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ON THE PHASE RECONSTRUCTION AND ANALYTICITY CONDITION

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1. Introduction

In some branches of physics (optics, quantum dispersion theory, communication theory) there occurs a problem which may be mathematically formulated as follows:

Let $g(\nu)$ be a physically important function of a real variable ν. Suppose $g(\nu)$ is not directly accesible to measurement, but the modulus of its Fourier transformation $\gamma(x)$ is measurable (x is a real variable). As $\gamma(x)$ is generally complex function $\gamma(x) = |\gamma(x)|\, e^{i\Phi(x)}$ it will be possible to obtain $g(\nu)$ as the inverse Fourier transformation of $\gamma(x)$ if we construct the phase $\Phi(x)$ of the function $\gamma(x)$.

The solving of this problem is trivial in the case that $g(\nu)$ is real and symmetric with respect to a fixed point ν_0, i.e. $g(\nu_0 + \nu) = g(\nu_0 - \nu)$. In this case

$$\gamma(x) = \int_{-\infty}^{+\infty} g(\nu)\, e^{i2\pi\nu x}\, d\nu = e^{i2\pi x\nu_0} \int_{-\Delta\nu}^{+\Delta\nu} g(\mu + \nu_0)\, e^{i2\pi\mu x}\, d\mu = e^{i2\pi x\nu_0}\, |\gamma(x)|, \quad (1)$$

where $2\Delta\nu$ is the width of the function $g(\nu)$. We see that in this case the phase of the function $\gamma(x)$ is $\Phi(x) = 2\pi x\nu_0$.

In the most of physically important problems $g(\nu)$ is an one-side function, i.e. we may write

$$g(\nu) \equiv 0 \qquad \nu < 0 \quad (2)$$

or a finite function, i.e.

$$g(\nu) \equiv 0 \qquad |\nu| > a \quad (3)$$

where a is a real number.

This conditions allowed us to continue the function $\gamma(x)$ analytically over the upper half complex plane eventually over the whole complex plane; then we can deduce (on the base of Cauchy integral) the relations between the real and imaginary part of the function $\gamma(x)$ (Hilbert transformations, dispersion relations). As it is necessary to obtain a relation between $|\gamma(x)|$ and phase $\Phi(x)$ for the solving of the up formulated problem, it is usual to apply the dispersion relations on the function $\ln \gamma(x) = \ln |\gamma(x)| + i\Phi(x)$ [9].

Let us quote some authors who worked in this region. J. S. Toll [1] has studied the question of a connection between causality and the dispersion relations (his work is attended more on the quantum theory of dispersion). E. Wolf [2] formulated this problem in optics as a problem of determination of the energetic spectrum from the measurement of the degree of coherence (i.e. the visibility of the interference patterns). Wolf showed that this problem is uniquely soluable in the case that $\gamma(z)$ has no complex zeros in the upper half plane (it is the case of the blackbody radiation [3]). The important step in the solving of this problem with respect to the possibility of complex zeros in the region of analyticity was given by introducing the most general form of unimodular analytical signal [4]. The problem of analyticity from the standpoint of Fourier formulation of the optical imaging theory with respect to the condition (3) is considered by O'Neil and Walther [5], [6] and the relation between the amplitude and phase effect of an optical system on the image of an object structure with respect to (3) was dealt with in [7]. The complex work on this problem was given by P. Roman and A. S. Marathay [8] where is this problem moreover transferred on a some nonlinear eigenvalue problem.

In this work we shall give another way of deducing the relation between $|\gamma(x)|$ and $\Phi(x)$ based on the solving of a some singular integral equation of the Cauchy type by the method given by N. I. Muschelišvili [10]. Then we shall deduce the general formula for the spectrum of the function $\gamma(x)$ [i.e. for $g(\nu)$] with the use of the general unimodular analytical signal which allowed us to expand $\gamma(z)$ as the product of a function $\gamma_0(z)$ which has no zeros in the region of analyticity and so called factors of Blaschke containing zero points. On the base of this formula we shall show that the requirement of real spectrum $g(\nu)$ leads to the symmetrical distribution of zero points with respect to the imaginary axis (it was shown by another way in [8]). At the end of this work we shall take interest in some connections between the unicity of the solution of this problem and moments of spectrum.

