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ON PERMUTABLE QUADRATIC INVERSIONS IN PLANE WITH TWO COMMON SINGULAR POINTS

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This text refers to the work of S. S. Subramanyam "On Permutable quadratic involutions" [1]. The author deals there with involutary quadratic transformations and conditions under which two or more involutions are permutable. In section 6.2 the author says that two quadratic inversions with two common singular points can be permutable if one of the common points is the centre for both inversions. Consequently any case is excluded where the centres of both inversions are different and the remaining two singular points of one inversion coincide with the remaining two singular points of the second inversion. But, even in such a case the quadratic inversions can be permutable.

§ 1.

There are two types of quadratic involutory transformations (shortly quadratic involutions):

A. A quadratic involution with three distinct singular points in which to each singular point there exists a corresponding principal line not passing throught it. This is a relationship of polar conjugated points with respect to the pencil of conics.

B. A quadratic involution with three distinct singular points in which to one singular point (called the centre) there exists a corresponding principal line not passing through it and to each of the remaining singular points corresponds a principal line passing through it. This is a relationship of points which are polar conjugated with respect to the conic of invariant points and collinear with a centre of involution. This type of quadratic involution is called the quadratic inversion.

Notation: $I(A)$ will be used to denote the transformation of the element A with the aid of the mapping I. The product $I_2 \,.\, I_1$ means to apply the involution I_1 first and I_2 next. $J_{(ABC)}$ implies a quadratic involution with the singular points A, B, C; if a quadratic inversion is in question, the centre is underlined: $I_{(\underline{A}BC)}$.

Theorem 1. A quadratic involution is uniquely determined if its singular points and a pair of self-corresponding points are given. (See [2]. page 616, theorem a) paragraph 257.)

Let us first pay attention to some general properties of the permutable point transformations which will be used for two quadratic inversions later on.

Lemma 1. Let A be an invariant point of the transformation T, and T' a transformation permutable with T for which the point A is not the singular point. Then $T'(A) = B$ will be an invariant point of the transformation T also.

$$\left.\begin{array}{l} T(A) = A \quad TT' = T'T \quad T'(A) = B \quad TT'(A) = T(B) \\ \qquad\qquad T'T(A) = T'(A) = B \end{array}\right\} \begin{array}{l}\text{that is} \\ T(B) = B.\end{array}$$

Lemma 2. The product of two involutions is an involution if, and only if both involutions are permutable with each other. Any two permutable involutions together with their product and the identity form the Klein's four-group.

$$II' = I'I \Leftrightarrow II' = (II')^{-1}.$$

Lemma 3. Let I and I' be two permutable involutions and a point C being singular for neither of them. Then for points $I(C) = A$ and $I'(C) = B$ holds true: $I.I'(A) = I'.I(A) = B$.

Lemma 4. Any transformation permutable with either of the two given transformations will be permutable together with their product regardless of the order.

$$T_1 . T_2 = T_2 . T_1 \quad T_1 . T_3 = T_3 . T_1 \quad T_1 . (T_2 . T_3) = (T_1 . T_2) . T_3 =$$
$$= (T_2 . T_1) . T_3 = T_2 . (T_1 . T_3) = T_2 . (T_3 . T_1) = (T_2 . T_3) . T_1 .$$

§ 3.

There exist two possible cases for a pair of permutable quadratic inversions with two common singular points. In the first case one of the common singular points is the centre of both inversions (see [1] page 177, par. 6.2). In the second case just the centres of both inversions are different and the remaining two singular points are for both inversions common. A pair of permutable quadratic inversions with a common centre and with one more singular common point is called position I, and a pair of permutable quadratic inversions with two common singular points, neither being the centre of either inversion, is called position II.

Theorem 2. Two quadratic inversions $I_{1(AB_1C)}$ and $I_{2(AB_2C)}$ are in position I if and only if the third singular point of one inversion is the invariant point of the other inversion.

Proof:

1. Let $I_1I_2 = I_2I_1$, $I_2I_1(B_1) = I_2(B_1C) = B_1C$. Thus there must be $I_1I_2(B_1) = B_1C$, that is $I_2(B_1) = I_1(B_1C) = B_1$. Analogous $I_1(B_2) = B_2$.
2. Let $C = (1, 0, 0)$, $A = (0, 0, 1)$, $B_1 = (0, 1, 0)$, $B_2 = (m_1, m_2, m_3)$; $I_2(B_1) = B_1$, $I_1(B_2) = B_2 \quad m_1m_2m_3 \neq 0$.

