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Jiří Kobza

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STABILITY OF THE SECOND DERIVATIVE LINEAR MULTISTEP FORMULAS

JIŘÍ KOBZA

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1. Introduction

A linear multistep formula (LMF)

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j y'_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j y''_{n+j} \tag{F}$$

useful in a numerical solution (with the step size h) of ordinary differential equations is fully characterized by the so called "characteristic polynomials"

$$\varrho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j, \qquad \sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j, \qquad \tau(\zeta) = \sum_{j=0}^{k} \gamma_j \zeta^j.$$

The formula (F) is defined to be of order p iff the constants C_i,

$$C_0 = \sum_{j=0}^{k} \alpha_j, \qquad C_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j,$$

$$C_i = \sum_{j=0}^{k} (\alpha_j j^i - i\beta_j j^{i-1} - i(i-1)\gamma_j j^{i-2})/i!, \qquad i = 2, 3, \ldots$$

satisfy $C_0 = C_1 = \ldots = C_p = 0$, $C_{p+1} \neq 0$.

The formula of order $p \geqq 1$ satisfies then $C_0 = \Sigma\alpha_j = \varrho(1) = 0$, $C_1 = \Sigma j\alpha_j - \Sigma\beta_j = \varrho'(1) - \sigma(1) = 0$ (the condition of consistency).

There are various concepts of the stability of formula (F) and their characteristics to be found in the literature.

Definition: *The formula (F) is said to be stable (in the sense of Dahlquist, zero-stable), if no root of $\varrho(\zeta)$ has its modulus greater than one, and if every root with modulus one is simple (Dahlquist* 1956, 1959).

It is well known (Dahlquist 1956, 1959; Henrici 1962) that this stability is a necessary condition for convergency (when $h \to 0$) of methods based on formulas (F).

In studying LMF with a fixed step h the so called "test equation" $y' = \lambda y$ is often used, whose solution $y(x) = C \exp(\lambda x)$ has the property $|y(x)| \to 0$ when $x \to +\infty$ for $\operatorname{Re}\lambda < 0$. Applying the formula (F) to this equation results in the difference equation

$$\sum_{j=0}^{k} (\alpha_j - \lambda h \beta_j - \lambda^2 h^2 \gamma_j)\, y_{n+j} = 0,$$

whose characteristic polynomial (with $\lambda h = q$) is

$$\pi(\zeta; q) = \varrho(\zeta) - q\sigma(\zeta) - q^2 \tau(\zeta). \qquad (\pi)$$

Here the roots $\pi_i(q)$, $i = 1(1)\,k$ of this so called "stability polynomial" determine the behaviour of the discrete solution $\{y_i\}$ obtained by means of formula (F) with the fixed h.

Definition: (*Odeh–Liniger,* 1971). *Given a domain D in the complex q-plane, a formula (F) is called A_D-stable if $q \in D$ implies $|\pi_i(q)| < 1$, $i = 1(1)\,k$ (i.e. if the discrete solution maintains the decreasing character of the continuous solution for $q \in D$).*

This definition includes several stability concepts of various authors, such as
A-stability (*Dahlquist*, 1963) *with* $D = \{q \in C : \operatorname{Re} q < 0\}$,
A(α)-stability (*Widlund*, 1967) *with* $D = \left\{q \in C : |\pi - \arg q| < \alpha,\ 0 < \alpha < \frac{\pi}{2}\right\}$,
A_0-stability (*Cryer*, 1973) *with* $D = \{q \in R : q < 0\}$,
A_∞-stability (*Odeh–Liniger*, 1971) *with* $D = \{q \in C : |q| > W > 0\}$,
stiff-stability (*Gear*, 1971) *with* $D = \{q \in C : \operatorname{Re} q < K < 0\} \cup$

$$\cup \{q \in C : K \leqq \operatorname{Re} q \leqq \alpha,\ \alpha > 0;\ -\Theta \leqq \operatorname{Im} q \leqq \Theta, \Theta > 0\}.$$

It is well known that it holds for LMF with $\tau(\zeta) \equiv 0$:

- the maximal order of the stable formula is $p = 2$ for k even and $k + 1$ for k odd (Dahlquist, 1956);
- A, $A(\alpha)$, A_0, A_∞-stable formulas are necessary implicit, i.e. $\beta_k \neq 0$ (Dahlquist 1963, Widlund 1967, Cryer 1973, Liniger 1975);
- the maximal order of an A-stable formula is $p = 2$ (Dahlquist, 1963);
- for all $\alpha \in \left(0, \frac{\pi}{2}\right)$ there exist $A(\alpha)$-stable formulas with $k = p = 3$, $k = p = 4$ (Widlund, 1967);
- there exist A_0-stable formulas of arbitrary high order (Cryer, 1973);

- the polynomials $\varrho(\zeta)$, $\sigma(\zeta)$ of the A_0-stable formula may have at most double zeros on the unit circle (Cryer, 1973);
- for $k \leqq 2$, the classes of the A- and A_0-stable formulas coincide (Zlámal, 1975).

Concerning the LMF with $\tau(\zeta) \not\equiv 0$, it is known that
- order p of the stable formula satisfies $p \leqq 2k + 2$ (Dahlquist, 1959);
- for $k = 1$ there exists the A-stable formula of order $p = 4$ (Loscalzo, 1969);
- the maximal order of an A-stable formula with $k = 2$ is $p = 5$; such a formula is given by

$$4(1 + 3x)(y_{n+2} - 2y_{n+1} + y_n) - 2(2 + 3x)\,h(y'_{n+2} - y'_n) +$$
$$+ h^2[(1 + x)\,y''_{n+2} + 2(1 - x)\,y''_{n+1} + (1 + x)\,y''_n] = 0, \qquad 0 \leqq x \in R,$$

(Genin, 1974); these formulas are not stable in the Dahlquist's sense.

The necessary and sufficient conditions for A-stability (as conditions on the coefficients β_i, γ_i) are discussed also by Liniger (1968) for $k \leqq 2$, $\tau(\zeta) \equiv 0$ and by Liniger—Willoughby (1970) for $k \leqq 2$, $\tau(\zeta) \not\equiv 0$. The connections between A_0- and A_∞-stability are also discussed by Liniger (1975).

The present paper deals with necessary and sufficient conditions for A, A_0, A_∞-stability of the formulas (F) with $k \leqq 2$. The main results are given in theorems 1, 2, 5, 6 and 7. It is shown here that for the formulas (F):
- the regions of A_0, A-stability for the one step formulas of the second order coincide, the region of A_∞-stability includes the next part of the q-plane;
- any one-step A_0-stable formula of the third order (with one exception) is also A- and A_∞-stable;
- the two-step formulas of the sixth order are not A_0, A_∞-stable;
- there are A_0, A_∞-stable two-step formulas of the fourth and fifth order which are stable—or more precisely: the A_0, A_∞-stable formula of the fourth order may be unstable (the double root $\zeta = 1$ of $\varrho(\zeta)$; see example 8 in 3.3), whereas the A_0, A_∞-stable formula of the fifth order is stable.