2. Reconstruction of the phase in the case that $\gamma(z)$ has no complex zeros

Let us suppose that the condition (2) is valid and that $|\gamma(z)|$ tends to zero at least as $|z|^{-1}$ for $|z| \to \infty$. Let moreover the function $\gamma(z)$ is quadraticly integrable on the real axis (this requirement is ensured by the finitness of energy, i.e. $\int\limits_0^\infty [g(\nu)]^2 \, \mathrm{d}\nu < \infty$ and by the equation $\int\limits_0^\infty [g(\nu)]^2 \, \mathrm{d}\nu = \int\limits_{-\infty}^{+\infty} |\gamma(x)|^2 \, \mathrm{d}x$). Then for the real and imaginary part of the function $\gamma(x)$ we may write the Hilbert transformations [11]

$$\operatorname{Re} \gamma(x) = \frac{1}{\pi} P \int\limits_{-\infty}^{+\infty} \frac{\operatorname{Im} \gamma(x')}{x' - x} \, \mathrm{d}x', \tag{4}$$

$$\operatorname{Im} \gamma(x) = -\frac{1}{\pi} P \int\limits_{-\infty}^{+\infty} \frac{\operatorname{Re} \gamma(x')}{x' - x} \, \mathrm{d}x'. \tag{5}$$

On the base of this relations we can get the relations between modulus $|\gamma(x)|$ and the phase $\Phi(x)$ of the function

$$\gamma(x) = |\gamma(x)|\, e^{i\Phi(x)}. \tag{6}$$

It is valid that

$$|\gamma(x)| = \sqrt{[\mathrm{Re}\,\gamma(x)]^2 + [\mathrm{Im}\,\gamma(x)]^2} \tag{7}$$

and

$$\mathrm{tg}\,\Phi(x) = \frac{\mathrm{Im}\,\gamma(x)}{\mathrm{Re}\,\gamma(x)}\,. \tag{8}$$

Substituting from (8) to (4) we obtain

$$\frac{\mathrm{Im}\,\gamma(x)}{\mathrm{tg}\,\Phi(x)} = \frac{1}{\pi} P \int\limits_{-\infty}^{+\infty} \frac{\mathrm{Im}\,\gamma(x')}{x' - x}\,\mathrm{d}x'. \tag{9}$$

This is a singular integral equation of the Cauchy type for the function $f(x) = \mathrm{Im}\,\gamma(x)$. We shall solve this equation with the aid of the method given by Muschelišvili [10]. We shall suppose all functions are satysfying the Lipschitz condition. First, we shall deduce the auxiliary formulas of Sochotzki—Plemelj:

Let us consider the function $\psi(z)$ analytical in the upper half plane including the real axis, which may be represented in the integral form

$$\psi(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{f(x')}{x' - z}\,\mathrm{d}x'. \tag{10}$$

Using the symbolical identity

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{1}{x \mp i\varepsilon} = P\left(\frac{1}{x}\right) \pm \pi i \delta(x), \tag{11}$$

where P denotes the principal value of Cauchy and $\delta(x)$ is Dirac's delta function, we obtain from (10) Sochotzki—Plemelj formulas

$$\psi^+(x) - \psi^-(x) = f(x), \tag{12}$$

$$\psi^+(x) + \psi^-(x) = \frac{1}{\pi i} P \int\limits_{-\infty}^{\infty} \frac{f(x')}{x' - x}\,\mathrm{d}x'. \tag{13}$$

where $\psi^+(x)$ and $\psi^-(x)$ are the boundary values of the function $\psi(z)$ on the real axis from the upper and lower half plane respectively.*)

Let us return now to the equation (9). We shall denote

$$A(x) = \frac{i}{\mathrm{tg}\,\Phi(x)}$$

*) If $f(x)$ satisfies the Lipschitz condition, $\psi(z)$ may be continually extended on the real axis from the upper and lower half plane.

so that we may write (9) in the form

$$A(x) f(x) + \frac{P}{\pi i} \int\limits_{-\infty}^{+\infty} \frac{f(x')}{x' - x} \, \mathrm{d}x' = 0 \,. \tag{14}$$

Substituting from (12) and (13) we obtain

$$A(x) \{\psi^+(x) - \psi^-(x)\} + \{\psi^+(x) + \psi^-(x)\} = 0. \tag{15}$$

Hence, if $\gamma(z)$ has no zeros in the upper half plane*) we get

$$[\ln \psi(x)]^+ - [\ln \psi(x)]^- = \ln \frac{A(x) - 1}{A(x) + 1} = i\, 2\Phi(x). \tag{16}$$

This equation will be satisfied with the analytical function

$$\ln \psi(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{i\, 2\Phi(x')}{x' - z} \, \mathrm{d}x' = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{\Phi(x')}{x' - z} \, \mathrm{d}x', \tag{17}$$

i.e.

$$\psi(z) = \mathrm{e}^{\frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{\Phi(x')}{x'-z} \mathrm{d}x'} \,. \tag{18}$$

From (12) with respect to (11) we have

$$f(x) = \psi^+(x) - \psi^-(x) = 2i \sin \Phi(x)\, \mathrm{e}^{\frac{1}{\pi} P \int\limits_{-\infty}^{+\infty} \frac{\Phi(x')}{x'-x} \mathrm{d}x'} \,. \tag{19}$$

