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Svatoslav Staněk

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ON AN APPLICATION OF THE GENERALIZED FLOQUET THEORY TO THE TRANSFORMATION OF THE EQUATION $y'' = q(t)y$ INTO ITS ASSOCIATED EQUATION

SVATOSLAV STANĚK

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Dedicated to Academician O. Borůvka on his 80th birthday

1. Introduction

O. Borůvka established in [1] functions X such that for every solution u of the both-sided oscillatory equation (q): $y'' = q(t)y$, $q \in C^2(\mathbf{R})$ ($\mathbf{R} = (-\infty, \infty)$), $q(t) < 0$ for $t \in \mathbf{R}$, the function $\sqrt{-q(t)}\,\dfrac{uX(t)}{\sqrt{|X'(t)|}}$ equals (on $\mathbf{R}$) to the derivative of a solution of (q). Such functions are called the dispersions (of the 4th kind) of (q). Let us say that the (generally complex) number τ is a characteristic multiplier of (q) relative to the dispersion X, when there exists a (nontrivial) solution u of (q) for which $\dfrac{uX(t)}{\sqrt{|X'(t)|}} = \tau\,\dfrac{u'(t)}{\sqrt{-q(t)}}$, $t \in \mathbf{R}$. The author proves in this paper that such a number τ is a root of a certain quadratic algebraic equation and is expressed in terms of phases and dispersions of (q).

2. Basic definitions, notations and relations

In what follows we investigate a differential equation

$$y'' = q(t)\,y, \tag{q}$$

where $q \in C^2(\mathbf{R})$, $q(t) < 0$ for $t \in \mathbf{R}$ being both-sided oscillatory on $\mathbf{R}$. The trivial solution of (q) will be excluded from our consideration.

Let u, v be independent solutions of (q). Say that a function $\alpha : \mathbf{R} \to \mathbf{R}$, $\alpha \in C^\circ(\mathbf{R})$ is the 1st phase of the basis (u, v) of (q) if

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)} \qquad \text{on } \mathbf{R} - \{t \in \mathbf{R}, v(t) = 0\}.$$

A function $\beta : \mathbf{R} \to \mathbf{R}$, $\beta \in C^\circ(\mathbf{R})$ is called the 2nd phase of the basis (u, v) of (q) if

$$\operatorname{tg} \beta(t) = \frac{u'(t)}{v'(t)} \qquad \text{on } \mathbf{R} - \{t \in \mathbf{R}, v'(t) = 0\}.$$

Functions α and β are called the 1st and 2nd phases of (q), respectively, when there exists a basis (u, v) of (q), where α and β are its 1st and 2nd phases respectively. Let α and β be 1st and 2nd phases of a basis (u, v) of (q) and $w = uv' - u'v$. Then $\operatorname{sign} \alpha' = \operatorname{sign} \beta'$ and

$$u(t) = \sigma \sqrt{|w|} \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}}, \qquad v(t) = \sigma \sqrt{|w|} \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}},$$

$$u'(t) = \sigma' \sqrt{|w \,.\, q(t)|} \frac{\sin \beta(t)}{\sqrt{|\beta'(t)|}}, \qquad v'(t) = \sigma' \sqrt{|w \,.\, q(t)|} \frac{\cos \beta(t)}{\sqrt{|\beta'(t)|}},$$

$$(\sigma, \sigma' = \pm 1).$$

In case of $\sigma = \sigma'$ we say that the phases α and β have the same signature.

The set of the 1st phases of the equation $y'' = -y$ will be written as $\mathfrak{E}$. For every $\varepsilon \in \mathfrak{E}$ we have: $\varepsilon(t + \pi) = \varepsilon(t) + \pi \,.\, \operatorname{sign} \varepsilon'$, $t \in \mathbf{R}$.

We set $q_1(t) := q(t) + \sqrt{-q(t)} \left(\frac{1}{\sqrt{-q(t)}} \right)''$ for $t \in \mathbf{R}$. The equation $(\mathrm{q}_1) : y'' = q_1(t) y$ is called the associated equation to (q). There holds the following relation between the sets of solutions of (q) and (q_1): If u is a solution of (q), then the function $\frac{u'(t)}{\sqrt{-q(t)}}$ is a solution of (q_1) (on $\mathbf{R}$) and vice verza: if u_1 is a solution of (q_1), then $u_1(t) \sqrt{-q(t)}$ is the derivative of the only one solution of (q). It immediately follows from this that every second phase of (q) is the first phase of (q_1).