2. One—step formula

$$y_{n+1} - y_n = h(\beta_0 y'_n + \beta_1 y'_{n+1}) + h^2(\gamma_0 y''_n + \gamma_1 y''_{n+1}). \qquad \text{(F 1)}$$

The conditions $C_i = 0$, $i = 1(1)5$ take here the form

$$\begin{array}{lllll} i = 1 & 1 = \beta_0 + \beta_1, & & & \\ i = 2 & 1 = & 2\beta_1 & + 2\gamma_0 & + 2\gamma_1, \\ i = 3 & 1 = & 3\beta_1 & & + 6\gamma_1, \\ i = 4 & 1 = & 4\beta_1 & & + 12\gamma_1, \\ i = 5 & 1 = & 5\beta_1 & & + 20\gamma_1. \end{array}$$

Solving the test equation $y' = \lambda y$ by applying formula (F 1) we get

$$y_{n+1} = [(1 + q\beta_0 + q^2\gamma_0)/(1 - q\beta_1 - q^2\gamma_1)]\, y_n.$$

The formula (F 1) is then A_D-stable iff

$$|1 + q\beta_0 + q^2\gamma_0|/|1 - q\beta_1 - q^2\gamma_1| < 1 \qquad \text{for all } q \in D.$$

2.1 It is known that **the formula of the maximal order** $p = 4$, with $\beta_0 = \beta_1 = \frac{1}{2}, \gamma_0 = -\gamma_1 = \frac{1}{2}$, i.e.

$$y_{n+1} - y_n = \frac{h}{2}(y'_n + y'_{n+1}) + \frac{h^2}{12}(y''_n + y_{n+1}), \qquad C_5 = -\frac{1}{720}, \qquad \text{(F 1,4)}$$

is A-stable $\left[A(\alpha)\text{-stable for all } \alpha \in \left(0, \frac{\pi}{2}\right), A_0\text{-stable}\right]$. However, this formula is not A_∞-stable for

$$\lim_{|q|\to\infty} |y_{n+1}/y_n| = \lim_{|q|\to\infty} \left|\left(1 + \frac{1}{2}q + \frac{1}{12}q^2\right)\Big/\left(1 - \frac{1}{2}q + \frac{1}{12}q^2\right)\right| = 1.$$

2.2 For **one-step formulas of the third order** with the parameter γ_1 we get from $C_i = 0$, $i = 1, 2, 3$—after some calculation—the general expression

$$y_{n+1} - y_n = h\left[\left(\frac{2}{3} + 2\gamma_1\right)y'_n + \left(\frac{1}{3} - 2\gamma_1\right)y'_{n+1}\right] + h^2\left[\left(\frac{1}{6} + \gamma_1\right)y''_n + \gamma_1 y''_{n+1}\right],$$

$$C_4 = -\left(\frac{1}{3} + 4\gamma_1\right)/24. \qquad \text{(F 1,3)}$$

Remark: a) for $\gamma_1 = -1/12$ we get $C_4 = 0$ and the formula (F 1,4) from 2.1;
b) for the formulas with a small local truncation error the values of γ_1 should be taken near the value $-1/12$.

Using the formula (F 1,3) in the test equation $y' = \lambda y$, we get the reccurrence relations for the discrete solution $\{y_n\}$

$$y_{n+1} = \left\{\left[1 + \left(\frac{2}{3} + 2\gamma_1\right)q + \left(\frac{1}{6} + \gamma_1\right)q^2\right]\Big/\left[1 + \left(-\frac{1}{3} + 2\gamma_1\right)q - \gamma_1 q^2\right]\right\} y_n;$$

$$q = \lambda h.$$

Theorem 1. *The formula (F 1,3) is*

a) A_∞*-stable iff* $\gamma_1 < -1/12$,
b) A*-stable iff* $\gamma_1 \leqq -1/12$,
c) A_0*-stable iff* $\gamma_1 \leqq -1/12$.

Proof. a) $\lim_{|q|\to\infty} |y_{n+1}/y_n| = \left|\left(\frac{1}{6} + \gamma_1\right)/\gamma_1\right| < 1$ iff $\gamma_1 < -1/12$.

b) The proof follows by using the maximal modulus principle when investigating the A-stability conditions on the imaginary axis; a similar result—but with somewhat another choice of the parameter—is given by Liniger—Willoughby (1970).

c) A_0-stability of the formula (F 1,3) for $\gamma_1 \leqq -1/12$ follows from its A-stability (the direct proof can be easily obtained). For $\gamma_1 > -1/12$, we have $\lim_{q \to -\infty} | y_{n+1}/y_n | > 1$; the A_0-stability condition for sufficiently small $q < 0$ is not fulfilled.

Corollaries

1. The formula (F 1,3) with $\gamma_1 \leqq -1/12$ is $A(\alpha)$-stable for all $\alpha \in \left(0, \frac{\pi}{2}\right)$.
2. Any A_0-stable formula (F 1,3) is A-stable; it is also A_∞-stable for $\gamma_1 < -1/12$.

Formula examples.

1. $y_{n+1} - y_n = \frac{h}{6}(y'_n + 5y'_{n+1}) - \frac{h^2}{12}(y''_n + 3y''_{n+1}), \qquad C_4 = \frac{1}{36}$.

2. $y_{n+1} - y_n = \frac{h}{3}(y'_n + 2y'_{n+1}) - \frac{1}{6}h^2 y''_{n+1}, \qquad C_4 = \frac{1}{72}$;

(here $\lim y_{n+1}/y_n = 0$—the so called L-stable formula).

3. $y_{n+1} - y_n = \frac{h}{15}(7y'_n + 8y'_{n+1}) + \frac{h^2}{30}(2y''_n - 3y''_{n+1}), \qquad C_4 = \frac{1}{360}$.

2.3 One-step formulas of the second order can be obtained by choosing the parameters β_0, γ_1 and calculating the remaining coefficients from the conditions $C_i = 0$, $i = 1, 2$. We obtain the formula

$$y_{n+1} - y_n = h[\beta_0 y'_n + (1 - \beta_0) y'_{n+1}] + h^2\left[\left(-\frac{1}{2} + \beta_0 - \gamma_1\right) y''_n + \gamma_1 y''_{n+1}\right],$$

$$C_3 = \frac{1}{6}(-2 + 3\beta_0 - 6\gamma_1). \qquad \text{(F 1,2)}$$

Remark: a) for $\beta_0 = \frac{1}{2}$, $\gamma_1 = 0$ we obtain the trapezoidal rule known to be A, A_0-stable but not A_∞-stable;

b) we get the formulas of the third order with the parameter for $\beta_0 = 2\gamma_1 + \frac{2}{3}$.