With regard to the fact that we are solving the homogeneous equation, a solution (19) is determined apart from the constant. We shall choose this constant so that for $\Phi = \dfrac{\pi}{2}$ it would be $f(x) = 1$. Hence

$$f(x) = \mathrm{Im}\, \gamma(x) = \sin \Phi(x)\, \mathrm{e}^{\frac{1}{\pi} P \int\limits_{-\infty}^{+\infty} \frac{\Phi(x')}{x'-x} \mathrm{d}x'} \tag{20}$$

and from (8)

$$\mathrm{Re}\, \gamma(x) = \cos \Phi(x)\, \mathrm{e}^{\frac{1}{\pi} P \int\limits_{-\infty}^{+\infty} \frac{\Phi(x')}{x'-x} \mathrm{d}x'} \,. \tag{21}$$

With respect to (7) we have finally

$$|\, \gamma(x) \,| = \mathrm{e}^{\frac{1}{\pi} P \int\limits_{-\infty}^{+\infty} \frac{\Phi(x')}{x'-x} \mathrm{d}x'} \,. \tag{22}$$

*) In this case the function $\ln \dfrac{A(x) - 1}{A(x) + 1}$ is not changed after the circulation around the conture and hence it is unambiguous

This relation allowed us to compute the amplitude of the function $\gamma(x)$ when the phase $\Phi(x)$ is known. For the solving of the above formulated problem it is necessary to have the relation which permits to compute the phase when the amplitude is known. Such expression may be obtained by the inversion of (22). On the base of the Sochotzki—Plemelj formulas the validity of this relation may be proved: If for the functions $G(x)$ and $H(x)$ satisfying the Lipschitz condition holds

$$G(x) = \frac{1}{\pi i} P \int_{-\infty}^{+\infty} \frac{H(x')}{x' - x} \mathrm{d}x' \tag{23}$$

then

$$H(x) = \frac{1}{\pi i} P \int_{-\infty}^{+\infty} \frac{G(x')}{x' - x} \mathrm{d}x' \tag{24}$$

On the base of these relations it follows from (22)

$$\frac{1}{i} \ln |\gamma(x)| = \frac{1}{\pi i} P \int_{-\infty}^{+\infty} \frac{\Phi(x')}{x' - x} \mathrm{d}x' \tag{25}$$

and finally

$$\Phi(x) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\ln |\gamma(x')|}{x' - x} \mathrm{d}x' . \tag{26}$$

This is the required expression for the determination of the phase when the modulus is known.

3. The phase reconstruction in the case of existence of complex zeros of the function $\gamma(z)$

The formula (26) gives us the phase only in the case that the function $\gamma(z)$ has no zeros in the upper half plane, because, if some zeros occured, the function $\ln |\gamma(z)|$ would have the singularities and (26) would not be valid. It was shown in [4], that the most general function regular in the upper half plane and unimodular on the real axis can be represented by

$$A_n(z) = \mathrm{e}^{i\,2\pi c z} \prod_{k=1}^{n} B_k(z) \tag{27}$$

where c is a real nonnegative constant, $B_k(z)$ is the Blaschke-factor defined by

$$B_k(z) = \frac{z - z_k}{z - z_k^*} \tag{28}$$

where z_k is an arbitrary point in the upper half of the complex z-plane. On the base of (27) we can express the general function $\gamma(z)$ having zeros in the points

z_k of the upper half plane as the product of a function $\gamma_0(z)$ which has no zeros and for which the results of the preceeding section are valid, and the function $A_n(z)$ which is determined by the positions of zeros, i.e.

$$\gamma(z) = \gamma_0(z)\, e^{i\,2\pi cz} \prod_{k=1}^{n} \frac{z - z_k}{z - z_k^*}\,. \tag{29}$$

On the real axis it will hold

$$|\,\gamma(x)\,| = |\,\gamma_0(x)\,|, \tag{30}$$

but the phase of the function $\gamma(x)$ will be given

$$\Phi(x) = \Phi_0(x) + \sum_{k=1}^{n} \arg \frac{x - z_k}{x - z_k^*} + 2\pi cx \tag{31}$$

where $\Phi_0(x)$ is so called minimal phase determined by the expression (26). Therefore we may also write

$$\Phi(x) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\ln |\,\gamma(x')\,|}{x' - x}\, dx' + \sum_{k=1}^{n} \arg \frac{x - z_k}{x - z_k^*} + 2\pi cx\,. \tag{32}$$

The last term on the right causes the shifting of the whole spectrum $g(\nu)$ on the constant value c; hence it does not affect the spectral profile and may be dropped from our considerations. The second term on the right of (32) shows us that our task will be uniquely soluable only in the case that we shall know the positions of all the zero points z_k of the function $\gamma(z)$. As $\gamma(z)$ is the function which we want to réconstruct, we cannot know a priori the positions of its zeros. On the other hand we shall show in the next that some restrictions on the zeros 'distribution may be derived on the basis of some physical assumptions.