Let $n \neq 0$ be an integer and $x_0 \in \mathbf{R}$. Further let u be a solution of (q) and $u'(x_0) = 0$. Let $\psi_n(x_0)$ $(\omega_n(x_0))$ stand for the n^{th} zero of $u'(u)$ lying to the right or to the left from the point x_0 according as n is an positive or an negative integer. The functions ψ_n, ω_n $(n = \pm 1, \pm 2, \ldots)$ are defined on $\mathbf{R}$, $\psi_n(\mathbf{R}) = \mathbf{R}$, $\omega_n(\mathbf{R}) = \mathbf{R}$. The functions ψ_n and ω_n are respectively called the central dispersions of the 2nd and 4th kind of (q) with the index n. We set $\psi_0(t) \equiv t$.

Let α and β be respectively a 1st and a 2nd phases of a basis (u, v) of (q). Then

$$\beta \psi_n(t) = \beta(t) + n\pi \,.\, \operatorname{sign} \beta'$$

and if $0 < \beta(t) - \alpha(t) < \pi$ for $t \in \mathbf{R}$, then

$$\alpha\omega_n(t) = \beta(t) + \frac{1}{2}((2n - \operatorname{sign} n)\operatorname{sign}\beta' - 1)\pi \qquad (n = \pm 1, \pm 2, \ldots).$$

Between the functions ψ_k and ω_n it holds

$$\omega_n \psi_k = \begin{cases} \omega_{n+k} & \text{for } n > 0, n + k > 1 \quad \text{and} \quad n < 0, n + k \leqq -1; \\ \omega_{n+k-1} & \text{for } n > 0, n + k > 0; \\ \omega_{n+k+1} & \text{for } n < 0, n + k > 0. \end{cases} \tag{1}$$

A function $X \in C^3(\mathbf{R})$, $X' \neq 0$ representing a solution of the nonlinear differential equation

$$\sqrt{|X'|}\left(\frac{1}{\sqrt{|X'|}}\right)'' + X'^2 \,.\, q(X) = q_1(t) \tag{qq$_1$}$$

is called the dispersion (of the 4th kind) of (q). Let α and β be respectively a first and a second phase of (q). Then $\alpha^{-1}\mathfrak{E}\beta := \{\alpha^{-1}\varepsilon\beta, \varepsilon \in \mathfrak{E}\}$ is the set of all dispersions of (q).

Let X be a dispersion of (q) and let u be a solution of (q). Then there exists a solution v of (q) for which we have

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = \frac{v'(t)}{\sqrt{-q(t)}}, \qquad t \in \mathbf{R}.$$

The functions ω_n ($n = \pm 1, \pm 2, \ldots$) are solutions of (qq$_1$) and therefore they are also the dispersions of (q) and we have for every solution u of (q):

$$\frac{u\omega_n(t)}{\sqrt{\omega_n'(t)}} = (-1)^n \frac{u'(t)}{\sqrt{-q(t)}}, \qquad t \in \mathbf{R}. \tag{2}$$

All the above definitions and properties were stated and proved in [1] and [2].

3. Preparatory lemmas

Let X be a dispersion (of the 4th kind) of (q) and let u, v be independent solutions of (q). Then $\frac{uX(t)}{\sqrt{|X'(t)|}}$, $\frac{vX(t)}{\sqrt{|X'(t)|}}$ are independent solutions of (q$_1$) and thus

$$\begin{aligned} \frac{uX(t)}{\sqrt{|X'(t)|}} &= a_{11}\frac{u'(t)}{\sqrt{-q(t)}} + a_{12}\frac{v'(t)}{\sqrt{-q(t)}}, \\ \frac{vX(t)}{\sqrt{|X'(t)|}} &= a_{21}\frac{u'(t)}{\sqrt{-q(t)}} + a_{22}\frac{v'(t)}{\sqrt{-q(t)}}, \end{aligned} \tag{3}$$

where $\det a_{ij} \neq 0$ and a_{ij} $(i, j = 1, 2)$ are real numbers. Let y be such a solution of (q), where

$$\frac{yX(t)}{\sqrt{|X'(t)|}} = \tau \cdot \frac{y'(t)}{\sqrt{-q(t)}}, \qquad t \in \mathbf{R},$$

and τ is a (generally complex) number. Then τ is a root of the quadratic equation

$$\varrho^2 - (a_{11} + a_{22})\,\varrho + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{4}$$

The coefficients of the above equation do not depend on the choice of the independent solutions u, v of (q).