Applying formula (F 1,2) to the test equation $y' = \lambda y$, we obtain the reccurrence relation for the discrete solution

$$y_{n+1} = \left\{\left[1 + \beta_0 q + \left(-\frac{1}{2} + \beta_0 - \gamma_1\right) q^2\right] / [1 - (1 - \beta_0) q - \gamma_1 q^2]\right\} y_n, \quad q = \lambda h.$$

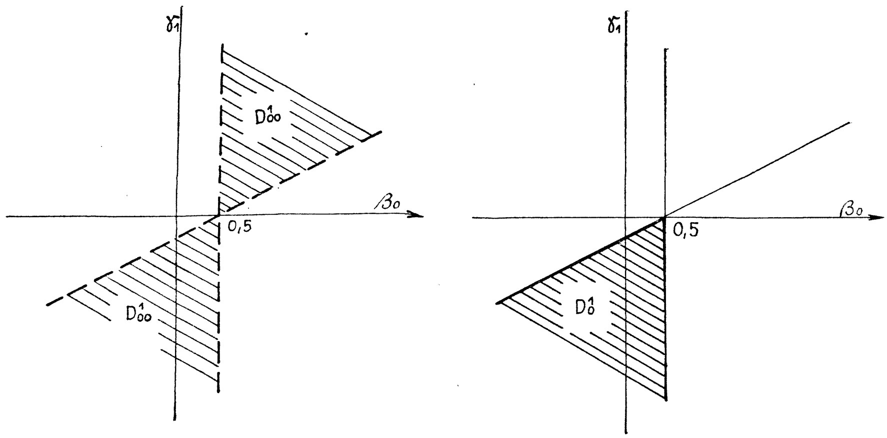
Theorem 2. *The formula (F 1,2) is*

a) A_∞*-stable iff* $(\beta_0, \gamma_1) \in D^1_\infty = \left\{(\beta_0, \gamma_1) : \left[\beta_0 < \frac{1}{2},\ \gamma_1 < -\frac{1}{4} + \frac{1}{2}\beta_0\right]\right.$ or $\left.\left[\beta_0 > \frac{1}{2},\ \gamma_1 > -\frac{1}{4} + \frac{1}{2}\beta_0\right]\right\}$;

b) A-stable iff $(\beta_0, \gamma_1) \in D_0^1 = \left\{(\beta_0, \gamma_1) : \gamma_1 \leqq 0, \frac{1}{2} + 2\gamma_1 \leqq \beta_0 \leqq \frac{1}{2}\right\}$;

c) A_0*-stable iff it is A-stable.*

Remark: the regions D_∞^1, D_0^1 are given in fig. 1.



Proof. a) $\lim\limits_{|q| \to \infty} | y_{n+1}/y_n | = | \gamma_0/\gamma_1 | = \left| \left(-\frac{1}{2} + \beta_0 - \gamma_1\right)/\gamma_1 \right| < 1$ iff $-1 < \left(-\frac{1}{2} + \beta_0 - \gamma_1\right)/\gamma_1 < 1$; this inequality is valid exactly in the domain D_∞^1 given above.

b) Using the maximum modulus principle, we can investigate only the fulfillment of the A-stability condition on $\operatorname{Re} q = 0$. We find it fulfilled exactly for $(\beta_0, \gamma_1) \in D_0^1$. This result—also with somewhat another choice of parameters related linear to β_0, γ_1—is given by Liniger—Willoughby (1970).

c) The A_0-stability of formulas (F 1,2) for $(\beta_0, \gamma_1) \in D_0^1$ follows from their A-stability. We obtain the region D_0^1 in investigating the conditions of A_0-stability in case $\gamma_1 \leqq 0$, $\beta_0 \leqq 1$ (the denominator in the reccurrence relation is positiv for $(\beta_0, \gamma_1) \in D_0^1$). In the remaining three possible casses $(\beta_0 \leqq 1, \gamma_1 > 0)$, $(\beta_0 > 1, \gamma_1 \leqq 0)$, $(\beta_0 > 1, \gamma_0 > 0)$ the denominator may have negative roots q and the fraction may be unbounded in the neighbourhood of such roots. Denoting the denominator considered by $\varphi(q) = 1 - (1 - \beta_0)\, q - \gamma_1 q^2$, its roots by $q_i = [1 - \beta_0 + (-1)^i \sqrt{\Delta}]/(-2\gamma_1)$, $i = 1, 2$, $\Delta = (1 - \beta_0)^2 + 4\gamma_1$, we obtain:

1. With $(\beta_0 > 1, \gamma_1 > 0)$ or $(\beta_0 \leqq 1, \gamma_1 > 0)$ one root q_i is positive, the second is negative; in the neighbourhood of the negative root the condition $| y_{n+1}/y_n | < 1$ does not hold.

2. With $\gamma_1 \leqq 0$, $\beta_0 > 1$ the condition of A_0-stability is not valid

– because of disturbing the condition $\beta_0 \leqq \frac{1}{2}$ if $\Delta \neq 0$,

– because of the negative double root of $\varphi(q)$ if $\Delta = 0$ (the fraction $|y_{n+1}/y_n|$ is unbounded in the neighbourhood of this root).

Formula examples

1. $y_{n+1} - y_n = hy'_{n+1} - \frac{1}{2}h^2 y''_{n+1}$, $C_3 = 1/6$ (Taylor's formula)

2. $y_{n+1} - y_n = hy'_{n+1} - \frac{1}{6}h^2(y''_n + 2y''_{n+1})$, $C_4 = 1/24$

3. $y_{n+1} - y_n = \frac{h}{5}(2y'_n + 3y'_{n+1}) + \frac{h^2}{30}(y''_n - 2y''_{n+1})$, $C_4 = 1/120$

4. $y_{n+1} - y_n = \frac{h}{2}(y'_n + y'_{n+1}) + \frac{h^2}{2}(y''_n - y''_{n+1})$, $C_3 = 5/12$

3. Two-step formula of the type considered is

$$y_{n+2} - (1 + a)\, y_{n+1} + ay_n =$$
$$= h(\beta_0 y'_n + \beta_1 y'_{n+1} + \beta_2 y'_{n+2}) + h^2(\gamma_0 y''_n + \gamma_1 y''_{n+1} + \gamma_2 y''_{n+2}). \tag{F 2}$$

Applying (F 2) to the test equation $y' = \lambda y$ we obtain a difference equation whose characteristic polynomial ("stability polynomial" of (F 2)) is

$$\pi(\zeta; q) \equiv \varrho(\zeta) - q\sigma(\zeta) - q^2\tau(\zeta) =$$
$$= (1 - \beta_2 q - \gamma_2 q^2)\,\zeta^2 - (1 + a + \beta_1 q + \gamma_1 q^2)\,\zeta + (a - \beta_0 q - \gamma_0 q^2). \tag{π 2}$$

A_∞-stability

Theorem 3. *a) (Implicitness of the A_∞-stable formulas).*
When (F 2) is A_∞-stable, $\tau(\zeta) \equiv 0$ implies $\beta_2 \neq 0$, $\tau(\zeta) \not\equiv 0$ implies $\gamma_2 \neq 0$.
b) We have in case of $\tau(\zeta) \equiv 0$: (F 2) is A_∞-stable iff $|\sigma_i| < 1$, $i = 1(1)\,k$;
in case of $\tau(\zeta) \equiv 0$: (F 2) is A_∞-stable iff $|\tau_i| < 1$, $i = 1(1)\,k$,
where $\sigma_i[\tau_i]$ are the roots of the polynomial $\sigma(\zeta)$ $[\tau(\zeta)]$.

