that  $u = \int_0^\infty f_t dt$ ,  $m$ -a.e.

Let  $\mathcal{P}$  be the cone of  $m$ -purely excessive functions with respect to  $\mathbb{P}$  and  $\mathcal{I}mV$  be the cone of  $m$ -potential functions. It is clear that  $\mathcal{I}mV \subseteq \mathcal{R} \subseteq \mathcal{P}$ . In this paper we are interested in the converse inclusion. We extend some results already obtained under the assumption of the existence of a reference measure. Also, we give an integral representation of the mutual energy function.

*Keywords:* semigroup, weak duality, exit law

*Classification:* 31D05, 60J45

### 1. Introduction

Let  $X$  be a Lusin metrizable topological space with its Borel tribe  $\mathcal{B}$ . We denote by  $\mathcal{B}^+$  the cone of nonnegative Borel functions on  $X$  and by  $\mathcal{M}$  the class of  $\sigma$ -finite positive measures on  $(X, \mathcal{B})$ .

In the sequel, let  $\mathbb{P} := (P_t)_{t>0}$  and  $\hat{\mathbb{P}} := (\hat{P}_t)_{t>0}$  be two submarkovian measurable semigroups on  $(X, \mathcal{B})$ , in weak duality with respect to a fixed measure  $m \in \mathcal{M}$ , namely,

$$(1) \quad \int P_t f(x)g(x)m(dx) = \int f(x)\hat{P}_t g(x)m(dx), \quad \forall t > 0, \quad \forall f, g \in \mathcal{B}^+.$$

The potential kernels  $V := \int_0^\infty P_t dt$  and  $\hat{V} := \int_0^\infty \hat{P}_t dt$  are assumed to be proper and satisfy the unicity of charges, that is, for each  $\mu, \nu \in \mathcal{M}$

$$(2) \quad \text{if } \mu\hat{V} = \nu\hat{V} \in \mathcal{M} \text{ then } \mu = \nu.$$

Throughout this paper we denote by  $\mathcal{F}$  the set of Borel nonnegative functions which are finite  $m$ -a.e. and by  $\mathcal{E}$  the cone of functions  $u \in \mathcal{F}$  which are excessive with respect to  $\mathbb{P}$ .

Let  $\mathcal{R}$  be the cone of functions  $u \in \mathcal{E}$  such that there exists an  $m$ -exit law  $(f_t)_{t>0}$  for  $\mathbb{P}$  satisfying

$$u = \int_0^\infty f_t dt, \quad m\text{-a.e.}$$

Denote by  $\mathcal{P} := \{u \in \mathcal{E} : \inf_{t \rightarrow \infty} P_t u = 0 \text{ } m\text{-a.e.}\}$ , the cone of purely excessive functions with respect to  $\mathbb{P}$  and  $\mathcal{Im}V := \{u \in \mathcal{E} : u = V\varphi \text{ } m\text{-a.e. for some } \varphi \in \mathcal{B}^+\}$ .

Each object related to  $\hat{\mathbb{P}}$  is equipped with “ $\hat{\phantom{x}}$ ”, so we define as above  $\hat{\mathcal{E}}$  and  $\hat{\mathcal{R}}$ . Finally, we denote by  $\prec$  the strong  $m$ -domination order defined by  $\mathcal{E}$ .

Obviously, we have the natural inclusion  $\mathcal{R} \subseteq \mathcal{P}$ , while in [10] Hmissi proved that  $\mathcal{R} = \mathcal{P}$  provided that  $\varepsilon_x P_t \ll m$  for each  $x \in X$  and  $t > 0$  (where  $\varepsilon_x$  denotes the Dirac mass at  $x$ ) and in [12] he showed that if  $\mathbb{P}$  is a lattice semigroup then  $\mathcal{R} = \mathcal{Im}V$ . Furthermore, Hmissi gave an example of a nonlattice semigroup for which  $\mathcal{R} = \mathcal{Im}V$ .

The first purpose of this paper is to give necessary and sufficient conditions on  $\hat{\mathbb{P}}$  such that  $\mathcal{R} = \mathcal{P}$ . More precisely, we prove the following

**Theorem 1.** *The following statements are equivalent:*

- (i)  $\mathcal{R} = \mathcal{P}$ ;
- (ii) for every  $\mu \in \mathcal{M}$ , if  $\mu \hat{V} \ll m$  then  $\mu \hat{P}_t \ll m$  for all  $t > 0$ .