Equation (4) and its roots are called respectively the *characteristic equation* of (q) and the *characteristic multipliers* of (q) *relative to the dispersion X.*

Let n be an integer, $n \neq 0$, and let ω_n be the central dispersion of the 4th kind of (q) with the index n. Let us say that x, $x \in \mathbf{R}$, denotes a *number of type n* of (q) relative to the dispersion X if $X(x) = \omega_n(x)$.

In what follows let α and β represent respectively the 1st and 2nd phases of a basis (u, v) of (q) with $0 < \beta(t) - \alpha(t) < \pi$ for $t \in \mathbf{R}$.

Lemma 1. *Let* $\operatorname{sign} X' = 1$. *Then all numbers of* (q) *relative to the dispersion X (if any) are of the same type.*

Proof. Let $X = \alpha^{-1}\varepsilon\beta$, $\varepsilon \in \mathfrak{E}$. Then $\operatorname{sign} \varepsilon' = 1$. Let x and y be respectively numbers of the type n and m of (q) relative to the dispersion X, $n \neq m$; $X(x) = \omega_n(x)$, $X(y) = \omega_m(y)$. This yields $\varepsilon\beta(x) = \beta(x) + \frac{1}{2}((2n - \operatorname{sign} n) \operatorname{sign} \beta' - 1)\,\pi$, $\varepsilon\beta(y) = \beta(y) + \frac{1}{2}((2m\text{-}\operatorname{sign} m) \operatorname{sign} \beta' - 1)\,\pi$. The first equality gives further $t + \frac{1}{2}((2n\text{-}\operatorname{sign} n) \operatorname{sign} \beta' - 1)\,\pi - \pi < \varepsilon(t) < t + \frac{1}{2}((2n\text{-}\operatorname{sign} n) \operatorname{sign} \beta' - 1)\,\pi + \pi$ for $t \in \mathbf{R}$. From this we get $\beta(y) + \frac{1}{2}((2n\text{-}\operatorname{sign} n) \operatorname{sign} \beta' - 1)\,\pi - \pi < \varepsilon\beta(y) = \beta(y) + \frac{1}{2}((2m\text{-}\operatorname{sign} m) \operatorname{sign} \beta' - 1)\,\pi < \beta(y) + \frac{1}{2}((2n\text{-}\operatorname{sign} n) \operatorname{sign} \beta' - 1)\,\pi + \pi$. Hence $-1 < \left((m - n) + \frac{1}{2}(\operatorname{sign} m - \operatorname{sign} n)\right) \operatorname{sign} \beta' < 1$ and $|m - n + \frac{1}{2}(\operatorname{sign} m\text{-}\operatorname{sign} n)| < 1$. If $\operatorname{sign} m = \operatorname{sign} n$, then $|m - n| < 1$, that is $n = m$. If $\operatorname{sign} m \neq \operatorname{sign} n$, then $|\operatorname{sign} m - \operatorname{sign} n| = 2$, $|m - n| = |m| + |n| > 2$ which contradicts $|m - n + \frac{1}{2}(\operatorname{sign} m - \operatorname{sign} n)| < 1$.

Corollary 1. *Let* $\operatorname{sign} X' = 1$. *If there exists a number of type n of* (q) *relative to the dispersion X, then*

$$
\begin{aligned}
&\omega_{n-1}(t) < X(t) < \omega_{n+1}(t) && \textit{for } n \neq \pm 1; \\
&\omega_{-1}(t) < X(t) < \omega_2(t) && \textit{for } n = 1; \\
&\omega_{-2}(t) < X(t) < \omega_1(t) && \textit{for } n = -1.
\end{aligned} \tag{5}
$$

Proof. Let there be a number of type n of (q) relative to the dispersion X. If (5) does not hold, then from the continuity of the function X we get the existence of a number of type m of (q) relative to the dispersion X, $n \neq m$, which conflicts with Lemma 1.

Lemma 2. *Let* sign $X' = 1$ *and let x be a number of type n of* (q) *relative to the dispersion X. Then $\psi_i(x)$ are numbers of type n of* (q) *relative to the dispersion X for every integer i.*

