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A MODIFICATION OF THE STURM'S THEOREM ON SEPARATING ZEROS OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF THE 2ND ORDER

MIROSLAV LAITOCH

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Let us consider a linear nonhomogeneous differential equation of the 2nd order

$$(r) \qquad y'' - q(t) \,.\, y = r(t),$$

where the functions $q \in C^{(2)}$, $q < 0$, $r \in C^{(0)}$ represent the functions of the variable t in an open interval j. We shall suppose the solutions of the corresponding homogenesus equation

$$(q) \qquad y'' = q(t) \,.\, y$$

to be oscillatory towards both end points of the interval j (see [1], p. 4).

Trivial solutions of the differential equation (q) will be excluded from our considerations. The symbols (r), (q) denote either the given differential equation or the set of solutions of the differential equation in question.

The set of all real numbers will be denoted by $\boldsymbol{R}$.

Concurrently with O. Borůvka we make use of the following concepts: conjugate numbers with respect to the differential equation (q) (see [1], p. 14 and onwards), the basic and the n-th central dispersions corresponding to the differential equation (q) (see [1], p. 106 and onwards).

Conjugate numbers: Let $t \in j$ be arbitrary and let u, $v \in (q)$ be arbitrary solutions such that $u(t) = 0$, $v'(t) = 0$. We call a number $x \in j(x \neq t)$ as being conjugate with the number t with respect to the differential equation (q) and more precisely of the 1st kind, 2nd kind, 3rd kind, 4th kind according as $u(x) = 0$, $v'(x) = 0$, $u'(x) = 0$, $v(x) = 0$.

We observe that the number $x(\neq t)$ is a conjugate number with t of the 1st, 2nd, 3rd or 4th kind according as it is a zero of the function u, v', u', v respectively. If the number x is the n-th zero ($n = 1, 2, \ldots$) of this function lying on the left or right of t, then we call it the n-th left or right conjugate number with t.

Dispersions: Let φ, ψ, χ, ω be the fundamental dispersions of the first, second, third and fourth kinds corresponding to the differential equation (q). For $n = 0, \pm 1, \pm 2, \ldots$ let φ_n and ψ_n be the n-th central dispersions of the first and second kinds respectively. For $n = \pm 1, \pm 2, \ldots$ let χ_n and ω_n be the n-th central dispersions of the third and fourth kinds, respectively. Thereby $\varphi_1 = \varphi$, $\psi_1 = \psi$, $\chi_1 = \chi$, $\omega_1 = \omega$; $\varphi_0 \equiv \psi_0 \equiv t$.

From the definition of the conjugate numbers we get the following

Lemma 1. *Let $t_0 \in j$ and let $u, v \in (q)$ be particular solutions with $u(t_0) = 0$, $v'(t_0) = 0$. Then for $n = 0, \pm 1, \pm 2, \ldots$ we have*

$$u[\varphi_n(t_0)] = 0 \quad \text{and} \quad u(t) \neq 0 \quad \text{for } t \in j, t \neq \varphi_n(t_0),$$
$$v[\psi_n(t_0)] = 0 \quad \text{and} \quad v'(t) \neq 0 \quad \text{for } t \in j, t \neq \psi_n(t_0)$$

and for $n = \pm 1, \pm 2, \ldots$ *we have*

$$u'[\chi_n(t_0)] = 0 \quad \text{and} \quad u'(t) \neq 0 \quad \text{for } t \in j, t \neq \chi_n(t_0),$$
$$v[\omega_n(t_0)] = 0 \quad \text{and} \quad v(t) \neq 0 \quad \text{for } t \in j, t \neq \omega_n(t_0).$$

The properties of solutions of the differential equation (r): From the theory of linear differential equations ([3], p. 216) we know the following

Lemma 2. *If v is an arbitrary particular solution of the differential equation (r), then its general solution y represents the sum of this particular solution and of the general solution u of (q), that is $y = u + v$.*

Let us recall the following information of

Lemma 3: *If v_1, $v_2 \in (r)$ are particular solutions, then the function $v_2 - v_1$* is a particular solution of the differential equation (q).