4. The influence of some physical conditions on the positions of zeros of $\gamma(z)$

We shall study in this part, how the distribution of zeros will be affected by the natural physical assumptions that the spectrum $g(\nu)$ is real and nonnegative. This question was investigated by another way in [8].

The condition that the spectrum is real

$$g(\nu) = g^*(\nu) \tag{33}$$

may be formulated with the aid of the function $\gamma(x)$ as the relation of crossing-symmetry

$$\gamma(x) = \gamma^*(-x) \tag{34}$$

which express the function $\gamma(x)$ for negative values of the argument with the aid of the positive values.

To determine, how will this conditions affect on the spectrum $g(\nu)$ let us write (the exponential term is dropped)

$$\gamma(x) = \gamma_0(x) \prod_{k=1}^{n} \frac{x - z_k}{x - z_k^*} \tag{35}$$

and for the Fourier transformation of this function we have

$$g(\nu) = \int_{-\infty}^{+\infty} \gamma_0(x) \prod_{k=1}^{n} \frac{x - z_k}{x - z_k^*} \, e^{-i2\pi\nu x} \, dx. \tag{36}$$

If we write

$$\gamma_0(x) = \int_{0}^{+\infty} g_0(\nu) \, e^{i2\pi\nu x} \, d\nu \tag{37}$$

we obtain from (36)

$$g(\nu) = \int_{0}^{+\infty} g_0(\mu) \, d\mu \int_{-\infty}^{+\infty} \prod_{k=1}^{n} \frac{x - z_k}{x - z_k^*} \, e^{i2\pi x(\mu-\nu)} dx. \tag{38}$$

Now it will be necessary to compute the integral according to x. This integral (denoted J) we shall compute with the use of the Cauchy theorem. Let us write

$$\frac{x - z_k}{x - z_k^*} = 1 + \frac{z_k^* - z_k}{x - z_k^*} = 1 + \alpha_k(x), \tag{39}$$

where

$$\alpha_k(x) = \frac{z_k^* - z_k}{x - z_k^*}. \tag{40}$$

Then

$$\prod_{k=1}^{n} \frac{x - z_k}{x - z_k^*} = \prod_{k=1}^{n} [1 + \alpha_k(x)] = 1 + \frac{1}{1!} \sum_{j=1}^{n} \alpha_j(x) +$$

$$+ \frac{1}{2!} \sum^{n}\sum^{n}_{j \neq l} \alpha_j(x) \, \alpha_l(x) + \ldots + \frac{1}{n!} \underbrace{\sum^{n}\sum^{n} \ldots \sum^{n}}_{\substack{j \neq l \neq \ldots \neq s \\ n}} \underbrace{\alpha_j(x) \, \alpha_l(x) \, \ldots \, \alpha_s(x)}_{n} \tag{41}$$

We shall suppose firstly that $\mu > \nu$. With this assumption it is necessary to enclose the integration conture over the upper half plane; of course here the function under the integral sign has no singularities and therefore $J = 0$ in this case.

Let us consider now the case $\mu \leq \nu$. From (41) we have

$$J = \int_{-\infty}^{+\infty} \prod_{k=1}^{n} [1 + \alpha_k(x)]\, e^{i 2\pi x(\mu-\nu)}\, dx =$$

$$= \delta(\mu - \nu) + \int_{-\infty}^{+\infty} \Bigg[\sum_{j=1}^{n} \alpha_j(x) + \frac{1}{2!} \sum_{j \neq l}^{n} \sum^{n} \alpha_j(x)\, \alpha_l(x) + \dots$$

$$\dots + \frac{1}{n!} \underbrace{\sum^{n} \sum^{n} \dots \sum^{n}}_{\substack{j \neq l \neq \dots \neq s \\ n}} \underbrace{\alpha_j(x)\, \alpha_l(x) \dots \alpha_s(x)}_{n} \Bigg] e^{i 2\pi x(\mu-\nu)}\, dx. \tag{42}$$