Proof. Let $X = \alpha^{-1}\varepsilon\beta$, $\varepsilon \in \mathfrak{E}$. Then sign $\varepsilon' = 1$. Further $X(x) = \omega_n(x)$, thus $\varepsilon\beta(x) = \beta(x) + \frac{1}{2}((2n\text{-sign } n) \text{ sign } \beta' - 1)\pi$. Let i be an integer. Then $X\psi_i(x) = \alpha^{-1}\varepsilon\beta\psi_i(x) = \alpha^{-1}\varepsilon(\beta(x) + ni\pi \,.\, \text{sign } \beta') = \alpha^{-1}(\varepsilon\beta(x) + ni\pi \,.\, \text{sign } \beta') = \alpha^{-1}\left[\beta(x) + \frac{1}{2}((2n + 2i - \text{sign } n) \text{ sign } \beta' - 1)\pi\right] = \alpha^{-1}\left[\beta(x) + \frac{1}{2}((2(n + i) - \text{sign } (n + i)) \text{ sign } \beta' - 1 + (\text{sign } (n + i) - \text{sign } n) \text{ sign } \beta')\pi\right]$. The rest of the proof we will break up into five parts:

(i) if sign $(n + i) =$ sign n, then $X\psi_i(x) = \alpha^{-1}\left[\beta(x) + \frac{1}{2}((2(n + i) - \text{sign } (n + i)) \text{ sign } \beta' - 1)\pi\right] = \omega_{n+i}(x)$,

(ii) if sign $(n + i) = 0$ and sign $n = -1$, then $X\psi_i(x) = \alpha^{-1}\left[\beta(x) + \frac{1}{2}(\text{sign } \beta' - 1)\pi\right] = \omega_1(x)$,

(iii) if sign $(n + i) = 0$ and sign $n = 1$, then $X\psi_i(x) = \alpha^{-1}\left[\beta(x) - \frac{1}{2}(\text{sign } \beta' + 1)\pi\right] = \omega_{-1}(x)$,

(iv) if sign $(n + i) = 1$ and sign $n = -1$, then $X\psi_i(x) = \alpha^{-1}\left[\beta(x) + \frac{1}{2}((2(n + i + 1) - \text{sign } (n + i + 1)) \text{ sign } \beta' - 1)\pi\right] = \omega_{n+i+1}$,

(v) if sign $(n + i) = -1$ and sign $n = 1$, then $X\psi_i(x) = \alpha^{-1}\left[\beta(x) + \frac{1}{2}((2(n + i - 1) - \text{sign } (n + i - 1)) \text{ sign } \beta' - 1)\pi\right] = \omega_{n+i-1}$.

This and (1) yields $X\psi_i(x) = \omega_n\psi_i(x)$. Consequently $\psi_i(x)$ are the numbers of type n of (q) relative to the dispersion X for every integer ι.

Lemma 3. *Let* sign $X' = -1$. *Then there exists to every integer n, $n \neq 0$ precisely one number of type n of* (q) *relative to the dispersion X.*

Proof. The proof immediately follows from the fact that for every integer n, $n \neq 0$, $\omega_n(t)$ is an increasing function on $\mathbf{R}$, $\omega_n(\mathbf{R})$, $X(\mathbf{R}) = \mathbf{R}$ and by our assumption X is a decreasing function on $\mathbf{R}$.

Lemma 4. *Let $x \in \mathbf{R}$ and let u, v be independent solutions of* (q) *satisfying the initial conditions $u(x) = 1$ $u'(x) = 0$, $v(x) = 0$, $v'(x) = 1$. Then*

$$\varrho^2 + \left[\frac{1}{\sqrt{-q(t)}}\left(u'X(x)\,.\,\sqrt{|X'(x)|}\,.\,\text{sign}\,X' - \frac{1}{2}\frac{X''(x)}{X'(x)\sqrt{|X'(x)|}}\,.\,uX(x) + \right.\right.$$
$$\left.\left. + \frac{1}{2}\frac{q'(x)}{q(x)\sqrt{|X'(x)|}}\,.\,uX(x)\right) - \sqrt{\left|\frac{q(x)}{X'(x)}\right|}\,.\,vX(x)\right]\varrho + \text{sign}\,X' = 0 \qquad (6)$$

is the characteristic equation of (q) *relative to the dispersion X.*

Proof. Let the assumptions of Lemma 4 be fulfilled. Then (3) holds for the solutions u, v of (q), where $\det a_{ij} \neq 0$. Writing x for t in (3) we obtain

$$a_{12} = \sqrt{\left|\frac{q(x)}{X'(x)}\right|}\,.\,uX(x), \qquad a_{22} = \sqrt{\left|\frac{q(x)}{X'(x)}\right|}\,.\,vX(x).$$