The properties of conjugate numbers of the 1st and 2nd kinds make it possible to express the following theorems:

Theorem 1. *Let $t_0 \in j$, $v_0 \in \boldsymbol{R}$ be arbitrary numbers. Let $v_1, v_2 \in (r)$ be arbitrary particular solutions. If $v_1(t_0) = v_2(t_0) = v_0$, then we have $v_1[\varphi_n(t_0)] = v_2[\varphi_n(t_0)]$ and $v_1(t) \neq v_2(t)$ for $t \in j$, $t \neq \varphi_n(t_0)$ holding for every $n = 0, \pm 1, \pm 2, \ldots$.*

Proof. Let us set $u = v_2 - v_1$ for $t \in j$. Then $u \in (q)$. At the point t_0 we have $u(t_0) = v_2(t_0) - v_1(t_0) = 0$. According to Lemma 1 we have $u[\varphi_n(t_0)] = 0$ and $u(t) \neq 0$ for $t \in j$, $t \neq \varphi_n(t_0)$ holding for every $u = 0, \pm 1, \pm 2, \ldots$. Consequently $0 = u[\varphi_n(t_0)] = v_2[\varphi_n(t_0)] - v_1[\varphi_n(t_0)]$ and $0 \neq u(t) = v_2(t) - v_1(t)$ for $t \in j$, $t \neq \varphi_n(t_0)$ and from this we obtain the statement of the theorem.

Convention: By the common points of the two different solutions (of the first derivatives of solutions) of the differential equation (r) or (q) we mean the common points of the graphs of those solutions (of the graphs of the first derivatives of those solutions).

Corollary 1. *Let $t_0 \in j$, $v_0 \in \boldsymbol{R}$ be arbitrary numbers. All solutions $v \in (r)$ for which $v(t_0) = v_0$ possess precisely the points $[\varphi_n(t_0), v[\varphi_n(t_0)]]$ in common, where $n = 0, \pm 1, \pm 2, \dots$.*

Theorem 2. *Let $t_0 \in j$, $v_0' \in \boldsymbol{R}$ be arbitrary numbers. Let $v_1, v_2 \in (r)$ be arbitrary particular solutions. If $v_1'(t_0) = v_2'(t_0) = v_0'$, then we have $v_1'[\psi_n(t_0)] = v_2'[\psi_n(t_0)]$ and $v_1'(t) \neq v_2'(t)$* for $t \in j$, $t \neq \psi_n(t_0)$ *holding for every $n = 0, \pm 1, \pm 2, \dots$.*

Proof. Let us put $u = v_2 - v_1$ for $t \in j$. Then $u \in (q)$. At the point t_0 we have $u'(t_0) = v_2'(t_0) - v_1'(t_0) = 0$. According to Lemma 1 we have $u'[\psi_n(t_0)] = 0$ and $u'(t) \neq 0$ for $t \in j$, $t \neq \psi_n(t_0)$ holding for every $n = 0, \pm 1, \pm 2, \dots$. Consequently we have $0 = u'[\psi_n(t_0)] = v_2'[\psi_n(t_0)] - v_1[\psi_n(t_0)]$ and $0 \neq u'(t) = v_2'(t) - v_1'(t)$ for every $t \in j$, $t \neq \psi_n(t_0)$ and from this we obtain the statement of the theorem.