The integration conture must be enclosed over the lower half plane now and with the use of the residual theorem we obtain

$$J = \delta(\mu - \nu) - 2\pi i \sum_{j=1}^{n} (z_j^* - z_j)\, e^{i 2\pi z_j^*(\mu-\nu)} -$$

$$- \frac{1}{2!} 2\pi i \sum_{j \neq l}^{n} \sum^{n} (z_j^* - z_j)\, \alpha_l(z_j^*)\, e^{i 2\pi z_j^*(\mu-\nu)} -$$

$$- \frac{1}{2!} 2\pi i \sum_{j \neq l}^{n} \sum^{n} \alpha_j(z_l^*)\, (z_l^* - z_l)\, e^{i 2\pi z_l^*(\mu-\nu)} - \dots$$

$$\dots - \frac{1}{n!} 2\pi i \underbrace{\sum^{n} \sum^{n} \dots \sum^{n}}_{\substack{j \neq l \neq \dots \neq s \\ n}} \underbrace{(z_j^* - z_j)\, \alpha_l(z_j^*) \dots \alpha_s(z_j^*)}_{n}\, e^{i 2\pi z_j^*(\mu-\nu)} - \dots$$

$$\dots - \frac{1}{n!} 2\pi i \underbrace{\sum^{n} \sum^{n} \dots \sum^{n}}_{\substack{j \neq l \neq \dots \neq s \\ n}} \underbrace{\alpha_j(z_s^*) \dots (z_s^* - z_s)}_{n}\, e^{i 2\pi z_s^*(\mu-\nu)}. \tag{43}$$

Hence

$$J = \delta(\mu - \nu) - 2\pi i \sum_{j=1}^{n} (z_j^* - z_j)\, e^{i 2\pi z_j^*(\mu-\nu)} \Bigg\{ 1 + \frac{1}{1!} \sum_{l \neq j}^{n} \alpha_l(z_j^*) + \dots$$

$$\dots + \frac{1}{(n-1)!} \sum^{n}_{\substack{l \neq \dots \neq s \\ l, \dots, s \neq j}} \dots \sum^{n} \underbrace{\alpha_l(z_j^*) \dots \alpha_s(z_j^*)}_{n-1} \Bigg\} =$$

$$= \delta(\mu - \nu) - 2\pi i \sum_{j=1}^{n} (z_j^* - z_j)\, e^{i 2\pi z_j^*(\mu-\nu)} \prod_{k \neq j}^{n} [1 + \alpha_k(z_j^*)] \tag{44}$$

and we have the result

$$J = \delta(\mu - \nu) - 2\pi i \sum_{j=1}^{n} (z_j^* - z_j)\, e^{i2\pi z_j^*(\mu-\nu)} \prod_{k \neq j}^{n} \frac{z_j^* - z_k}{z_j^* - z_k^*}. \tag{45}$$

The substitution (45) into (38) gives us

$$g(\nu) = g_0(\nu) - 2\pi i \sum_{j=1}^{n} (z_j^* - z_j)\, e^{-i2\pi z_j^* \nu} \prod_{k \neq j}^{n} \frac{z_j^* - z_k}{z_j^* - z_k^*} \int_0^{\nu} g_0(\mu)\, e^{i2\pi z_j^* \mu}\, d\mu. \tag{46}$$

The formulas (45) and (46) are the generalisation of the formulas (2.5) and (2.6) from [8]. It is obvious that according to the original assumption we have again $g(\nu) = 0$ for $\nu < 0$.

For the next considerations let us denote

$$S\{z_j\} = i(z_j^* - z_j) \prod_{k \neq j}^{n} \frac{z_j^* - z_k}{z_j^* - z_k^*} = A_j + iB_j, \qquad (j = 1, 2, \ldots n) \tag{47}$$

where A_j, B_j are real. From (47) it may be verified that as long as for every j there exists $l \neq j$ so that $z_j = -z_l^*$ then

$$S\{z_j\} = S^*\{z_l\} \qquad (j \neq l) \tag{48}$$

is valid, i.e.

$$A_j = A_l, \qquad B_j = -B_l \qquad (j \neq l). \tag{49}$$

The condition (33) gives us with the use of (46) and (47)

$$\sum_{j=1}^{n} S\{z_j\}\, e^{-i2\pi z_j^* \nu} \int_0^{\nu} g_0(\mu)\, e^{i2\pi z_j^* \mu}\, d\mu = \sum_{j=1}^{n} S^*\{z_j\}\, e^{2\pi i z_j \nu} \int_0^{\nu} g_0^*(\mu)\, e^{-i2\pi z_j \mu}\, d\mu. \tag{50}$$

According to (48) it is obvious that this identity will be fulfilled if for every j on the left there exists l on the right so that $z_j = -z_l^*$. It may be noted that we shall gain the same result by putting $\operatorname{Im} g(\nu) = 0$ directly from (46). If we write $z_j = a_j + ib_j$ (a_j, b_j real) it must hold

$$\operatorname{Im} \left\{ \sum_{j=1}^{n} (A_j + iB_j)\, e^{-2\pi\nu b_j} (\cos 2\pi\nu a_j - i \sin 2\pi\nu a_j) \int_0^{\nu} g_0(\mu)\, e^{2\pi\mu b_j} (\cos 2\pi\mu a_j + \right.$$

$$\left. + i \sin 2\pi\mu a_j)\, d\mu \right\} = 0 \tag{51}$$

and consequently

$$\sum_{j=1}^{n} e^{-2\pi\nu b_j} \left\{ (A_j \cos 2\pi\nu a_j + B_j \sin 2\pi\nu a_j) \int_0^{\nu} g_0(\mu)\, e^{2\pi\mu b_j} \sin 2\pi\mu a_j\, d\mu + \right.$$

$$\left. + (B_j \cos 2\pi\nu a_j - A_j \sin 2\pi\nu a_j) \int_0^{\nu} g_0(\mu)\, e^{2\pi\mu b_j} \cos 2\pi\mu a_j\, d\mu \right\} = 0. \tag{52}$$