By differentiating the equalities of (3) and writing x for t, we get after some evident modifications:

$$a_{11} = -\frac{1}{\sqrt{-q(x)}}\left[u'X(x)\,.\,\sqrt{|X'(x)|}\,.\,\text{sign}\,X' - \frac{1}{2}\frac{X''(x)}{X'(x)\sqrt{|X'(x)|}}\,.\,uX(x) + \right.$$
$$\left. + \frac{1}{2}\frac{q'(x)}{q(x)\sqrt{|X'(x)|}}\,.\,uX(x)\right],$$

$$a_{22} = -\frac{1}{\sqrt{-q(x)}}\left[v'X(x)\,.\,\sqrt{|X'(x)|}\,.\,\text{sign}\,X' - \frac{1}{2}\frac{X''(x)}{X'(x)\sqrt{|X'(x)|}}\,.\,vX(x) + \right.$$
$$\left. + \frac{1}{2}\frac{q'(x)}{q(x)\sqrt{|X'(x)|}}\,.\,vX(x)\right].$$

Then

$$a_{11} + a_{22} = -\frac{1}{\sqrt{-q(x)}}\left[u'X(x)\,.\,\sqrt{|X'(x)|}\,.\,\text{sign}\,X' - \right.$$
$$- \frac{1}{2}\frac{X''(x)}{X'(x)\sqrt{|X'(x)|}}\,.\,uX(x) +$$

$$+ \frac{1}{2} \frac{q'(x)}{q(x)\sqrt{|X'(x)|}} \, . \, uX(x)\Bigg] + \sqrt{\left|\frac{q(x)}{X'(x)}\right|} \, . \, vX(x)$$

and

$$a_{11}a_{22} - a_{12}a_{21} =$$

$$= -\frac{1}{\sqrt{-q(x)}}\Bigg[u'X(x) \, . \, \sqrt{|X'(x)|} \, . \, \text{sign}\, X' - \frac{X''(x)}{2X'(x)\sqrt{|X'(x)|}} \, . \, uX(x) +$$

$$+ \frac{q'(x)}{2q(x)\sqrt{|X'(x)|}} \, . \, uX(x)\Bigg]\sqrt{\left|\frac{q(x)}{X'(x)}\right|} \, . \, vX(x) +$$

$$+ \frac{1}{\sqrt{-q(x)}}\Bigg[v'X(x) \, . \, \sqrt{|X'(x)|} \, . \, \text{sign}\, X' - \frac{X''(x)}{2X'(x)\sqrt{|X'(x)|}} \, . \, vX(x) +$$

$$+ \frac{q'(x)}{2q(x)\sqrt{|X'(x)|}} \, . \, vX(x)\Bigg]\sqrt{\left|\frac{q(x)}{X'(x)}\right|} \, . \, uX(x) =$$

$$= (v'X(x) \, . \, uX(x) - u'X(x) \, . \, vX(x)) \, . \, \text{sign}\, X' = \text{sign}\, X'.$$

On substituting the above results into (4) we get (6).

Corollary 2. *If* (q) *relative to the dispersion X has complex characteristic multipliers, then they are equal to $e^{\pm a\pi i}$, where $0 < a < 1$ and* sign $X' = 1$.

Proof. The proof of Corollary 2 proceeds analogous to that of Corollary 4 in [3].

4. Main results

Theorem 1. *Let $0 < a < 1$. Then $e^{\pm a\pi i}$ are characteristic multipliers of* (q) *relative to the dispersion X iff there exists an integer n, a first phase α and a second phase β relative to the same basis of* (q) *having the same signature, for which*

$$\alpha X(t) = \beta(t) + (a + 2n)\,\pi, \qquad t \in \mathbf{R}. \tag{7}$$

Proof. Let $0 < a < 1$ and $e^{\pm a\pi i}$ are the characteristic multipliers of (q) relative to the dispersion X. Then there exist independent solutions u, v of (q):

$$\begin{aligned} \frac{uX(t)}{\sqrt{|X'(t)|}} &= \cos a\pi \, . \, \frac{u'(t)}{\sqrt{-q(t)}} + \sin a\pi \, . \, \frac{v'(t)}{\sqrt{-q(t)}} \\ \frac{vX(t)}{\sqrt{|X'(t)|}} &= -\sin a\pi \, . \, \frac{u'(t)}{\sqrt{-q(t)}} + \cos a\pi \, . \, \frac{v'(t)}{\sqrt{-q(t)}} \end{aligned} \tag{8}$$