Corollary 2. *Let $t_0 \in j$, $v_0' \in \boldsymbol{R}$ be arbitrary numbers. All solutions $y \in (r)$ whose first derivatives satisfy the condition $v'(t_0) = v_0'$ have the property by which their derivatives v' possess precisely the points $[\varphi_n(t_0), v'[\psi_n(t_0)]]$ in common,* where $n = 0, \pm 1, \pm 2, \dots$

For convenience, we introduce the following definitions:

Let $t_0 \in j$; $v_0, v_0' \in \boldsymbol{R}$ represent arbitrary numbers and $v \in (r)$ is a particular solution for which $v(t_0) = v_0$ or $v'(t_0) = v_0$. Let $\varphi_n(t)$ and $\psi_n(t)$ stand for the n-th central dispersions of the 1st and 2nd kindsr espectively, corresponding to the differential equation (q), where $n = 0, +1, +2, \dots$

Definition 1. The sets of all points $[\varphi_n(t_0), v[\varphi_n(t_0)]]$ and $[\psi_n(t_0), v'[\psi_n(t_0)]]$ for $n = 0, +1, +2, \dots$ will be called respectively the systems of knots of the 1st and 2nd kinds, corresponding to the differential equation (r) and to the initial conditions $(t_0; v_0)$ and $(t_0; v_0')$ and will be symbolized by $\mathscr{S}(t_0; v_0)$ and $\mathscr{T}(t_0; v_0')$, respectively.

It should be noted that the sets of knots $\mathscr{S}(t_0; v_0)$ and $\mathscr{T}(t_0; v_0')$ are uniquely determined by any point fıom the sets of points $[\varphi_n(t_0), v\,[\varphi_n(t_0)]]$ and $[\psi_n(t_0), v'[\psi_n(t_0)]]$, discussed by definition 1.

Let $\mathscr{S}(t_0; v_0)$ and $\mathscr{T}(t_0; v_0')$ be the systems of knots of the 1st and 2nd kinds appropriate to the differential equation (r) and to the initial conditions $(t_0; v_0)$ and $(t_0; v_0')$, respectively. Let $t_1, t_2 \in j$; $v_1, v_2, v_1', v_2' \in \boldsymbol{R}$ be such numbers that the points $[t_1, v_1], [t_2, v_2] \in \mathscr{S}(t_0; v_0)$, and $[t_1, v_1'], [t_2, v_2'] \in \mathscr{T}(t_0; v_0')$.

Definition 2. The points $[t_1, v_1]$, $[t_2, v_2]$ and $[t_1, v_1']$, $[t_2, v_2']$ will be called respectively the neighbouring knots of the 1st and 2nd kinds corresponding to the differential equation (r) and to the initial conditions $(t_0; v_0)$ and $(t_0; v_0')$ if the numbers t_1 and t_2 are the neighbouring numbers of the 1st and 2nd kinds corresponding to the differential equation (q), respectively.

Let $t_0 \in j$; $v_0, v_0' \in \boldsymbol{R}$ be arbitrary numbers.

Definition 3. By a bundle of solutions of the 1st or 2nd kind appropriate to the differential equation (r) and to the initial condition $(t_0; v_0)$ or $(t_0; v_0')$ we mean all solutions $v \in (r)$ satisfying the condition $v(t_0) = v_0$ or whose first derivatives satisfy the condition $v'(t_0) = v_0'$.

Let us remark that the bundle of solutions of the 1st or 2nd kind appropriate to the differential equation (r) and to the initial condition $(t_0; v_0)$ or $(t_0; v_0')$ is uniquely determined by any knot from the system of knots of the 1st kind $\mathscr{S}(t_0; v_0)$ or of the 2nd kind $\mathscr{T}(t_0; v_0')$.

Theorem 3. *Let $\mathscr{S}(t_0; v_0)$ be a system of knots of the 1st kind appropriate to the differential equation (r) and to the initial* condition $(t_0; v_0)$. *Let $t_1, t_2 \in j$, $t_1 < t_2$; $v_1, v_2 \in \boldsymbol{R}$ be such numbers, where the points $[t_1, v_1]$, $[t_2, v_2]$ are two neighbouring knots of the 1st kind from the system $\mathscr{S}(t_0, v_0)$.* Let $v \in (r)$ *be such a solution for which* $v(t_0) = (v_0)$.