It is obvious according to (49) that for every j there must exist l so that $a_j = -a_l$, $b_j = b_l$, i.e. again $z_j = -z_l^*$. Hence zero points must be distributed symmetricly with respect to the imaginary axis or must lay on this axis. According to representation (29) it means that the poles corresponding to these zeros of the function $\gamma(z)$ in the lower half plane are distributed symmetricly with respect to the imaginary axis, too. Besides this, there may exist other poles $\gamma(z)$ in the lower half plane, we have no information about.

We have seen that the requirement the spectrum is real admitted the existence of zero points in the upper half plane including the imaginary axis. We shall see now that the requirement of nonnegative spectrum $g(\nu) \geqq 0$ excludes zeros on the imaginary axis. Let us suppose that $\gamma(z)$ has the zero in the point $z = ia$ ($a > 0$ is real). Then the Fourier transformation gives us

$$\gamma(ia) = \int\limits_0^\infty g(\nu)\, \mathrm{e}^{-2\pi\nu a}\, \mathrm{d}\nu > 0 \tag{53}$$

and consequently $z = ia$ cannot be the zero point.

The condition that the spectrum is nonnegative leads moreover to some nonlinear eigenvalue problem as it was shown in [8].

5. The unicity of the phase reconstruction problem and moments of spectrum

In quantum theory of decay there occurs a problem of the similar type we have formulated here. Chalfin [12] studied the connection between the first order moment of spectrum and soluability of this problem. We shall try to generalise and apply his methods on our optical case.

Let us consider the function $\gamma(x) = M(x)\, \mathrm{e}^{iN(x)}$ where M, N are real functions. Let it be possible to continue analyticaly this function over the upper half plane, let on the real axis the relation of crossing-symmetry

$$\gamma(x) = \gamma^*(-x) \qquad [M(x) = M(-x), \quad N(x) = -N(-x)] \tag{54}$$

be valid, let $\gamma(0) = 1$, $0 < M(x) \leqq 1$ and moreover let $|\ln \gamma(z)| \underset{|z| \to \infty}{\leqq} A\, |z|^l$, $l < 2n - 1$; $|\ln \gamma(z)| \underset{|z| \to 0}{\leqq} B\, |z|^l$, $l > 2n - 1$, $(A, B > 0)$.

We shall compute the integral

$$J = \frac{1}{2\pi} \int\limits_C \frac{\ln \gamma(z)}{z^{2n}}\, \mathrm{d}z, \qquad (n \geqq 1), \tag{55}$$

where the integration conture is formed by the real axis and a half circle with the center in the origin and with a radius R. With respect to the above formulated assumptions the integral over the half-circle vanishes with $R \to \infty$; with the use of (54) we have

$$J = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\ln \gamma(x)}{x^{2n}} \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\ln M(x)}{x^{2n}} \mathrm{d}x + \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{N(x)}{x^{2n}} \mathrm{d}x =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\ln M(x)}{x^{2n}} \mathrm{d}x = \frac{1}{\pi} \int_{0}^{\infty} \frac{\ln M(x)}{x^{2n}} \mathrm{d}x . \tag{56}$$

With the aid of the integration per partes we obtain

$$J = -\frac{1}{2\pi} \frac{\ln \gamma(z)}{(2n-1)\, z^{2n-1}}\bigg|_{C} + \frac{1}{2\pi(2n-1)} \int_{C} \frac{\gamma'(z)}{\gamma(z)\, z^{2n-1}} \mathrm{d}z. \tag{57}$$

The first term here is equal to $-i \dfrac{n(R)}{(2n-1)\, R^{2n-1}}$ where $n(R)$ is the number of zero points of the function $\gamma(z)$ inside the region determined by C. For $R \to \infty$ this expression vanishes. Using the residual theorem and the theorem on the number of zeros and poles (we suppose the zeros are single) we have from (57) with $R \to \infty$

$$J = \frac{i}{2(2n-1)!} \left[\frac{\gamma'(z)}{\gamma(z)} \right]_0^{(2n-2)} + \frac{i}{2n-1} \sum_r \frac{1}{z_r^{2n-1}}, \tag{58}$$

where the summation is taken over the all zero points z_r of the function $\gamma(z)$ in the upper half plane. According to (56)

$$\frac{i}{2(2n-1)!} \left[\frac{\gamma'(z)}{\gamma(z)} \right]_0^{(2n-2)} + \frac{i}{2n-1} \sum_r \frac{1}{z_r^{2n-1}} = \frac{1}{\pi} \int_0^{\infty} \frac{\ln M(x)}{x^{2n}} \mathrm{d}x . \tag{59}$$