Let $\alpha \in C^0(\mathbf{R})$, $\beta \in C^0(\mathbf{R})$ be functions for which $\text{tg}\, \alpha(t) = \frac{u(t)}{v(t)}$ on $\mathbf{R} - \{t \in \mathbf{R}, v(t) = 0\}$

and $\operatorname{tg} \beta(t) = \dfrac{u'(t)}{v'(t)}$ on $\mathbf{R} - \{t \in \mathbf{R},\ v'(t) = 0\}$. Then α is a first phase and β is a second phase of the basis (u, v) of (q). We choose α and β to have the same signature. From (8) we get $\operatorname{tg} \alpha X(t) = \operatorname{tg}(\beta(t) + a\pi)$. There exists therefore an integer k where $\alpha X(t) = \beta(t) + (a + k)\pi$. We prove that k is an even number. First $u(t) = \dfrac{c}{\sqrt{|\alpha'(t)|}} \sin \alpha(t)$, $v(t) = \dfrac{c}{\sqrt{|\alpha'(t)|}} \cos \alpha(t)$, $\dfrac{u'(t)}{\sqrt{-q(t)}} = \dfrac{c}{\sqrt{|\beta'(t)|}} \sin \beta(t)$, $\dfrac{v'(t)}{\sqrt{-q(t)}} = \dfrac{c}{\sqrt{|\beta'(t)|}} \cos \beta(t)$, where c is an appropriate constant. We have further

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = \frac{c \sin \alpha X(t)}{\sqrt{|\alpha' X(t) \,.\, X'(t)|}} = \frac{c}{\sqrt{|(\alpha X(t))'|}} \sin(\beta(t) + (a + k)\pi) =$$

$$= (-1)^k \frac{c}{\sqrt{|\beta'(t)|}} \sin(\beta(t) + a\pi)$$

From (7) we have

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = \cos a\pi \, \frac{c \,.\, \sin \beta(t)}{\sqrt{|\beta'(t)|}} + \sin a\pi \, \frac{c \,.\, \cos \beta(t)}{\sqrt{|\beta'(t)|}} = \frac{c}{\sqrt{|\beta'(t)|}} \sin(\beta(t) + a\pi).$$

Therefore $(-1)^k = 1$ and k is an even number ($k = 2n$).

Let there be a first phase α and a second phase β of a basis (u_1, v_1) of (q) having the same signature, for which (7) applies with $0 < a < 1$. Put $u(t) = \dfrac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $v(t) = \dfrac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}}$ for $t \in \mathbf{R}$. Then u, v are independent solutions of (q) and

$$\frac{u'(t)}{\sqrt{-q(t)}} = \frac{\sin \beta(t)}{\sqrt{|\beta'(t)|}}, \qquad \frac{v'(t)}{\sqrt{-q(t)}} = \frac{\cos \beta(t)}{\sqrt{|\beta'(t)|}}.$$

From

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = \frac{\sin \alpha X(t)}{\sqrt{|\alpha' X(t) \,.\, X'(t)|}} = \frac{\sin(\beta(t) + (a + 2n)\pi)}{\sqrt{|\beta'(t)|}} =$$

$$= \cos a\pi \, \frac{u'(t)}{\sqrt{-q(t)}} + \sin a\pi \, \frac{v'(t)}{\sqrt{-q(t)}},$$

$$\frac{vX(t)}{\sqrt{|X'(t)|}} = \frac{\cos \alpha X(t)}{\sqrt{|\alpha' X(t) \,.\, X'(t)|}} = \frac{\cos(\beta(t) + (a + 2n)\pi)}{\sqrt{|\beta'(t)|}} =$$

$$= -\sin a\pi \, \frac{u'(t)}{\sqrt{-q(t)}} + \cos a\pi \, \frac{v'(t)}{\sqrt{-q(t)}},$$

it follows that

$$\varrho^2 - 2\varrho \cos a\pi + 1 = 0$$

is the characteristic equation of (q) relative to the dispersion X and $e^{+a\pi i}$ are its roots.

Remark 1. Phases α and β in Theorem 1 may be chosen to be $0 < \beta(t) - \alpha(t) < \pi$ for $t \in \mathbf{R}$.

Corollary 3. *The equation* (q) *relative to the dispersion X has real characteristic multipliers iff there exists numbers of type n of* (q) *relative to the dispersion X.*

Proof. Let τ be a real characteristic multiplier of (q) relative to the dispersion X. Then there exists a (nontrivial real) solution u of (q):

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = \tau \, . \, \frac{u'(t)}{\sqrt{-q(t)}}, \qquad t \in \mathbf{R}. \tag{9}$$

Now, on our assumption, the equation (q) is both-sided oscillatory on $\mathbf{R}$. Let $u'(x_0) = 0$. Then it follows from (9) that $uX(x_0) = 0$ and thus there exists an integer n, $n \neq 0$: $X(x_0) = \omega_n(x_0)$ and x_0 is a number of type n of (q) relative to the dispersion X.