If $\bar{v} \in (r)$ is a solution not passing through these knots, then there exists precisely one number τ in the interval (t_1, t_2) such that $[\tau, v(\tau)] = [\tau, \bar{v}(\tau)]$.

Proof. It follows from our assumption that $v(t_i) \neq \bar{v}(t_i)$, $i = 1, 2$. Consequently the function $u = v(t) - \bar{v}(t)$ is a solution of the differential equation (q), for which $u(t_i) \neq 0$, $i = 1, 2$. According to the Sturm theorem ([3], p. 276) the solution u has precisely one zero in the interval (t_1, t_2), which we will mark by τ. Thus $0 = u(\tau) = v(\tau) - \bar{v}(\tau)$ and from this we get $v(\tau) = \bar{v}(\tau)$. Hence the point $[\tau, v(\tau)] = [\tau, \bar{v}(\tau)]$ is the only common point of the solution $v, \bar{v} \in (r)$ for $t_1 < t < t_2$.

Corollary 3. *The solutions $v, \bar{v} \in (r)$ discussed in Theorem 3 belong to the same bundle of solutions of the 1st kind appropriate to the differential equation (r) and to the initial condition $(\tau; v_\tau)$, where $v_\tau = v(\tau) = \bar{v}(\tau)$.*

By an analogous method to that used above we can prove the Theorem below if we apply the statement of the Sturm's theorem to the associated equation of the differential equation (q) ([1], page 6).

Theorem 4. *Let $\mathscr{T}(t_0; v_0')$ be a system of knots of the 2nd kind appropriate to the differential equation (r) and to the initial condition $(t_0; v_0')$. Let $t_1, t_2 \in j$, $t_1 < t_2$; $v_1', v_2' \in \boldsymbol{R}$ be such numbers that the points $[t_1, v_1']$, $[t_2, v_2']$ are the two neighbouring knots of the 2nd kind from the system $\mathscr{T}(t_0; v_0')$. Let $v \in (r)$ be such a solution for which $v'(t_0) = v_0'$.*

If $\bar{v} \in (r)$ is a solution whose first derivative $\bar{v}'$ is not passing through these knots, then there exists precisely one number τ in the interval (t_1, t_2), such that $[\tau, v'(\tau)] = [\tau, \bar{v}'(\tau)]$.

Corollary 4. *The solutions $v, \bar{v} \in (r)$ discussed by Theorem 4 belong to the same bundle of solutions of the 2nd kind corresponding to the differential equation (r) and to the initial condition $(\tau; v_\tau')$, where $v_\tau' = v'(\tau) = \bar{v}'(\tau)$.*

With respect to our assumption that the right side of the differential equation (q) is $r \in C^{(0)}$ only, the preceding considerations are valid even under the assumption that $r(t) = 0$ in j without changing the statements of the theorems. Theorems 3 and 4 give thus

1. in case of $r(t) \equiv 0$ in j

the generalization of the Sturm's theorem on separating zeros of solutions or of zeros of the first derivatives of solutions of a 2nd order linear homogeneous differential equation (q), and

2. in case of $r(t) \equiv 0$ in j; $v_0 = 0$ or $v'_0 = 0$
the Sturm's theorem on separating zeros or of zeros of the first derivatives of the solutions of a 2nd order linear homogenous differential equation (q).

We now apply the Theorem on bilinear relations between the solutions of the differential equation (q) ([1], p. 24) which enables us on taking account of Lemma 3, to express the following theorem for three particular solutions of the differential equation (r).