As an application we obtain an integral representation of the mutual energy.

The second purpose of this paper is to give necessary and sufficient conditions on  $\hat{\mathbb{P}}$  for  $\mathcal{R} = \mathcal{Im}V \cap V(\mathbb{M})$  (where  $V(\mathbb{M})$  will be defined later). More precisely, we have the following result:

**Theorem 2.** *The following statements are equivalent:*

- (i)  $\mathcal{Im}V = \mathcal{R} \cap V(\mathbb{M})$ ;
- (ii) for every  $\mu \in \mathcal{M}$ , if  $\mu \hat{P}_t \ll m$  for all  $t > 0$  then  $\mu \ll m$ .

In Section 3 we suppose that the semigroup  $\mathbb{P}$  is defined by a vaguely continuous convolution semigroup on  $X = \mathbb{R}^n$  ( $n \geq 1$ ). Then we prove the existence of the largest element of  $\mathcal{R}$  in the strong  $m$ -domination order which is strongly  $m$ -dominated by a given function  $u \in \mathcal{P}$ . Namely,

**Theorem 3.** *Let  $u \in \mathcal{P}$ . Then there exist a function  $r(u) \in \mathcal{R}$  such that  $r(u) \prec u$  and for each function  $v \in \mathcal{R}$  satisfying  $v \prec u$ , we have  $v \prec r(u)$ .*

Note that the above result has been established by Hmissi ([11], Theorem 2.4) under the assumption of existence of a reference measure.

## 2. Characterization of $\mathcal{R}$

### 2.1 Exit laws in weak duality.

For the coming definitions we refer to [1], [3], [8] and [9].

**Definition 1.** (i) A family  $(f_t)_{t>0} \subset \mathcal{F}$  is called an  $m$ -exit law for  $\mathbb{P}$  if

$$(3) \quad P_s f_t = f_{s+t} \text{ } m\text{-a.e. for each } s, t > 0.$$

(ii) Two  $m$ -exit laws  $(f_t)_{t>0}$  and  $(g_t)_{t>0}$  are equivalent if  $f_t = g_t$   $m$ -a.e. for each  $t > 0$ .

**Remark 1.** Let  $(f_t)_{t>0}$  be an  $m$ -exit law for  $\mathbb{P}$  such that  $u := \int_0^\infty f_t dt \in \mathcal{F}$ . Then  $P_t u = V f_t$   $m$ -a.e. for each  $t > 0$  and by [9, (6.19)] there exists  $v \in \mathcal{E}$  such that  $u = v$   $m$ -a.e.

**Definition 2.** A function  $u \in \mathcal{E}$  will be called the potential of a measure  $\mu \in \mathcal{M}$  if  $u \cdot m = \mu \hat{V}$ , and we write  $u = V(\mu)$  (see [8, (3.5)]).

In the sequel, we denote

$$\mathbb{M} := \{ \mu \in \mathcal{M} : u \cdot m = \mu \hat{V} \text{ with } u \in \mathcal{E} \}$$

and

$$V(\mathbb{M}) := \{ V(\mu) : \mu \in \mathbb{M} \}.$$

**Remark 2.** (i)  $\mathcal{I}mV \subseteq \mathcal{R} \cap V(\mathbb{M})$  and  $\mathcal{R} \subseteq \mathcal{P}$ .

(ii) Let  $\mathbb{P}$  be the heat semigroup on  $\mathbb{R}^{n+1}$ . Then we have  $\mathcal{I}mV \neq \mathcal{R} \cap V(\mathbb{M})$  and  $\mathcal{R} \neq \mathcal{P}$ .

We recall that an entrance law for  $\hat{\mathbb{P}}$  is a family  $(\mu_t)_{t>0} \subset \mathcal{M}$  such that

$$(4) \quad \mu_t \hat{P}_s = \mu_{s+t} \text{ for each } s, t > 0.$$

To prove Theorem 1, we need the following representation theorem (see K. Janssen [13]).

**Theorem 4.** Any purely excessive measure  $\mu$  can be uniquely decomposed as the integral  $\mu = \int_0^\infty \mu_t dt$  of an entrance law  $(\mu_t)_{t>0}$  with respect to  $\mathbb{P}$ .