Let $e^{\pm a\pi i}$, $0 < a < 1$, be complex characteristic multipliers of (q) relative to the dispersion X. Assume that for an $x_0 \in \mathbf{R}$ and for an integer j, $j \neq 0$ we have $X(x_0) = \omega_j(x_0)$. According to Theorem 1 there exists a first phase α and a second phase β of a basis (u, v) of (q) having the same signature for which (7) applies and $0 < \beta(t) - \alpha(t) < \pi$ for $t \in \mathbf{R}$. Let $X = \alpha^{-1}\varepsilon\beta$, $\varepsilon \in \mathfrak{E}$. Then $\alpha X(x_0) = \varepsilon\beta(x_0) = \beta(x_0) + (a + 2n)\pi = \alpha\omega_j(x_0) = \beta(x_0) + \frac{1}{2}((2j - \operatorname{sign} j) \operatorname{sign} \beta' - 1)\pi$. From this we get $2(a + 2n) = (2j - \operatorname{sign} j) \operatorname{sign} \beta' - 1$ contrary to $0 < a < 1$.

Theorem 2. *Let x be a number of type n of* (q) *relative to the dispersion X. Then*

$$(-1)^n \sqrt{\frac{|X'(x)|}{\omega_n'(x)}} \, . \, \operatorname{sign} X', \qquad (-1)^n \sqrt{\frac{\omega_n'(x)}{|X'(x)|}},$$

are the characteristic multipliers of (q) *relative to the dispersion X.*

Proof. Let x be a number of type n of (q) relative to the dispersion X; $X(x) = \omega_n(x)$. Let u, v be independent solutions of (q) satisfying the initial conditions $u(x) = 1$, $u'(x) = 0$, $v(x) = 0$, $v'(x) = 1$. Then $uX(x) = u\omega_n(x) = 0$ and we have from (2) the relations

$$\frac{u\omega_n(t)}{\sqrt{\omega_n'(t)}} = (-1)^n \frac{u'(t)}{\sqrt{-q(t)}}, \qquad \frac{v\omega_n(t)}{\sqrt{\omega_n'(t)}} = (-1)^n \frac{v'(t)}{\sqrt{-q(t)}}. \tag{10}$$

From this, it follows $vX(x) = v\omega_n(x) = (-1)^n \sqrt{-\frac{\omega_n'(x)}{q(x)}}\, v'(x) = (-1)^n \sqrt{-\frac{\omega_n'(x)}{q(x)}}$.

By differenting the first equality in (10) we obtain

$$u'\omega_n(t)\sqrt{\omega_n'(t)} + u\omega_n(t)\left(\frac{1}{\sqrt{\omega_n'(t)}}\right)' =$$

$$= (-1)^{n+1} u(t)\sqrt{-q(t)} + (-1)^n u'(t)\left(\frac{1}{\sqrt{-q(t)}}\right)'$$

and writing x for t yields

$$u'X(x)\sqrt{\omega_n'(x)} = (-1)^{n+1}\sqrt{-q(x)},$$

$$u'X(x) = (-1)^{n+1}\sqrt{-\frac{q(x)}{\omega_n'(x)}}.$$

Then the characteristic equation (6) of (q) relative to the dispersion X can be written as

$$\varrho^2 - \left[(-1)^n\sqrt{\frac{|X'(x)|}{\omega_n'(x)}}\,.\,\operatorname{sign} X' + (-1)^n\sqrt{\frac{\omega_n'(x)}{|X'(x)|}}\right]\varrho + \operatorname{sign} X' = 0. \quad (11)$$

Since (11) has the roots $(-1)^n\sqrt{\frac{|X'(x)|}{\omega_n'(x)}}\,.\,\operatorname{sign} X'$ and $(-1)^n\sqrt{\frac{|X'(x)|}{\omega_n'(x)}}$, these numbers are the characteristic multipliers of (q) relative to the dispersion X. This completes the proof of Theorem 2.