Theorem 5. *For three particular solutions v_1, v_2, v_3 of the differential equation (r) with $v_1 - v_3 \neq k.(v_2 - v_3)$ in the interval j, where $k \neq 0$ is a constant, the following equalities hold at two different points $t, x \in j$:*

$$\text{(I)}\ [v_1(t) - v_3(t)]\,.\,[v_2(x) - v_3(x)] = [v_2(t) - v_3(t)]\,.\,[v_1(x) - v_3(x)],$$
$$\text{(II)}\ [v'_1(t) - v'_3(t)]\,.\,[v'_2(x) - v'_3(x)] = [v'_2(t) - v'_3(t)]\,.\,[v'_1(x) - v'_3(x)],$$
$$\text{(III)}\ [v_1(t) - v_3(t)]\,.\,[v'_2(x) - v'_3(x)] = [v_2(t) - v_3(t)]\,.\,[v'_1(x) - v'_3(x)],$$
$$\text{(IV)}\ [v'_1(t) - v'_3(t)]\,.\,[v_2(x) - v_3(x)] = [v'_2(t) - v'_3(t)]\,.\,[v_1(x) - v_3(x)],$$

if and only if

in (I) *t, x are* 1*-conjugate numbers with respect to (q),*
in (II) *t, x are* 2*-conjugate numbers with respect to (q),*
in (III) *x is a* 3*-conjugate number with t with respect to (q),*
in (IV) *x is a* 4*-conjugate number with t with respect to (q).*

Proof.

(I) Let the bilinear relation (I) hold between two distinct numbers $t, x \in j$. Then the linear equations with the unknowns c_1, c_2:

$$c_1[v_1(t) - v_3(t)] + c_2[v_2(t) - v_3(t)] = 0,$$
$$c_1[v_1(x) - v_3(x)] + c_2[v_2(x) - v_3(x)] = 0$$

are satisfied for appropriate constants $c_1, c_2, c_1^2 + c_2^2 \neq 0$ since – as it follows from (I) – the determinant of this system vanishes. The numbers t and x represent zeros of solutions $y = c_1(v_1 - v_3) + c_2(v_2 - v_3)$ of the differential equation (q).

If, conversely, t and x are conjugate numbers of the 1st kind with respect to (q), then $t \neq x$ and there is a solution $y = c_1(v_1 - v_3) + c_2(v_2 - v_3)$ of the differential equation (q) vanishing at t and x, where $c_1^2 + c_2^2 \neq 0$. This is:

$$c_1[v_1(t) - v_3(t)] + c_2[v_2(t) - v_3(t)] = 0,$$
$$c_1[v_1(x) - v_3(x)] + c_2[v_2(x) - v_3(x)] = 0.$$

From this follows with respect to the condition $c_1^2 + c_2^2 \neq 0$, the bilinear relation (I).

We proceed analogously in proving relations (II), (III) and (IV).

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SOUHRN

MODIFIKACE STURMOVY VĚTY O ODDĚLOVÁNÍ NULOVÝCH BODŮ ŘEŠENÍ LINEÁRNÍ DIFERENCIÁLNÍ ROVNICE 2. ŘÁDU

MIROSLAV LAITOCH

Uvažujme lineární diferenciální rovnici 2. řádu s pravou stranou

$$(r) \qquad y'' - q(t) . y = r(t),$$

kde funkce $q \in C^{(2)}$, $q < 0$, $r \in C^{(0)}$ jsou funkce proměnné t v otevřeném intervalu j. O příslušné rovnici zkrácené

$$(q) \qquad y'' = q(t) . y$$

předpokládáme, že její řešení k oběma krajním bodům intervalu j oscilují.

Značíme φ resp. ψ, χ, ω základní centrální disperzi 1. resp. 2., 3., 4. druhu příslušnou k diferenciální rovnici (q). Pro $n = 0, \pm 1, \pm 2, \dots$ značíme φ_n, resp. ψ_n n-tou centrální disperzi 1., resp. 2. druhu a pro $n = \pm 1, \pm 2, \dots$ značíme χ_n, resp. ω_n n-tou centrální disperzi 3., resp. 4. druhu. Přitom je $\varphi_1 = \varphi, \psi_1 = \psi, \chi_1 = \chi, \omega_1 = \omega$.