In case of $\tau(\zeta) \equiv 0$ the proof was given by Liniger (1975) and Odeh–Liniger (1971). The proof in case of $\tau(\zeta) \not\equiv 0$ is analogous:

a) for the normed polynomial $\pi(\zeta; q)$ with $\gamma_2 = 0 (\gamma_0 \neq 0)$ we have $\lim_{|q|\to\infty} |(a - \beta_0 q - \gamma_0 q^2)/(1 - \beta_2 q)| = +\infty$; both roots $\pi_i(q)$ couldn't be bounded for $|q| \to +\infty$.

b) The proof follows from the continuous dependency of $\pi_i(q)$ on q and from the fact that $\lim_{|q|\to\infty} \pi_i(q) = \tau_i$, $i = 1, 2$.

In what follows we shall several times use the special case of the so called Schur–Cohen criterion (which can be derived from the Hurwitz criterion – see for example lit. [1]).

Lemma 1. *All the roots of the quadratic polynomial with real coefficients* $a_2x^2 + a_1x + a_0$, $a_2 > 0$ *lie inside the unit circle iff* $a_2 + a_1 + a_0 > 0$; $a_2 - a_0 > 0$; $a_2 - a_1 + a_0 > 0$.

Theorem 4. *The formula* $(F\,2)$ *is* A_∞*-stable iff*

$$\gamma_0 + \gamma_1 + \gamma_2 > 0, \quad \gamma_0 - \gamma_1 + \gamma_2 > 0, \quad \gamma_2 - \gamma_0 > 0 \quad \textit{for } \gamma_2 > 0,$$
$$\gamma_0 + \gamma_1 + \gamma_2 < 0, \quad \gamma_0 - \gamma_1 + \gamma_2 < 0, \quad \gamma_2 - \gamma_0 < 0 \quad \textit{for } \gamma_2 < 0.$$

The proof follows easily from lemma 1 and theorem 3.

A-stability. Genin (1974) showed a necessary and sufficient condition for A-stability of a more general type of formulas (higher derivatives); it doesn't possess the form of the condition for the coefficients of the formula. Jackson – Kenue (1974) showed A-stability of the formula derived by Enright (1974)

$$y_{n+2} - y_{n+1} = \frac{h}{48}(-y'_n + 20y'_{n+1} + 29y'_{n+2}) - \frac{h^2}{8}y''_{n+2}, \qquad C_5 = -7/1440.$$

No fuller analytic investigations in A-stability of formulas (F2) are known to the author for the time being.

A_0-stability. The expression $1 - \beta_2 q - \gamma_2 q^2$ in $(\pi 2)$ is positive for $q \to 0$; when $a_2 \to 0$, the modulus of one of the roots of the quadratic polynomial $a_2x^2 + a_1x + a_0$ tends to infinity, so that the positivity of the expression $1 - \beta_2 q - \gamma_2 q^2$ is necessary for A_0-stability of (F2). The necessary and sufficient conditions of A_0-stability of a consistent formula (F2) can be written using lemma 1 and some calculations in the form

A. $1 - \beta_2 q - \gamma_2 q^2 > 0$,
B. $(a - 1)\,q - (\gamma_0 + \gamma_1 + \gamma_2)\,q^2 > 0$,
C. $1 - a + (\beta_0 - \beta_2)\,q + (\gamma_0 - \gamma_2)\,q^2 > 0$,
D. $2(1 + a) - (\beta_2 - \beta_2 + \beta_0)\,q - (\gamma_2 - \gamma_1 + \gamma_0)\,q^2 > 0$.

Lemma 2. *Let* $ax^2 + bx + c$ *be a quadratic polynomial with real coefficients,* $a^2 + b^2 + c^2 > 0$; *then*

$$ax^2 + bx + c > 0 \quad \textit{for all } x < 0 \quad \textit{iff} \quad a \geqq 0, \quad c \geqq 0, \quad (b \leqq 0 \ \textit{or} \ b^2 - 4ac < 0).$$

Proof is elementary.

Applying lemma 2 we can rewrite the necessary and sufficient conditions of A_0-stability $A - D$ thus

A. $\gamma_2 \leqq 0$; $(\beta_2 \geqq 0$ or $\beta_2^2 - 4\gamma_2 < 0)$;
B. $1 + a \geqq 0$; $\gamma_0 + \gamma_1 + \gamma_2 \leqq 0$; except $a = 1$, $\gamma_0 + \gamma_1 + \gamma_2 = 0$;
C. $1 - a \geqq 0$; $\gamma_0 - \gamma_2 \geqq 0$; $[\beta_2 - \beta_0 \geqq 0$ or $(\beta_2 - \beta_0)^2 - 4(1 - a)(\gamma_0 - \gamma_2) < 0]$; *with the exception of* $a = 1$, $\gamma_0 = \gamma_2$, $\beta_0 = \beta_2$.

D. $1 + a \geqq 0;\ \gamma_2 - \gamma_1 + \gamma_0 \leqq 0;$
$[\beta_2 - \beta_1 + \beta_0 \geqq 0$ or $(\beta_2 - \beta_1 + \beta_0)^2 + 8(1 + a)\ (\gamma_2 - \gamma_1 + \gamma_0) < 0]$;
with the exception of $a = 1, \gamma_2 - \gamma_1 + \gamma_0 = 0$.

Remark. The conditions of A_∞-stability are included in the A_0-stability conditions (see Theorem 4).