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Souhrn

UŽITÍ ZOBECNĚNÉ FLOQUETOVY TEORIE PŘI TRANSFORMACI DIFERENCIÁLNÍ ROVNICE $y'' = q(t)\,y$ DO JEJÍ PRŮVODNÍ ROVNICE

SVATOSLAV STANĚK

V práci jsou vyšetřovány diferenciální rovnice typu

$$y'' = q(t)\,y, \qquad q \in C^2(\mathbf{R}), \qquad q(t) < 0 \qquad \text{pro } t \in \mathbf{R}. \tag{q}$$

které jsou oboustranně oscilatorické na $\mathbf{R}$. Buď $X \in C^3(\mathbf{R})$, $X' \neq 0$, řešení (na $\mathbf{R}$) nelineární diferenciální rovnice

$$\sqrt{|X'|}\left(\frac{1}{\sqrt{|X'|}}\right)'' + X'^2 \,.\, q(X) = q_1(t),$$

kde $q_1(t) := q(t) + \sqrt{-q(t)}\left(\dfrac{1}{\sqrt{-q(t)}}\right)''$, $t \in \mathbf{R}$. Řekneme, že (obecně komplexní) číslo τ je charakteristickým kořenem rovnice (q) při X, jestliže existuje (netriviální) řešení u rovnice (q) pro něž $\dfrac{uX(t)}{\sqrt{|X'(t)|}} = \tau \,.\, \dfrac{u'(t)}{\sqrt{-q(t)}}$, $t \in \mathbf{R}$. Taková čísla τ jsou kořeny jisté kvadratické algebraické rovnice. Nechť $0 < a < 1$. Pak $e^{\pm a\pi i}$ jsou (komplexní) charakteristické kořeny rovnice (q) při X právě když existuje celé číslo n a první fáze α a druhá fáze β příslušné k nějaké bázi řešení rovnice (q), které mají stejnou signaturu, pro něž

$$\alpha X(t) = \beta(t) + (a + 2n)\,\pi, \qquad t \in \mathbf{R}.$$

Nechť $\omega_n(t)$ je centrální disperse 4. druhu rovnice (q) s indexem n. Rovnice (q) při X má reálné charakteristické kořeny právě když existuje číslo $x \in \mathbf{R}$ a celé číslo n: $X(x) = \omega_n(x)$. V tomto případě $(-1)^n\sqrt{\dfrac{|X'(x)|}{\omega_n'(x)}}\,.\,\operatorname{sign} X'$ a $(-1)^n\sqrt{\dfrac{\omega_n'(x)}{|X'(x)|}}$ jsou charakteristické kořeny rovnice (q) při X.

Резюме

ПРИМЕЧАНИЕ ОБОБЩЕННОЙ ТЕОРИИ ФЛОКЕ ПРИ ПРЕОБРАЗОВАНИИ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y'' = q(t)y$ В СОПРОВОЖДАЮЩЕЕ УРАВНЕНИЕ

СВАТОСЛАВ СТАНЕК

В работе исследуются колеблющиеся на **R** дифференциальные уравнения типа

$$y'' = q(t)\,y,\ q \in C^2(\mathbf{R}),\ q(t) < 0 \text{ для } t \in \mathbf{R}. \qquad \text{(q)}$$

Пусть $X \in C^3(\mathbf{R})$, $X' \neq 0$ решение (на **R**) нелинейного дифференциального уравнения

$$\sqrt{|X'|}\left(\frac{1}{\sqrt{|X'|}}\right)'' + X'^2 \,.\, q(X) = q_1(t),$$

где $q_1(t) := q(t) + \sqrt{|q(t)|}\left(\frac{1}{\sqrt{|q(t)|}}\right)''$, $t \in \mathbf{R}$. Говорим, что (вообще комплексное) число τ является характеристическим корнем уравнения (q) относительно X если существует (нетривиальное) решение u уравнения (q) выполняющее соотношение $\frac{uX(t)}{\sqrt{|X'(t)|}} = \tau \frac{u'(t)}{\sqrt{|q(t)|}}$, $t \in \mathbf{R}$. Такие числа τ представляют корни некоторого квадратического алгебраического уравнения. Пусть $0 < a < 1$. Тогда $e^{\pm a\pi i}$ (комплексные) характеристические корни уравнения (q) относительно X только в сулчае, когда существует целое число n, первая фаза α и вторая фаза β некоторой базы уравнения (q), которые имеют одинаковую сигнатуру, так что

$$\alpha X(t) = \beta(t) + (a + 2n)\,\pi, \quad t \in \mathbf{R}.$$

Пусть $\omega_n(t)$ центральная дисперсия 4-го рода уравнения (q) с индексом n. Уравнение (q) относительно X имеет действительные характеристические корни только в том случае, когда существует число $x \in \mathbf{R}$ и целое число n для которых $X(x) = \omega_n(x)$. В этом случае $(-1)^n \sqrt{\frac{|X'(x)|}{\omega_n'(x)}} \,.\, \operatorname{sign} X'$ и $(-1)^n \sqrt{\frac{\omega_n'(x)}{|X'(x)|}}$ являются характеристическими корнями уравнения (q) относительно X.