Then $I_{1(AB_1C)}$ will be expressed by:

$$\begin{aligned} \varrho x_1 &= m_1^2 x_2' x_3' \\ \varrho x_2 &= m_2 m_3 x_1' x_2' \\ \varrho x_3 &= m_2 m_3 x_1' x_3' \end{aligned} \tag{1}$$

and $I_{2(AB_2C)}$ by:

$$\begin{aligned} \varrho x_1 &= m_1 x_2'(m_3 x_1' - m_1 x_3') \\ \varrho x_2 &= m_3 x_2'(m_2 x_1' - m_1 x_2') \\ \varrho x_3 &= m_3 x_3'(m_2 x_1' - m_1 x_2') \end{aligned} \tag{2}$$

Combining $I_2 I_1$ and $I_1 I_2$ gives in both cases the same result, namely a quadratic inversion $I_{3(B_1B_2C)}$ which can be represented by:

$$\begin{aligned} \varrho x_1 &= m_1 x_3'(m_2 x_1' - m_1 x_2') \\ \varrho x_2 &= m_2 x_2'(m_3 x_1' - m_1 x_3') \\ \varrho x_3 &= m_3 x_3'(m_3 x_1' - m_1 x_3') \end{aligned} \tag{3}$$

All the three inversions I_1, I_2, I_3 and the identity E form the Klein's four-group.

Theorem 3. Two quadratic inversions $I_{1(ABC_1)}$ and $I_{2(ABC_2)}$ are in the position II if and only if $I_1(C_2) = I_2(C_1) = D_3$ holds true. The product of both inversions gives the quadratic involution $J_{3(ABD_3)}$.

Proof:

1. If P is an arbitrary point of the line AB, different from poits A, B, then $I_2 I_1(P) = I_2(C_1)$ and $I_1 I_2(P) = I_1(C_2)$ hence, if $I_1 I_2 = I_2 I_1$ then $I_1(C_2) = I_2(C_1)$.
2. If for any given inversion I_1 the inversion I_2 changes $I_1(C_2)$ and C_1, then $I_2 I_1$ and $I_1 I_2$ are quadratic involutions with the same singular points A, B, D_3 possessing the points C_1 and C_2 as a self-corresponding pair; consequently $I_2 I_1 = I_1 I_2 = J_{3(ABD_3)}$. (Theorem 1.)

Let $A = (0, 0, 1)$, $B = (0, 1, 0)$, $C_1 = (1, 0, 0)$, $C_2 = (m_1, m_2, m_3)$, then the inversion I_1 can be expressed by:

$$\begin{aligned} \varrho x_1 &= a x_2' x_3' \\ \varrho x_2 &= x_1' x_2' \qquad a \neq 0 \\ \varrho x_3 &= x_1' x_3' \end{aligned} \tag{4}$$

In the inversion $I_{2(ABC_2)}$ the point $B = (0, 1, 0)$ must correspond to the line BC_2 having the equation $m_3 x_1' - m_1 x_3' = 0$ and the point $A = (0, 0, 1)$ must correspond to the line AC_2 having the equation $m_2 x_1' - m_1 x_2' = 0$ that is, it holds:

$$\begin{aligned} \varrho x_1 &= b(m_2 x_1' - m_1 x_2')(m_3 x_1' - m_1 x_3'), \\ \varrho x_2 &= (m_2 x_1' - m_1 x_2')(a_1 x_1' - a_2 x_3'), \\ \varrho x_3 &= (m_3 x_1' - m_1 x_3')(b_1 x_1' - b_2 x_2'). \end{aligned} \tag{5}$$

This is a quadratic transformation if it holds $bm_1 \neq 0$, $a_1 m_1 - a_2 m_3 \neq 0$, $b_1 m_1 - b_2 m_2 \neq 0$. To the line AB corresponds the point $C_2 = (m_1, m_2, m_3)$, that is:

$$\begin{aligned} \varrho m_1 &= bm_1^2 x_2' x_3', & bm_1 \neq 0 \text{ that is } a_2 &= bm_2, \\ \varrho m_2 &= a_2 m_1 x_2' x_3, & b_2 &= bm_3. \\ \varrho m_3 &= b_2 m_1 x_2' x_3', \end{aligned}$$