3.1 Retaining the free parameter a in the formula (F2) and finding the coefficients $\beta_0, \beta_1, \beta_2; \gamma_0, \gamma_1, \gamma_2$ from the conditions $C_i = 0$, $i = 1(1)\,6$, i.e.

$$
\begin{aligned}
i &= 1, & 1 - a &= \beta_0 + \beta_1 + \beta_2 \\
i &= 2, & 3 - a &= 2\beta_1 + 4\beta_2 + 2\gamma_0 + 2\gamma_1 + 2\gamma_2 \\
i &= 3, & 7 - a &= 3\beta_1 + 12\beta_2 + 6\gamma_1 + 12\gamma_2 \\
i &= 4, & 15 - a &= 4\beta_1 + 32\beta_2 + 12\gamma_1 + 48\gamma_2 \\
i &= 5, & 31 - a &= 5\beta_1 + 80\beta_2 + 20\gamma_1 + 160\gamma_2 \\
i &= 6, & 63 - a &= 6\beta_1 + 192\beta_2 + 30\gamma_1 + 480\gamma_2 \\
i &= 7, & 127 - a &= 7\beta_1 + 448\beta_2 + 42\gamma_1 + 1344\gamma_2
\end{aligned}
$$

(to calculate C_7),

we obtain the **sixth-order formula**

$$
\begin{aligned}
& y_{n+2} - (1 - a)\, y_{n+1} + a y_n = \\
&= \frac{h}{240}\left[(101 - 11a)\, y'_{n+2} + 128(1 - a)\, y'_{n+1} + (11 - 101a)\, y'_n\right] + \\
&+ \frac{h^2}{240}\left[(-13 + 3a)\, y''_{n+2} + 40(1 + a)\, y''_{n+1} + (3 - 13a)\, y''_n\right], \\
& C_7 = \frac{1 - a}{9450}.
\end{aligned}
\tag{F2,6}
$$

Remarks 1. We obtain the seventh-order formula for $a = 1$ – see Jankovič (1965) – which is unstable (the double root $\zeta = 1$ of $\varrho(\zeta)$)

$$
y_{n+2} - 2y_{n+1} + y_n = \frac{3}{8} h(y'_{n+2} - y'_n) + \frac{h^2}{24}(y''_{n+2} - 8y_{n+1} + y_n),
$$

$$
C_8 = \frac{1}{60\,480}.
$$

2. We have the formula by Jackson-Kenue (1974) for $a = 0$

$$
y_{n+2} - y_{n+1} = \frac{h}{240}(11y'_n + 128y'_{n+1} + 101y'_{n+2}) + \frac{h^2}{240}(3y''_n + 40y''_{n+1} - 13y''_{n+2}),
$$

$$
C_7 = \frac{1}{9450}.
$$

3. It follows from the expression for $C_7 = (1 - a)/9450$ that by no choice of two values a', a'' of parameter a stable formulas with $C_7' = -C_7''$ (two-sided formulas) are obtainable.

Theorem 5. *The formula* (F2,6) *is not* A_0, A_∞ – *stable for any real value of parameter* a.

Proof. a) The polynomial $\tau(\zeta) = (-13 + 3a)\zeta^2 + 40(1 + a)\zeta + 3 - 13a$ doesn't satisfy the necessary and sufficient conditions of A_∞-stability given in Theorem 3 for any real value of a.

b) Investigating the necessary and sufficient conditions of A_0-stability (conditions $A - D$) yields after some elementary but lenghty calculations

A. $240 - q(101 - 11a) - q^2(-13 + 3a) > 0, \qquad \forall q < 0$ iff $a < 13/3$

B. $-q\left[1 - a + \frac{1}{8}q(1 + a)\right] > 0, \qquad \forall q < 0$ iff $a \leqq -1$

C. $1 - a - \frac{3}{8}q(1 + a) + \frac{1}{15}(1 - a)q^2 > 0, \qquad \forall q < 0$ iff $-1 \leqq a \leqq 1$

D. $2(1 + a) - \frac{1}{15}(a - 1)q + \frac{5}{24}(1 + a)q^2 > 0 \qquad \forall q < 0$ iff

$a > (-188 + 5\sqrt{15})/187 \approx -0{,}90179\ldots$

With respect to B, D there is no real value of a simultaneously satisfying the conditions $A - D$.

3.2 Keeping besides the parameter a another free parameter γ_2 in the formula (F2) we obtain the following values for the remaining coefficients from the conditions $C_i = 0$, $i = 1(1)5$:

$$\beta_0 = \frac{5}{24} - \frac{11}{24}a + 3\gamma_2, \qquad \gamma_0 = \frac{1}{15}(1 - a) + \gamma_2,$$

$$\beta_1 = \frac{8}{15}(1 - a), \qquad \gamma_1 = \frac{23}{60} + \frac{7}{60}a + 4\gamma_2,$$

$$\beta_2 = \frac{31}{120} - \frac{1}{120}a - 3\gamma_2, \qquad C_6 = \frac{1}{6!}\left(-\frac{47}{60} + \frac{17}{60}a - 30\gamma_2\right).$$

This finally leads to **the fifth-order two-step formula**

$$y_{n+2} - (1 + a)y_{n+1} + ay_n =$$
$$= h\left[\left(\frac{5}{24} - \frac{11}{24}a + 3\gamma_2\right)y_n' + \frac{8}{15}(1 - a)y_{n+1}' + \left(\frac{31}{120} - \frac{1}{120}a - 3\gamma_2\right)y_{n+2}'\right] +$$
$$+ h^2\left[\left(\frac{1}{15} - \frac{1}{15}a + \gamma_2\right)y_n'' + \left(\frac{23}{60} + \frac{7}{60}a + 4\gamma_2\right)y_{n+1}'' + \gamma_2 y_{n+2}''\right]. \quad \text{(F2,5)}$$

A_∞-stability. Substitution into the necessary and sufficient conditions of A_∞-stability (Theorem 4) results in

$$\left.\begin{array}{ll}\frac{1}{20}(9+a)+6\gamma_2>0 \\ \frac{1}{15}(a-1) & >0 \\ -\frac{1}{60}(19+11a)-2\gamma_2>0\end{array}\right\} \text{ if } \gamma_2 \geqq 0, \qquad \left.\begin{array}{ll}\frac{1}{20}(9+a)+6\gamma_2<0 \\ \frac{1}{15}(a-1) & <0 \\ -\frac{1}{60}(19+11a)-2\gamma_2<0\end{array}\right\} \text{ if } \gamma_2<0.$$

These conditions are satisfied exactly in the region

$$D_\infty^{2,5}=\left\{(a,\gamma_2): -1<a<1,\ -\frac{1}{120}(19+11a)<\gamma_2<-\frac{1}{120}(9+a)\right\}$$

given in fig. 2.

A_0-stability. The necessary and sufficient conditions of the A_0-stability for (F2,5) can be written (after some calculation) in the form

A. $\gamma_2 \leqq 0$;
$$\left\{\frac{1}{120}(31-a)-3\gamma_2 \geqq 0 \text{ or } 9\gamma_2^2+\frac{1}{20}(49+a)\gamma_2+\frac{1}{14\,400}(31-a)^2<0\right\};$$

B. $1-a \geqq 0$; $\frac{1}{20}(9+a)+6\gamma_2 \leqq 0$; $(a,\gamma_2) \neq \left(1,-\frac{1}{12}\right)$;

C. $1-a \geqq 0$; $(a,\gamma_2) \neq \left(1,\frac{1}{12}\right)$;

$$\left\{\frac{1}{20}(1+9a)-6\gamma_2 \geqq 0 \text{ or } 36\gamma_2^2-\frac{3}{5}(1+9a)\gamma_2-\frac{1}{1200}(77a^2+694a+317)<0\right\};$$

D. $1+a \geqq 0$; $\frac{1}{60}(19+11a)+2\gamma_2 \geqq 0$;

$$\left\{a-1 \geqq 0 \text{ or } \left[120\gamma_2>-19-11a+\frac{(a-1)^2}{30(a+1)} \text{ if } a>-1\right]\right\}.$$