Hence separating the real and imaginary parts we have

$$-\frac{1}{2(2n-1)!} \operatorname{Im} \left[\frac{\gamma'(z)}{\gamma(z)} \right]_0^{(2n-2)} + \frac{1}{2n-1} \sum_r \frac{\operatorname{Im} z_r^{2n-1}}{|z_r|^{2(2n-1)}} = \frac{1}{\pi} \int_0^{\infty} \frac{\ln M(x)}{x^{2n}} \mathrm{d}x \tag{60}$$

and

$$\frac{1}{2(2n-1)!} \operatorname{Re} \left[\frac{\gamma'(z)}{\gamma(z)} \right]_0^{(2n-2)} + \frac{1}{2n-1} \sum_r \frac{\operatorname{Re} z_r^{2n-1}}{|z_r|^{2(2n-1)}} = 0. \tag{61}$$

But it holds

$$\frac{\gamma'(x)}{\gamma(x)} = \frac{M'(x)}{M(x)} + iN'(x). \tag{62}$$

As $\dfrac{M'(x)}{M(x)}$ is an odd, $N'(x)$ an even function, is

$$\operatorname{Re} \left[\frac{\gamma'(z)}{\gamma(z)} \right]_0^{(2n-2)} = \left[\frac{M'(z)}{M(z)} \right]_0^{(2n-2)} = 0, \qquad \operatorname{Im} \left[\frac{\gamma'(z)}{\gamma(z)} \right]_0^{(2n-2)} = N^{(2n-1)}(0). \tag{63}$$

Moreover also $\ln \gamma(0) = 0$. With the use of (63) we get from (60), (61) and (62)

$$-\frac{N^{(2n-1)}(0)}{2(2n-1)!} + \frac{1}{2n-1} \sum_r \frac{\operatorname{Im} z_r^{2n-1}}{|z_r|^{2(2n-1)}} = \frac{1}{\pi} \int_0^\infty \frac{\ln M(x)}{x^{2n}} \, dx, \tag{64}$$

$$\sum_r \frac{\operatorname{Re} z_r^{2n-1}}{|z_r|^{2(2n-1)}} = 0 \tag{65}$$

As $z_k = -z_l^*$, $(k \neq l)$ it holds

$$\operatorname{Re} z_k^{2n-1} = -\operatorname{Re} z_l^{2n-1}, \qquad |z_k| = |z_l| \tag{66}$$

and we see that the condition (65) requires the symmetrical distribution of zeros with respect to the imaginary axis.

Let us compute now the $(2n-1)^{st}$ order moment of spectrum of the function $\ln \gamma(x)$, i.e.

$$\mu_{2n-1} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \nu^{2n-1} \left(\int_{-\infty}^{+\infty} \ln \gamma(x) \, e^{-i\nu x} \, dx \right) d\nu. \tag{67}$$

After the changing of the order of integration we have

$$\mu_{2n-1} = \int_{-\infty}^{+\infty} \ln \gamma(x) \, dx \, \frac{1}{2\pi} \int_{-\infty}^{+\infty} \nu^{2n-1} \, e^{-i\nu x} \, d\nu = i^{2n-1} \int_{-\infty}^{+\infty} \delta^{(2n-1)}(x) \ln \gamma(x) \, dx \tag{68}$$

where $\delta^{(2n-1)}(x)$ is the $(2n-1)^{st}$ order derivative of the Dirac's delta function. The integration per partes of (68) gives us

$$\mu_{2n-1} = \frac{(-1)^{n+1}}{i} \int_{-\infty}^{+\infty} \frac{\gamma'(x)}{\gamma(x)} \, \delta^{(2n-2)}(x) \, dx = \frac{(-1)^{n+1}}{i} \left[\frac{\gamma'(x)}{\gamma(x)} \right]_0^{(2n-2)} \tag{69}$$

With respect to (63)

$$\mu_{2n-1} = (-1)^{n+1} N^{(2n-1)}(0) \tag{70}$$

Instead of (64) we are getting

$$\frac{1}{\pi} \int_0^\infty \frac{\ln M(x)}{x^{2n}} \, dx = (-1)^n \frac{\mu_{2n-1}}{2(2n-1)!} + \frac{1}{2n-1} \sum_r \frac{\operatorname{Im} z_r^{2n-1}}{|z_r|^{2(2n-1)}}. \tag{71}$$

This expression gives us the relation among the behaviour of the modulus of the function at the surroundings of zero and at infinity, the moment of the spectrum of the function $\ln \gamma(x)$ and the number and a distribution of zero points of $\gamma(z)$ in the upper half plane.