If we wish to obtain an involution, it must hold furthermore $a_1 = b_1$. Hence, the inversion I_2 is given by equations:

$$\begin{aligned} \varrho x_1 &= b(m_2 x_1' - m_1 x_2')(m_3 x_1' - m_1 x_3'), \\ \varrho x_2 &= (m_2 x_1' - m_1 x_2')(a_1 x_1' - bm_2 x_3'), \\ \varrho x_3 &= (m_3 x_1' - m_1 x_3')(a_1 x_1' - bm_3 x_2'), \end{aligned} \tag{6}$$

$$\begin{aligned} I_1(C_2) &= (am_2 m_3, m_1 m_2, m_1 m_3), \\ I_2(C_1) &= (bm_2 m_3, a_1 m_2, a_1 m_3), \end{aligned} \qquad I_1(C_2) = I_2(C_1) \Rightarrow \begin{cases} b = a, \\ a_1 = m_1. \end{cases}$$

Thus, the inversion I_2 will be expressed as follows:

$$\begin{aligned} \varrho x_1 &= a(m_2 x_1' - m_1 x_2')(m_3 x_1' - m_1 x_3'), \\ \varrho x_2 &= (m_2 x_1' - m_1 x_2')(m_1 x_1' - am_2 x_3'), \qquad am_1 \neq 0, \\ \varrho x_3 &= (m_3 x_1' - m_1 x_3')(m_1 x_1' - am_3 x_2'), \qquad \Delta = m_1^2 - am_2 m_3 \neq 0. \end{aligned} \tag{7}$$

Compounding $I_2 I_1 = I_1 I_2$:

$$\begin{aligned} \varrho x_1 &= (m_1 x_1' - am_2 x_3')(m_1 x_1' - am_3 x_2'), \\ \varrho x_2 &= (m_1 x_1' - am_2 x_3')(m_2 x_1' - m_1 x_2'), \qquad am_1 \neq 0, \\ \varrho x_3 &= (m_1 x_1' - am_3 x_2')(m_3 x_1' - m_1 x_3'), \qquad \Delta \neq 0. \end{aligned} \tag{8}$$

This is the quadratic involution $J_{3(ABD_3)}$, where $D_3 = (am_2 m_3, m_1 m_2, m_1 m_3)$.

Theorem 4. Two quadratic inversions $I_{1(ABC_1)}$ and $I_{2(ABC_2)}$ are in the position II if and only if the polars of points C_1 and C_2 are pairwise coincident with respect to the conics of invariant points σ_1, σ_2.

Proof: The conic σ_1 of invariant points of the inversion I_1 can be represented by equation:

$$x_1^2 + ax_2 x_3 = 0. \tag{9}$$

The conic σ_2 of invariant points of the inversion I_2 (starting from relation (6)) can be represented by equation:

$$a_1 x_1^2 - bm_3 x_1 x_2 - bm_2 x_1 x_3 - bm_1 x_2 x_3 = 0. \tag{10}$$

The polar of the point C_1 with respect to σ_1 and that of point C_2 with respect to σ_2 can by represented by equation:

$$x_1 = 0. \tag{11}$$

The polar of the point C_2 with respect to σ_1 by:

$$2m_1x_1 - am_3x_2 - am_2x_3 = 0, \tag{12}$$

and the polar of the point C_1 with respect to σ_2 by:

$$2a_1x_1 - bm_3x_2 - bm_2x_3 = 0. \tag{13}$$

Since one pair of polars (11) coincides in any case, we prove the theorem for polars (12) and (13) only.

1. Let I_1 and I_2 be in the position II. Then $b = a, a_1 = m_1$ hence the polars (12) and (13) coincide.

2. Let the polars (12) and (13) coincide. Then $a_1 = km_1$, $b = ka$, by substitution into (6) we obtain (7). Thus, I_1 and I_2 are permutable.