Solving the quadratic inequalities we get

A. $\gamma_2 \leqq 0$; $\left\{\frac{1}{120}(31-a)-3\gamma_2 \geqq 0\right.$ or

$$\left.[-49-a-4\sqrt{10(9+a)}<360\gamma_2<-49-a+4\sqrt{10(9+a)} \text{ if } a>-9]\right\};$$

B. $1-a \geqq 0$; $\frac{1}{20}(9+a)+6\gamma_2 \leqq 0$: $(a,\gamma_2) \neq \left(1,-\frac{1}{12}\right)$;

C. $1-a \geqq 0$; $(a,\gamma_2) \neq \left(1,\frac{1}{12}\right)$;

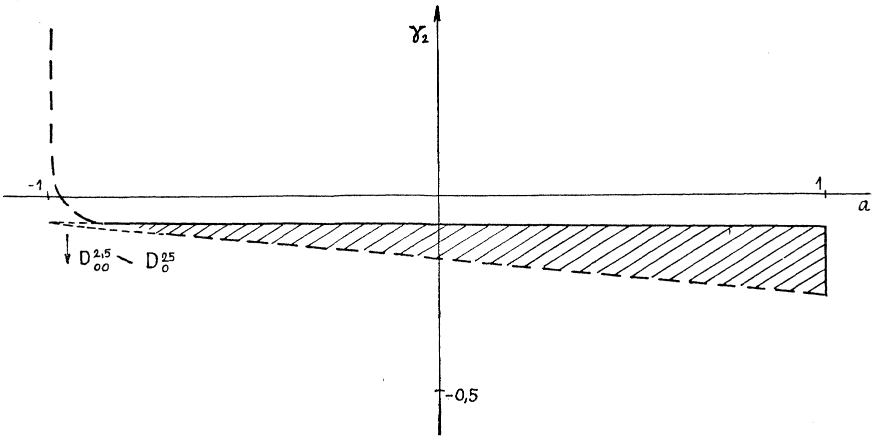
$$\left\{\frac{1}{20}(1+9a)-6\gamma_2 \geqq 0 \text{ or } \left[3(1+9a)-2\sqrt{3}\sqrt{80a^2+187a+80} < 360\gamma_2 < \right.\right.$$

$$< 3(1+9a)+2\sqrt{3}\sqrt{80a^2+187a+80} \text{ if } 160a \notin (-187-\sqrt{9369},$$

$$\left.\left. -187+\sqrt{9369}), \text{ i.e. if } a \notin (-1{,}77\ldots, -0{,}56\ldots)\right]\right\};$$

D. $1+a \geqq 0$; $\frac{1}{60}(19+11a)+2\gamma_2 \geqq 0$;

$$\left\{a-1 \geqq 0 \text{ or } \left[120\gamma_2 > -19-11a+\frac{(a-1)^2}{30(a+1)} \text{ if } a > -9\right]\right\}.$$



A detailed analysis shows that the conditions $A - D$ are simultaneously satisfied exactly in the region $D_0^{2,5}$ of the (a, γ_2) – plane defined by the relations

$$D_0^{2,5} = \left\{(a, \gamma_2) : -0.890\,833\ldots \leqq a \leqq 1, -19-11a+\frac{(a-1)^2}{30(a+1)} < 120\gamma_2 \leqq -9-a\right\}$$

except for $(a, \gamma_2) = (1, -1/12)$. The region $D_0^{2,5}$ together with the region $D_\infty^{2,5}$ are given in fig. 2. As can be seen now these two regions differs in the neighbourhood of their boundaries only (see the defining conditions). So we have proved

Theorem 6. *The A_∞, A_0-stability regions of the formula* (F2,5) *in the (a, γ_2)-plane are exactly the regions $D_\infty^{2,5}$, $D_0^{2,5}$ defined above; none of these is a subset of the other. The A_∞-stability of the formula* (F2,5) *implies its stability* (*in the sense of Dahlquist*).

Examples of the A_∞, A_0-stable formulas.

1. $a = 0$; $\gamma_2 = -0{,}1$; $C_6 = 11/7200$

$$y_{n+2} - y_{n+1} = \frac{h}{120}(-11y'_n + 64y'_{n+1} + 67y'_{n+2}) - \frac{h^2}{60}(2y''_n + y''_{n+1} + 6y''_{n+2}),$$

2. $a = 1/2$; $\gamma_2 = -0{,}1$; $C_0 = 25/14400$

$$y_{n+2} - \frac{3}{2} y_{n+1} + \frac{1}{2} y_n = \frac{h}{240}(-77y'_n + 64y'_{n+1} + 133y'_{n+2}) + \\ + \frac{h^2}{120}(-8y''_n + 5y''_{n+1} - 12y''_{n+2}),$$

3. $a = 0$; $\gamma_2 = -1/8$; $C_6 = 89/21600$

$$y_{n+2} - y_{n+1} = \frac{h}{30}(-5y'_n + 16y'_{n+1} + 19y'_{n+2}) - \frac{h^2}{120}(7y''_n + 14y''_{n+1} + 15y''_{n+2})$$

4. $a = -1/2$; $\gamma_2 = -0{,}1$; $C_6 = 83/28800$;

$$y_{n+2} - \frac{1}{2} y_{n+1} - \frac{1}{2} y_n = \frac{h}{80}(11y'_n + 64y'_{n+1} + 45y'_{n+2}) - \frac{h^2}{40}(3y''_{n+1} + 4y''_{n+2}),$$

5. $a = -0{,}8$; $\gamma_2 = -0{,}07$; $C_6 = 109/72000$;

$$y_{n+2} - 0{,}2y_{n+1} - 0{,}8y_n = \frac{h}{200}(73y'_n + 129y'_{n+1} + 95y'_{n+2}) + \\ + \frac{h^2}{100}(5y''_n + y''_{n+1} - 7y''_{n+2}).$$

3.3 Retaining the free parameters a, β_1, γ_1 in the formula (F2) and calculating the remaining coefficients from the relations $C_i = 0$, $i = 1(1)4$, we obtain

$$\beta_0 = \frac{9}{48} - \frac{39}{48}a - \frac{1}{2}\beta_1 + \frac{3}{4}\gamma_1, \qquad \gamma_0 = \frac{5}{48} - \frac{11}{48}a - \frac{1}{4}\beta_1 + \frac{1}{4}\gamma_1,$$

$$\beta_2 = \frac{39}{48} - \frac{9}{48}a - \frac{1}{2}\beta_1 - \frac{3}{4}\gamma_1, \qquad \gamma_2 = \frac{11}{48} + \frac{5}{48}a + \frac{1}{4}\beta_1 + \frac{1}{4}\gamma_1,$$

$$C_5 = \frac{1}{360}(8 - 8a - 15\beta_1),$$

which leads to **the fourth order two-step formula**

$$y_{n+2} - (1 + a)\, y_{n+1} + a y_n = h\Bigg[\left(\frac{9}{48} - \frac{39}{48}a - \frac{1}{2}\beta_1 + \frac{3}{4}\gamma_1\right) y_n + \\ + \beta_1 y'_{n+1} + \left(\frac{39}{48} - \frac{9}{48}a - \frac{1}{2}\beta_1 - \frac{3}{4}\gamma_1\right) y'_{n+2}\Bigg] + \\ + h^2\Bigg[\left(\frac{5}{48} - \frac{11}{48}a - \frac{1}{4}\beta_1 + \frac{1}{4}\gamma_1\right) y''_n + \gamma_1 y''_{n+1} + \\ + \left(-\frac{11}{48} + \frac{5}{48}a + \frac{1}{4}\beta_1 + \frac{1}{4}\gamma_1\right) y''_{n+2}\Bigg].$$