If $n = 1$ we have according to (69)

$$\mu_1 = -i\gamma'(0) \tag{72}$$

because $\gamma(0) = 1$ and for the 1^{st} order moment of the spectrum of $\gamma(x)$ it holds

$$\omega_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \nu \left(\int_{-\infty}^{+\infty} \gamma(x) \mathrm{e}^{-i\nu x} \mathrm{d}x \right) \mathrm{d}\nu = \int_{-\infty}^{+\infty} \gamma(x) \mathrm{d}x \, \frac{1}{2\pi} \int_{-\infty}^{+\infty} \nu \, \mathrm{e}^{-i\nu x} \, \mathrm{d}\nu =$$

$$= i \int_{-\infty}^{+\infty} \gamma(x) \, \delta'(x) \, \mathrm{d}\nu = -i\gamma'(0) \tag{73}$$

and consequently $\omega_1 = \mu_1$ and from (71) the relation follows given in [12]

$$\frac{1}{\pi} \int_0^\infty \frac{\ln M(x)}{x^2} \mathrm{d}x = -\frac{\omega_1}{2} + \sum_r \frac{\operatorname{Im} z_r}{|z_r|^2} . \tag{74}$$

From this relation with respect to $0 < M(x) \leqq 1$ and $\operatorname{Im} z_r > 0$ it follows that the minimal value of the moment ω_1 is

$$\omega_1^{\min} = -\frac{2}{\pi} \int_0^\infty \frac{\ln M(x)}{x^2} \mathrm{d}x > 0 \tag{75}$$

and

$$\omega_1 = -\frac{2}{\pi} \int_0^\infty \frac{\ln M(x)}{x^2} \mathrm{d}x + 2 \sum_r \frac{\operatorname{Im} z_r}{|z_r|^2} \geqq \omega_1^{\min} \tag{76}$$

is valid.

If we reconstruct the phase of the function $\gamma(x)$ with the use of the dispersion relation under the assumption that the 1^{st} order moment of the function $\gamma(x)$ will be minimal, the function $\gamma(z)$ will not have zeros in the upper half plane and the reconstruction is unique. The phase of the function $\gamma(x)$ is determined by (26).

The existence of the integral (74) leads to the existence of the moment ω_1 and vice versa.

6. Conclusion

In this paper we have given the general solution of the problem of the phase reconstruction. We have shown that this solution was unique only in the case that the function $\gamma(z)$ has no zeros in the upper half plane. If there are some zeros of $\gamma(z)$ the resulting phase depends on the positions of these zeros. We have studied how the natural physical condition of real and nonnegative spectrum leads to the requirement of a symmetrical distribution of zeros according to the imaginary axis and nonexistence zeros on the imaginary axis. It seems to be true that it will not be possible to determine the positions of zeros without any other physical information. It is possible that these information might be given by measurements of moments of spectrum. This question was not yet investigated in detail.

On the other hand there exists another way for finding some additional restrictions on the function $\gamma(z)$ in the optical case [when $\gamma(x)$ is the optical autocorrelation function] from some knowledge of the statistical fluctuations in the beam as it was suggested by Wolf and Mandel [2], [13] in a connection with the analysis of the experimental results [14]. From the knowledge of the radiation mechanism of the light source it would be possible to gain the information on zeros [for example Kano and Wolf have proved [3] that for the blackbody radiation the function $\gamma(z)$ has no zeros in the upper half plane].

There exists still one possibility of solving this problem. As it was mentioned at the end of the part 4, this problem may be transferred on some nonlinear eigenvalue problem. The solution of this mathematical task leads then to defining some regions in the upper half plane in which no zeros can occure.

At present it cannot be decided which of the methods given above will be more succesful. In every case it is obvious that this is the problem, a definite solution of which would have certain importance not only in optics but in many other branches of physics.

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SHRNUTÍ

REKONSTRUKCE FÁZE A PODMÍNKA ANALYTIČNOSTI

JAN PEŘINA A JOSEF TILLICH

V práci byl studován problém rekonstrukce fáze fyzikálně významné funkce $\gamma(z)$ z její amplitudy na základě analytičnosti této funkce v horní polorovině. Řešení tohoto problému dovoluje určit energetické spektrum z naměřených hodnot kontrastu interferenčních proužků. Bylo ukázáno, že problém je řešitelný jednoznačně jen v případě že funkce $\gamma(z)$ nemá nuly v horní polorovině. Dále byla diskutována otázka vlivu některých fyzikálních podmínek na rozložení nulových bodů a souvislost jednoznačné řešitelnosti problému s požadavky kladenými na momenty spektra.