Since it holds:

$$m_1(x_1^2 - ax_2x_3) + (m_1x_1^2 - am_3x_1x_2 - am_2x_1x_3 + am_1x_2x_3) = \\ = x_1(2m_1x_1 - am_3x_2 - am_2x_3).$$

and the polar of (11) passes through the points A, B, the polar of (12) must pass through the points M, N, i.e. through another two common points of conics σ_1, σ_2. Thereby is determined another pair of quadratic inversions in the position II, namely, inversions $I'_{(MNC_1)}$ and $I''_{(MNC_2)}$.

§ 4.

We can readily find further inversions permutable with inversions I_1 and I_2, respectively. We now ask whether there exists such an inversion I_3 permutable with both of the two permutable inversions or whether there exist even more such inversions mutually pairwise permutable.

Theorem 5. There cannot exist more than *four quadratic inversions mutually pairwise permutable in the position II, of which any three form an Abelian group of order eight.*

Proof: The inversion $I_{3(ABC_3)}$ with the centre $C_3 = (y_1, y_2, y_3)$ permutable with I_1 has according to § 3. the form:

$$\begin{aligned} \varrho x_1 &= a(y_2x'_1 - y_1x'_2)(y_3x'_1 - y_1x'_3), & ay_1 &\neq 0, \\ \varrho x_2 &= (y_2x'_1 - y_1x'_2)(y_1x'_1 - ay_2x'_3), & y_1^2 - ay_2y_3 &\neq 0, \\ \varrho x_3 &= (y_3x'_1 - y_1x'_3)(y_1x'_1 - ay_3x'_2). \end{aligned} \tag{7'}$$

If I_3 is to be permutable with I_2 it must hold according to theorem 3.: $I_3(C_2) = I_2(C_3)$.

$$I_3(C_2) = (a(y_2m_1 - y_1m_2) . (y_3m_1 - y_1m_3); (y_2m_1 - y_1m_2) . (y_1m_1 - ay_2m_3); \\ (y_3m_1 - y_1m_3) . (y_1m_1 - ay_3m_2)),$$

$$I_2(C_3) = (a(m_2y_1 - m_1y_2) . (m_3y_1 - m_1y_3); (m_2y_1 - m_1y_2) . (m_1y_1 - am_2y_3); (m_3y_1 - m_1y_3) . (m_1y_1 - am_3y_2)),$$

$$I_3(C_2) = I_2(C_3) \Leftrightarrow \begin{Bmatrix} y_1m_1 - ay_2m_3 = am_2y_3 - m_1y_1 \\ y_1m_1 - ay_3m_2 = am_3y_2 - m_1y_1 \end{Bmatrix} \Leftrightarrow$$

$$\Leftrightarrow 2m_1y_1 - am_3y_2 - am_2y_3 = 0,$$

that is, the point C_3 is a point of the line $2m_1x_1 - am_3x_2 - am_2x_3 = 0$, which is the line (12). With the exception of intersections with the conics σ_1 and σ_2 M, N and the intersection K with the line AB, each point of the line (12) is the centre of the quadratic inversion I_3 which is in the position II with respect to both inversions I_1 and I_2. For the points $M = (am_2m_3; m_2(m_1 + \sqrt{\Delta}); m_3(m_1 - \sqrt{\Delta}))$, $N = (am_2m_3; m_2(m_1 - \sqrt{\Delta}); m_3(m_1 + \bar{\Delta}))$ we obtain no involutory mapping; for the point $K = (O, m_2, - m_3)$ we get a linear mapping, a harmonic homology whose centre is the point K and its axis is the line C_1C_2.

Thus the centre of another inversion permutable with the foregoing three inversions will lie on the line (12) and on the polar of the point C_3 with respect to the conic σ_2 ($\equiv$ the polar of the point C_2 with respect to the conic σ_3), or on the polar of the point C_1 with respect to σ_3 ($\equiv$ the polar of the point C_3 with respect to σ_1). However all the three lines will intersect in the single point C_4 because every cubic containing any eight simple intersections of two cubics will contain the ninth one as well. (see [2], page 423, theorem a, par. 170.) Let

$$\sigma_1 \cap \sigma_2 = \{A, B, M, N\}$$
$$\sigma_1 \cap \sigma_3 = \{A, B, P, Q\}$$
$$\sigma_2 \cap \sigma_1 = \{A, B, R, S\},$$

then the cubic $k_1 = \sigma_1 . RS$, $k_2 = \sigma_2 . PQ$, $k_3 = \sigma_3 . MN$.
The common points of the cubics k_1 and k_2 are the points A, B, M, N, P, Q, R, S and the intersection of the line-pairs PQ, RS. As the cubic k_3 likewise contains the first eight points, it must evidently contain also the ninth one lying on the line MN.