We have $C_5 = 0$, $C_6 = (1 + a - 6\gamma_1)/720$ for $\beta_1 = 8(1 - a)/15$; especially we get the sixth order formula from 3.1 for $\gamma_1 = (1 + a)/6$ and $\beta_1 = 8(1 - a)/15$.

A_∞-stability. The necessary and sufficient conditions of the A_∞-stability (Theorem 4) take for (F2,4) the form

$$\left.\begin{aligned} -11 + 5a + 12\beta_1 + 12\gamma_1 &> 0 \\ -\frac{1}{8}(1+a) \qquad + \frac{3}{2}\gamma_1 &> 0 \\ -\frac{1}{3}(1-a) + \frac{1}{2}\beta_1 \qquad &> 0 \\ -\frac{1}{8}(1+a) \qquad - \frac{1}{2}\gamma_1 &> 0 \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} -11 + 5a + 12\beta_1 + 12\gamma_1 &< 0, \\ -\frac{1}{8}(1+a) \qquad + \frac{3}{2}\gamma_1 &< 0, \\ -\frac{1}{3}(1-a) + \frac{1}{2}\beta_1 \qquad &< 0, \\ -\frac{1}{8}(1+a) \qquad - \frac{1}{2}\gamma_1 &< 0, \end{aligned}\right.$$

These conditions are satisfied exactly in the region

$$D_\infty^{2,4} = \left\{(a, \beta_1, \gamma_1) : a > -1, \beta_1 < \frac{2}{3}(1-a), -\frac{1}{4}(1+a) < \gamma_1 < \frac{1}{12}(1+a)\right\}$$

drawn in fig. 3 with $a = 0$ and $a = 1$. When $a \to -1_+$, the boundary of $D_\infty^{2,4}$ degenerates in a half-line $\beta_1 < 4/3$; for $a = -1$ the $D_\infty^{2,4}$ is empty. We obtain A_∞-stable formulas of the fifth order for $a \in (-1, 1)$, $\beta_1 = 8(1-a)/15$ and $-\frac{1}{4}(1+a) < \gamma_1 < \frac{1}{12}(1+a)$; the formula corresponding to $\gamma_1 = (1+a)/6$ is not A_∞-stable, as already shown in 3.1.

A_0-stability. The necessary and sufficient conditions of the A_0-stability of formula (F2,4) can be written on substituting into $A - D$ as follows

A. $-11 + 5a + 12\beta_1 + 12\gamma_1 \leqq 0$;

$$\left[39 - 9a - 24\beta_1 - 36\gamma_1 \geqq 0 \text{ or } \left(\frac{39}{48} - \frac{9}{48}a - \frac{1}{2}\beta_1 - \frac{3}{4}\gamma_1\right)^2 - \left(-\frac{11}{12} + \frac{5}{12}a + \beta_1 + \gamma_1\right) < 0\right];$$

B. $1 + a \geqq 0$; $-\frac{1}{8}(1+a) + \frac{3}{2}\gamma_1 \leqq 0$: $(a, \beta_1, \gamma_1) \neq \left(1, \beta_1, \frac{1}{6}\right)$ with arbitrary β_1;

C. $1 + a \geqq 0$; $\frac{1}{3}(1-a) - \frac{1}{2}\beta_1 \geqq 0$; $(a, \beta_1, \gamma_1) \neq \left(1, 0, \frac{5}{6}\right)$;

$$\left\{\frac{5}{8}(1+a) - \frac{3}{2}\gamma_1 \geqq 0 \text{ or } \left(\frac{5}{8}(1+a) - \frac{3}{2}\gamma_1\right)^2 - 4(1-a)\left(\frac{1}{3}(1-a) - \frac{1}{2}\beta_1\right) < 0\right\};$$

D. $1 + a \geqq 0$; $-\frac{1}{8}(1 + a) - \frac{1}{2}\gamma_1 \leqq 0$; $(a, \beta_1, \gamma_1) \neq (-1, 1, 0)$;

$$\left\{1 - a - 2\beta_1 \geqq 0 \text{ or } (1 - a - 2\beta_1)^2 + 8(1 + a)\left[-\frac{1}{8}(1 + a) - \frac{1}{2}\gamma_1\right] < 0\right\}.$$

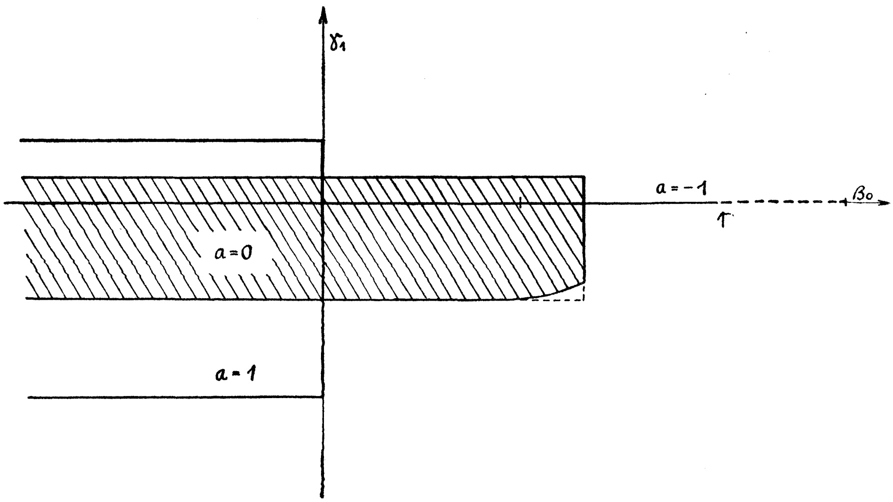
Discussing these conditions in more detail we find the region of A_0-stability relating to (F2,4) to be defined by

$$D_0^{2,4} = \left\{(a, \beta_1, \gamma_1) : -1 \leqq a \leqq 1, \beta_1 \leqq \frac{1}{2}(1 - a), -\frac{1}{4}(1 + a) \leqq \gamma_1 \leqq \frac{1}{12}(1 + a)\right\} \cup$$