Buď $\boldsymbol{R}$ množina všech reálných čísel.

Nechť $t_0 \in j$, $v_0, v_0' \in \boldsymbol{R}$ jsou libovolná čísla a $v \in (r)$ je partikulární řešení, pro něž $v'(t_0) = v_0$, resp. $v'(t_0) = v_0'$. Nechť φ_n, resp. ψ_n značí n-tou centrální dispersi 1. resp. 2. druhu příslušnou k diferenciální rovnici (q), kde $n = 0, \pm 1, \pm 2, \dots$

Množinu všech bodů $[\varphi_n(t_0), v[\varphi_n(t_0)]]$, resp. $[\varphi_n(t_0), v[\varphi_n(t_0)]]$ pro $n = 0, \pm 1, \pm 2, \dots$ nazýváme systémem uzlů 1., resp. 2. druhu příslušným k diferenciální rovnici (r) a k počáteční podmínce $(t_0; v_0)$, resp. $(t_0; v_0')$ a značíme $\mathscr{S}(t_0; v_0)$, resp. $\mathscr{T}(t_0; v_0)$.

Pomocí pojmu uzlů zobecňuje se Sturmova věta o oddělování nulových bodů řešení, resp. derivace řešení zkrácené rovnice (q).

РЕЗЮМЕ

МОДИФИКАЦИЯ ТЕОРЕМЫ ШТУРМА О НУЛЕВЫХ ТОЧКАХ РЕШЕНИЯ ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 2-ГО ПОРЯДКА

МИРОСЛАВ ЛАЙТОХ

Рассматривается линейного дифференциальное уравнение 2-го порядка неоднородное

$$(r) \qquad y'' - q(t) \,.\, y = r(t),$$

где функции $q \in C^{(2)}$, $q < 0$, $r \in C^{(0)}$ являются функциями переменного t на открытом интервале j. О соответствующем однородном уравнении

$$(q) \qquad y'' = q(t) \,.\, y$$

предполагаем, что его решения осцилируют к обоим концам интервала j.

$\varphi, \psi, \chi, \omega$ обозначают соответственно основную центральную дисперсию 1., 2., 3., 4. рода, соответствующую дифференциальному уравнению (q).

Для $n = 0, \pm 1, \pm 2 \ldots$ обозначаем φ_n, ψ_n соответственно n-тую центральную дисперсию 1., 2. рода и для $n = \pm 1, \pm 2, \ldots$ обозначаем χ_n, ω_n соответственно n-тую центральную дисперсию 3., 4. рода. Далее $\varphi_1 = \varphi$, $\psi_1 = \psi$, $\chi_1 = \chi$, $\omega_1 = \omega$ и $\boldsymbol{R}$-множество всех вещественных чисел.

Пусть $t_0 \in j$, v_0, $v'_0 \in \boldsymbol{R}$ есть произвольные числа и $v \in (r)$ частное решение, для которого $v(t_0) = v_0$, $v'(t_0) = v'_0$.

Пусть φ_n, ψ_n соответственно означают n-тую центральную дисперсию 1., 2. рода соответствующие дифференциальному уравнению (q), где $n = 0, \pm 1, \pm 2, \ldots$.

Множество точек $[\varphi_n(t_0), v(\varphi_n(t_0))]$, $[\psi_n(t_0)), v(\psi_n(t_0))]$ для $n = 0, \pm 1, \pm 2, \ldots$ называем соответственно системой узлов 1., 2. рода, соответствующая дифференциальному уравнению (r) и соответствующих начальному условию $(t_0; v_0)$, $(t_0; v'_0)$ и обозначаем соответственно $\mathscr{S}(t_0; v_0)$, $\mathscr{T}(t_0; v'_0)$.

При помощи понятия узлов обобщается теорема Штурма о взаимном разделении нулевых точек решения или производной решения однородного дифференциального уравнения (q).