Consequently, there now exists just another one inversion, permutable with those foregoing three. If $C_4 \neq K$, the inversion $I_{4(ABC_4)}$ is a quadratic one. $C_4 \neq K \Rightarrow C_3 \neq D_3$, $C_3 \neq M, N \Rightarrow C_4 \neq M, N$. The points C_3 and C_4 give rise to a point involution with invariant points M, N on the line (12). To the point K in this involution corresponds the intersection of line (12) with the connecting-line C_1, C_2, that is the point D_3.
For the results of the composition holds:

$$I_1I_2 = I_2I_1 = J_{3(ABD_3)},$$
$$I_1I_3 = I_3I_1 = J_{2(ABD_2)},$$
$$I_2I_3 = I_3I_2 = J_{1(ABD_1)}$$
$$I_1I_2I_3 = I_{4(ABC_4)}.$$

Further relations among the particular involutions are given in the Cayley table below:

E	I_1	I_2	I_3	I_4	J_1	J_2	J_3
I_1	E	J_3	J_2	J_1	I_4	I_3	I_2
I_2	J_3	E	J_1	J_2	I_3	I_4	I_1
I_3	J_2	J_1	E	J_3	I_2	I_1	I_4
I_4	J_1	J_2	J_3	E	I_1	I_2	I_3
J_1	I_4	I_3	I_2	I_1	E	J_3	J_2
J_2	I_3	I_4	I_1	I_2	J_3	E	J_1
J_3	I_2	I_1	I_4	I_3	J_2	J_1	E

The points C_1, C_2, C_3, C_4 form a complete quadrilateral; the points D_1, D_2, D_3 are its diagonal vertices. The side C_iC_j of this quadrilateral is the polar of the point C_k with respect to the conic σ_n and simultaneously the polar of the point C_n with respect to the conic σ_k, $i, j, k, n = 1, 2, 3, 4$.

It is possible to construct even more extensive groups of permutable quadratic involutions with two common singular points, neither of which is the centre of any quadratic inversion. Naturally then, not all the inversions are mutually pairwise permutable.

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[2] *B. Bydžovský:* Úvod do algebraické geometrie (Praha, JČMF 1948).

Shrnutí

O ZÁMĚNNÝCH ROVINNÝCH KVADRATICKÝCH INVERSÍCH SE DVĚMA SPOLEČNÝMI HLAVNÍMI BODY

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Práce ukazuje existenci a vlastnosti záměnných kvadratických inversí se dvěma společnými hlavními body. Nazveme poloha I. pár záměnných kvadratických inversí se společným středem a dalším jedním společným hlavním bodem. Dvě kvadratické inverse $I_{1(AB_1C)}$ a $I_{2(AB_2C)}$ jsou v poloze I. tehdy a jen tehdy, je-li třetí hlavní bod jedné inverse samodružným bodem inverse druhé. Složení obou inversí je opět kvadratická inverse $I_{3(B_1B_2C)}$, která je v poloze I. s každou z inversí I_1, I_2. Všechny tři spolu s identitou tvoří Kleinovu čtyřgrupu involucí.

Nazveme poloha II. pár záměnných kvadratických inversí se dvěma společnými hlavními body, z nichž žádný není středem některé z inversí. Dvě kvadratické inverse $I_{1(ABC_1)}$, $I_{2(ABC_2)}$ jsou v poloze II. tehdy a jen tehdy, platí-li $I_1(C_2) = I_2(C_1) = D_3$. Součinem obou inversí je kvadratická involuce $J_{3(ABD_3)}$. Existují nejvýše čtyři kvadratické inverse navzájem po dvou v poloze II., kde každé tři z těchto inversí generují abelovskou grupu řádu osm.