$$\cup \left\{(a, \beta_1, \gamma_1) : -1 < a \leqq 1, \frac{1}{2}(1 - a) \leqq \beta_1 \leqq \frac{2}{3}(1 - a), \right.$$

$$\left. \frac{1}{4}\left[\frac{(1 - a - 2\beta_1)^2}{1 + a} - 1 - a\right] < \gamma_1 \leqq \frac{1}{12}(1 + a)\right\},$$

except the points where $(a, \beta_1, \gamma_1) = \left(1, \beta_1, \frac{1}{4}\right)$ with arbitrary β_1. The interiors of $D_0^{2,4}$ and $D_\infty^{2,4}$ are identical for $a = 1$; they differ in the neighbourhood of $\beta_1 = 2/3$, $\gamma_1 = -1/2$ (left-lower corner in fig. 3) for $a \in (-1, 1)$. We have $D_0^{2,4} = \{(a, \beta_1, \gamma_1) : a = -1, \beta_1 < 1, \gamma_1 = 0\}$ for $a = -1$. The regions $D_0^{2,4}$ and $D_\infty^{2,4}$ with $a = -1; 0; 1$ are plotted in fig. 3. We obtain the fifth order A_0-stable formulas for $\beta_1 = 8(1 - a)/15$, $(a, \beta_1, \gamma_1) \in D_0^{2,4}$; the sixth order formula corresponding to $\gamma_1 = (1 + a)/6$ is not A_0-stable. We summarize our analysis in



Theorem 7. *The A_∞, A_0-stability regions for formulas* (F2,4) *in the (β_1, γ_1)-plane are exactly the regions $D_\infty^{2,4}$, $D_0^{2,4}$ described above. None is the subset of the other. For $a \in (-1, 1)$ the corresponding formulas are stable* (*in the sense of Dahlquist*). *There are*

the fifth order A_∞, A_0*-stable formulas* (F2,4) *corresponding to the points of* $D_\infty^{2,4}$, $D_0^{2,4}$ *with* $\beta_1 \approx 8(1-a)/15$.

Formula examples.

1. $a = 0$; $\beta_1 = 8/15$; $\gamma_1 = 0$; $C_6 = 1/720$ (A_0-stable, A_∞-stable)

$$y_{n+2} - y_{n+1} = \frac{h}{240}(-19y'_n + 128y'_{n+1} + 131y'_{n+2}) - \frac{h^2}{240}(7y''_n + 23y''_{n+2})$$

2. $a = 0$; $\beta_1 = 8/15$; $\gamma_1 = 1/12$; $C_6 = 1/1440$ (A_0-stable, not A_∞-stable)

$$y_{n+2} - y_{n+1} = \frac{h}{60}(-y'_n + 32y'_{n+1} + 29y'_{n+2}) + \frac{h^2}{120}(-y''_n + 10y''_{n+2} - 9y''_{n+2})$$

3. $a = 1/2$; $\beta_1 = 4/15$; $\gamma_1 = 0$; $C_6 = 1/480$ (A_0, A_∞-stable)

$$y_{n+2} - \frac{3}{2}y_{n+1} + \frac{1}{2}y_n =$$

$$= \frac{h}{480}(-169y'_n + 128y'_{n+1} + 281y'_{n+2}) - \frac{h^2}{480}(37y''_n + 53y''_{n+2})$$

4. $a = -1/2$; $\beta_1 = 4/5$; $\gamma_1 = 1/24$; $C_6 = 1/2880$ (A_0-stable, not A_∞-stable)

$$y_{n+2} - \frac{1}{2}y_{n+1} + \frac{1}{2}y_n = \frac{h}{40}(9y'_n + 32y'_{n+1} + 19y'_{n+2}) +$$

$$+ \frac{h^2}{240}(7y''_n + 10y''_{n+1} - 17y''_{n+2})$$

5. $a = 0$; $\beta_1 = 5/8$; $\gamma_1 = -6/25$; $C_5 = -11/2880$ (A_∞-stable, not A_0-stable)

$$y_{n+2} - y_{n+1} = \frac{h}{200}(-61y'_n + 125y'_{n+1} + 136y'_{n+2}) -$$

$$- \frac{h^2}{2400}(269y''_n + 576y''_{n+1} + 319y''_{n+2})$$

6. $a = 0$, $\beta_1 = 1/2$; $\gamma_1 = 0$; $C_5 = 1/720$ (A_0, A_∞-stable)

$$y_{n+2} - y_{n+1} = \frac{h}{16}(-y'_n + 9y'_{n+2}) - \frac{h^2}{48}(y''_n + 5y''_{n+2})$$

7. $a = 0$; $\beta_1 = 8/15$; $\gamma_1 = 1/6$; $C_7 = 1/9450$; (A_0, A_∞-unstable)

$$y_{n+2} - y_{n+1} = \frac{h}{240}(11y'_n + 128y'_{n+1} + 101y'_{n+2}) + \frac{h^2}{240}(3y''_n + 40y''_{n+1} - 13y''_{n+2})$$

(Jackson–Kenue, 1974; see 3.1, example 2); stable in the Dahlquist's sense.

8. We get A_0, A_∞-stable formula unstable in the sense of Dahlquist for $a = 1$, $\beta_1 = -1/4$, $\gamma_1 = 0$.

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Shrnutí

STABILITA DIFERENČNÍCH FORMULÍ S DRUHOU DERIVACÍ

Jiří Kobza

V předložené práci se studují nutné a postačující podmínky A, A_0, A_∞ — stability diferenčních formulí s druhými derivacemi tvaru (F). Ve větách 1—7 se dokazuje, že

— u jednokrokových formulí 2. řádu splývají oblasti A_0, A-stability, oblast A_∞-stability je širší;

— u jednokrokových formulí 3. řádu je každá (s jednou výjimkou) A_0-stabilní formule též A, A_∞-stabilní;

— existují A_0, A_∞-stabilní dvoukrokové formule 4. a 5. řádu, které jsou též stabilní podle Dahlquista;
— dvoukrokové formule 6. řádu nejsou A_0, A_∞-stabilní.

Резюме

УСТОЙЧИВОСТЬ МНОГОШАГОВЫХ ФОРМУЛ С ВТОРЫМИ ПРОИЗВОДНЫМИ

Иржи Кобза

В работе изучаются необходимые и достаточные условия A, A_0, A_∞ — устойчивости многошаговых формул с вторыми производными типа (F). В теоремах 1—7 показывается, что
— у одношаговых формул второго порядка совпадают области A_0, A — устойчивости, область A_∞ — устойчивости более широкая;
— у одношаговых формул третьего порядка всякая (с одним исключением) A_0 — устойчивая формула является тоже A, A_∞ — устойчивой;
— существуют A_0, A_∞ — устойчивые формулы четвертого и пятого порядка, устойчивые по Дальквисту;
— двухшаговые формулы шестого порядка A_0, A_∞ — неустойчивы.