PROOF OF THEOREM 1: (ii)  $\Rightarrow$  (i). Let  $u \in \mathcal{P}$ , then the measure  $u \cdot m$  is purely excessive with respect to  $\hat{\mathbb{P}}$ . So by Theorem 4 there exists a unique entrance law  $(\nu_t)_{t>0}$  for  $\hat{\mathbb{P}}$  such that

$$(5) \quad u \cdot m = \int_0^\infty \nu_t dt.$$

Using (1) and (4) it follows that  $\nu_t \hat{V} = (P_t u) \cdot m \leq u \cdot m$  for each  $t > 0$ . Hence,  $\nu_t = \nu_{\frac{t}{2}} \hat{P}_{\frac{t}{2}} \ll m$  for each  $t > 0$ .

Let  $(f_t)_{t>0} \subset \mathcal{F}$  be such that  $\nu_t = f_t \cdot m$  for each  $t > 0$ . Using again (1) and (4) we check that  $(f_t)_{t>0}$  is an  $m$ -exit law with respect to  $\mathbb{P}$ . We deduce by (5) that  $u \in \mathcal{R}$ .

(i)  $\Rightarrow$  (ii). Let  $\mu \in \mathcal{M}$  and  $V(\mu) \in \mathcal{P}$  be such that

$$(6) \quad \mu \hat{V} = V(\mu) \cdot m = \int_0^\infty (\mu \hat{P}_t) dt.$$

Let  $(f_t)_{t>0}$  be an  $m$ -exit law with respect to  $\mathbb{P}$  such that

$$(7) \quad V(\mu) \cdot m = \int_0^\infty (f_t \cdot m) dt.$$

From (3) and (1) we check that  $(f_t \cdot m)_{t>0}$  is an entrance law with respect to  $\hat{\mathbb{P}}$ . Therefore by (2), (6) and (7), we have  $\mu \hat{P}_t = f_t \cdot m$  for each  $t > 0$ .

Hence,  $\mu \hat{P}_t \ll m$  for each  $t > 0$ . □

PROOF OF THEOREM 2: (i)  $\Rightarrow$  (ii). Let  $\mu \in \mathcal{M}$  be such that  $\mu \hat{P}_t \ll m$  for each  $t > 0$ . Then there exists  $V(\mu) \in \mathcal{E}$  which satisfies

$$(8) \quad \mu \hat{V} = V(\mu) \cdot m.$$

Let  $(f_t)_{t>0} \subset \mathcal{F}$  such that  $\mu \hat{P}_t = f_t \cdot m$  for each  $t > 0$ . By (1) and (4) it is easy to check that  $(f_t)_{t>0}$  is an  $m$ -exit law with respect to  $\mathbb{P}$ , and from (8) it follows that  $V(\mu) \in \mathcal{R} \cap V(\mathbb{M})$ . Now by (i), there exists a function  $\varphi \in \mathcal{B}^+$  such that  $\mu \hat{V} = V(\mu) \cdot m = V(\varphi) \cdot m$ . Hence, using (1) and (2) we deduce that  $\mu \ll m$ .

(ii)  $\Rightarrow$  (i). Let  $u \in \mathcal{R} \cap V(\mathbb{M})$ . There exists an  $m$ -exist law for  $\mathbb{P}$ ,  $(f_t)_{t>0} \subset \mathcal{F}$ , and  $\mu \in \mathcal{M}$  such that

$$(9) \quad u \cdot m = \int_0^\infty (f_t \cdot m) dt = \mu \hat{V} = \int_0^\infty (\mu \hat{P}_t) dt.$$

Using again (2) we get  $\mu \hat{P}_t = f_t \cdot m$  for each  $t > 0$ .

Let  $\varphi \in \mathcal{B}^+$  satisfy  $\mu = \varphi \cdot m$ . Then we get  $u \in \mathcal{I}mV$  by (1). □

### 2.2 Mutual energy formula.

**Definition 3** (see [5, XII, 39]). For  $(u, v) \in \mathcal{E} \times \hat{\mathcal{E}}$ , the mutual energy  $E(u, v)$  is defined by

$$(10) \quad E(u, v) := \sup\{m(\varphi \cdot v), \quad \varphi \in \mathcal{B}^+, \quad V\varphi \leq u\}$$

$$(11) \quad = \sup\{m(u \cdot \psi), \quad \psi \in \mathcal{B}^+, \quad \hat{V}\psi \leq v\}.$$

**Remark 3.** Let  $u, u_1 \in \mathcal{E}$  and  $v, v_1 \in \hat{\mathcal{E}}$  be such that  $u = u_1$   $m$ -a.e. and  $v = v_1$   $m$ -a.e. Then, using (10) and (11) we deduce that  $E(u, v) = E(u_1, v_1)$ .

We recall briefly the following properties of  $E$  (cf. [5]).

**Proposition 1.**

- (a)  $E(Vf, v) = \int f(x)v(x)m(dx)$  for each  $f \in \mathcal{B}^+$  such that  $Vf \in \mathcal{E}$  and  $v \in \hat{\mathcal{E}}$ .
- (b) If  $(u_n)_n \subset \mathcal{E} \nearrow u \in \mathcal{E}$  then  $(E(u_n, v))_n \nearrow E(u, v)$  for each  $v \in \hat{\mathcal{E}}$ .
- (c)  $E(\alpha u_1 + \beta u_2, v) = \alpha E(u_1, v) + \beta E(u_2, v)$  for each  $u_1, u_2 \in \mathcal{E}$ ,  $v \in \hat{\mathcal{E}}$  and  $\alpha, \beta \geq 0$ .

**Proposition 2.** For each  $(u, v) \in \mathcal{R} \times \hat{\mathcal{R}}$  we have the following integral representation of the mutual energy:

$$E(u, v) = 2 \int_0^\infty \int_X f_t(x)g_t(x)m(dx) dt,$$

where  $(f_t)_{t>0}$  ( $(g_t)_{t>0}$ ) is an  $m$ -exit law for  $\mathbb{P}$  ( $\hat{\mathbb{P}}$ ) such that  $u = \int_0^\infty f_t dt$   $m$ -a.e. ( $v = \int_0^\infty g_t dt$   $m$ -a.e., respectively).

PROOF: Let  $t > 0$ . Since  $P_t u = V f_t$   $m$ -a.e. and  $\hat{P}_t v = \hat{V} g_t$   $m$ -a.e., we have by Remark 3 and Proposition 1,

$$\begin{aligned} E(P_t u, \hat{P}_t v) &= \int_X f_t(x) \hat{V} g_t(x) m(dx) \\ &= \int_0^\infty \int_X f_t(x) \hat{P}_s g_t(x) m(dx) ds \\ &= \int_0^\infty \int_X P_{\frac{s}{2}} f_t(x) \hat{P}_{\frac{s}{2}} g_t(x) m(dx) ds \\ &= \int_0^\infty \int_X P_s f_t(x) \hat{P}_s g_t(x) m(dx) ds \\ &= 2 \int_t^\infty \int_X f_s(x) g_s(x) m(dx) ds. \end{aligned}$$

Therefore by the monotone convergence theorem and Proposition 1, we have

$$E(u, v) = \sup_{t \rightarrow 0} E(P_t u, \hat{P}_t v) = 2 \int_0^\infty \int_X f_s(x) g_s(x) m(dx) ds.$$

□

**Remark 4.** Suppose that  $\varepsilon_x P_t \ll m$  and  $\varepsilon_x \hat{P}_t \ll m$  for each  $t > 0$ ,  $x \in X$ , and let  $G$  be the unique density of  $V$  and  $\hat{V}$  with respect to  $m$  (see [3, XII,72]). Let  $\mu, \nu \in \mathcal{M}$  be such that

$$G\mu := \int G(\cdot, y)\mu(dy) \in \mathcal{F} \quad \text{and} \quad \nu G := \int G(x, \cdot)\nu(dx) \in \mathcal{F}.$$

Then it is well known that  $E(G\mu, \nu G) = \iint G(x, y)\mu(dy)\nu(dx)$ . But since in general the cone  $\mathcal{R}$  contains strictly the cone  $\{G\mu : \mu \in \mathcal{M}, G\mu \in \mathcal{F}\}$  (see [10, 3.2(2)]) by Proposition 2 we have obtained an integral representation of the mutual energy for a wider class in  $\mathcal{E} \times \hat{\mathcal{E}}$ .

**2.3 Examples.**

**2.3.1 Compound Poisson.**

Let  $c > 0$  and  $M, N$  be two submarkovian kernels on  $(X, \mathcal{B})$  in weak duality with respect to a fixed  $\sigma$ -finite measure  $m$ . Let  $\mathbb{P}^N := (P_t^N)_{t>0}$  be the submarkovian semigroup defined for each  $t > 0$  by

$$P_t^N := e^{ct(N-I)} = e^{-ct} \sum_{k=0}^{\infty} \frac{(ct)^k}{k!} N^k.$$

Note that the triplet  $(\mathbb{P}^N, \mathbb{P}^M, m)$  satisfies (1) and (2). In this case, conditions (ii) of Theorem 2 and Theorem 3 are obviously satisfied. Hence  $\mathcal{I}mV = \mathcal{R} \cap V(\mathbb{M})$  and  $\mathcal{P} = \mathcal{R}$ .

**2.3.2 Semigroups in strong duality.**

Assume that  $\varepsilon_x V \ll m$  and  $\varepsilon_x \hat{V} \ll m$  for each  $x \in X$ . In this case, condition (ii) of Theorem 2 is equivalent to  $\varepsilon_x \hat{P}_t \ll m$  for each  $x \in X$  and  $t > 0$ . Therefore Theorem 2 extends Theorem 3.3 in [10].

**2.3.3 Lattice semigroups.**

Suppose that  $\mathbb{P}$  is a measurable lattice semigroup of kernels on  $(X, \mathcal{B})$ , i.e.

$$P_t|f| = |P_t f|, \quad \text{for } f \in \mathcal{B} \text{ and } t > 0.$$

Then, condition (ii) of Theorem (3) is an obvious consequence of [12, Proposition 2.2]. Hence  $\mathcal{I}mV = \mathcal{R} \cap V(\mathbb{M})$ , and so we find again the result given in [12, Corollary 2.5].

**2.3.4 Nearly symmetric semigroups.**

Assume that  $\mathbb{P}$  and  $\hat{\mathbb{P}}$  satisfy the sector condition (see [4]). Then using [8, Proposition 3.8] and [4, Theorem 5.1] we deduce that condition (ii) of Theorem 2 is valid and, therefore,  $\mathcal{R} = \mathcal{P}$ .

### 2.3.5 Uniform motion on $\mathbb{R}$ .

Let  $X = \mathbb{R}$  and  $\mathbb{P} := (P_t)_{t>0}$  the uniform motion on  $X$  defined by

$$P_t f(x) = f(x + t), \quad t > 0, x \in \mathbb{R} \text{ and } f \in \mathcal{B}^+(\mathbb{R}).$$

Then  $\mathcal{R} \neq \mathcal{P}$  by Theorem 1.

### 3. Decomposition of elements in $\mathcal{P}$

In this section, we suppose that  $X = \mathbb{R}^n (n \geq 1)$  and we denote by  $m$  the Lebesgue measure on  $\mathbb{R}^n$ . Let  $(\mu_t)_{t>0}$  be a convolution semigroup on  $\mathbb{R}^n$  (see [1]) and  $\mathbb{P} := (P_t)_{t>0}$  be the associated semigroup of submarkovian kernels defined by

$$(12) \quad P_t f(x) := \int f(x + y) \mu_t(dy), \quad t > 0, x \in \mathbb{R}^n, f \in \mathcal{B}^+.$$

In the sequel we suppose that  $\int_0^\infty \mu_t(f) dt < \infty$  for each  $f \in C_c^+$ .

If we denote by  $\hat{\mu}_t$  the centrally symmetric image of  $\mu_t$  and by  $\hat{P}_t$  the induced kernel given by formula (12), then  $\mathbb{P} := (P_t)_{t>0}$  and  $\hat{\mathbb{P}} := (\hat{P}_t)_{t>0}$  are in weak duality with respect to  $m$  and  $V, \hat{V}$  satisfy the unicity of charges.

In general  $\mathcal{R} \neq \mathcal{P}$ . So we shall investigate the existence of the largest element of  $\mathcal{R}$  in the strong  $m$ -domination order which is strongly  $m$ -dominated by a given function  $u \in \mathcal{P}$ .

**Definition 4.** A function  $v \in \mathcal{E}$  strongly  $m$ -dominates a function  $v \in \mathcal{E}$  and we write  $u \prec v$  if there exists  $w \in \mathcal{E}$  with  $v = u + w$   $m$ -a.e.

To prove Theorem 3 we need the following

**Lemma 1.** Let  $\mu \in \mathcal{M}$  and  $A_\mu := \{\nu \in \mathcal{M} : \nu \leq \mu \text{ and } \nu \hat{P}_t \ll m \forall t > 0\}$ . Put  $\nu_0 := \sup A_\mu$ . Then we have  $\nu_0 \in A_\mu$ .

PROOF: Let  $\mu \in \mathcal{M}$ .

(i) If  $\nu, \nu' \in A_\mu$  then since  $\sup(\nu, \nu') \leq \nu + \nu'$ , we get  $\sup(\nu, \nu') \in A_\mu$ .

(ii) If  $B \in \mathcal{B}$  is such that  $\mu(B) < \infty$ , then  $\sup_{\nu \in A_\mu} \nu(B) \leq \mu(B) < \infty$ .

Hence there exists a sequences  $(\nu_k) \subset A_\mu$  such that

$$(13) \quad \lim_{k \rightarrow \infty} \nu_k(B) = \sup_{\nu \in A_\mu} \nu(B).$$

Using (i) we can assume that  $(\nu_k)$  is nondecreasing in (13). Let  $\nu_\infty := \lim_{k \rightarrow \infty} \nu_k$ .

Then we have  $\nu_\infty \hat{P}_t \ll m$  for any  $t > 0$  and for each  $\nu \in A_\mu, 1_B \nu \leq 1_B \nu_\infty$ .



(iii) Write  $\mathbb{R}^n = \bigcup_{p=0}^{\infty} B_p$  with  $(B_p)_p \nearrow$  in  $\mathcal{B}$ ,  $p \rightarrow \infty$ , and  $\mu(B_p) < \infty$  for each  $p \in \mathbb{N}$ . Then using (ii), for each  $p \in \mathbb{N}$  there exists a sequence  $(\nu_{p,k})$  in  $A_\mu$  such that  $(\nu_{p,k})_k \nearrow (k \rightarrow \infty)$  and

$$\lim_{k \rightarrow \infty} \nu_{p,k}(B_p) = \sup_{\nu \in A_\mu} \nu(B_p).$$

Therefore  $\nu_0 = \lim_{k \rightarrow \infty} \nu_{k,k}$  and  $\nu_0 \in A_\mu$  by (ii).  $\square$

PROOF OF THEOREM 3: Let  $u \in \mathcal{P}$  and  $\mu \in \mathcal{M}$  be the unique measure (see [1, Theorem 16.7]) such that

$$(14) \quad u \cdot m = \mu \hat{V}.$$

Let  $A_\mu := \{\nu \in \mathcal{M} : \nu \leq \mu \text{ and } \nu \hat{P}_t \ll m \forall t > 0\}$  and  $\nu_0 := \sup A_\mu$ . Then it follows from Lemma 1 that  $\nu_0 \in A_\mu$ . Now consider  $(f_t)_{t>0} \subset \mathcal{F}$  such that

$$(15) \quad \nu_0 \hat{P}_t = f_t \cdot m \text{ for each } t > 0.$$

By (1) and (4) one can check that  $(f_t)_{t>0}$  is an  $m$ -exit law with respect to  $\mathbb{P}$ . Therefore, by integrating (15) and using Remark 1 there exists a function  $r(u) \in \mathcal{R}$  such that  $r(u) \cdot m = \nu_0 \hat{V}$ . Now since  $\nu_0 \leq \mu$ , it follows from (14) that  $r(u) \prec u$ . Finally, let  $v \in \mathcal{R}$  satisfy  $v \prec u$ . Then (see [1, Theorem 16.7]) there exist  $\sigma \in \mathcal{M}$  such that  $v \cdot m = \sigma \hat{V}$ . Using (2) we get  $\sigma \leq \nu_0$  and, hence,  $v \prec r(u)$ .  $\square$

**Acknowledgments.** I thank Professor M. Hmissi for useful discussions. I also thank the referee for his careful reading of the paper.

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(Received February 8, 2000, revised May 11, 2001)