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ON TRANSFORMATIONS OF BUNDLES OF SOLUTIONS OF A FOURTH ORDER ITERATED LINEAR DIFFERENTIAL EQUATION

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Consider a linear homogeneous differential equation of the *4th* order

$$y^{IV}(t) + 10[q(t)\, y'(t)]' + 3[3q^2(t) + q''(t)]\, y(t) = 0, \tag{1}$$

which arises in iterating the linear homogeneous differential equation of the *2nd* order

$$y''(t) + q(t)\, y(t) = 0, \tag{2}$$

where the function $q(t) \in \mathbf{C}^2(-\infty, +\infty)$, $q(t) > 0$ for all $t \in (-\infty, +\infty)$ and is such, that the differential equation (2) is oscillatory [to every $t \in (-\infty, +\infty)$ there exists an infinite number of zeros of an arbitrary nontrivial solution of (2) lying both to the left and to the right of the point t].

As we know, if $[u(t), v(t)]$ is a basis of (2), then every nontrivial solution of (1) is of the form

$$y(t) = \sum_{i=1}^{4} C_i u^{4-i}(t)\, v^{i-1}(t), \tag{3}$$

where $C_i \in \mathbf{R}$, $i = 1, \dots, 4$, $\sum_{i=1}^{4} C_i^2 > 0$, whereby it follows from the oscillatority of (2) and with respect to the evenness of the order of (1) that it is oscillatory, too. Therefore, for the sake of brevity, we call the differential equation (1) also oscillatory.

Since the differential equation (1) is of the fourth order, it may be seen that an arbitrary zero relative to any of its nontrivial solution is at most of multiplicity three. Hereafter, under a solution of (2) and (1) we will understand a nontrivial solution, only.

It was proved in *Lemma 1* [1] that if $t_0 \in (-\infty, +\infty)$ is an arbitrary firmly chosen point, then every solution $y(t)$ of the oscillatory differential equation (1) vanishing at t_0 is of the form

1. $y(t) = \sum_{i=1}^{3} C_i u^{4-i}(t) v^{i-1}(t)$, $C_3 \neq 0$,

   exactly if t_0 is a simple zero of the solution $y(t)$

2. $y(t) = \sum_{i=1}^{2} C_i u^{4-i}(t) v^{i-1}(t)$, $C_2 \neq 0$,

   exactly if t_0 is a double zero of the solution $y(t)$

3. $y(t) = C_1 u^3(t)$, $C_1 \neq 0$,

   exactly if t_0 is a triple zero of the solution $y(t)$, where $[u(t), v(t)]$ is such a basis of (2) that

$$u(t_0) = 0, \qquad v'(t_0) = 0. \tag{P}$$

In the theorem below we make an assertion on a mutual relation between two bundles having different types of solutions of the oscillatory differential equation (1). We will show that to every solution $y_0(t)$ relative to the twoparametric bundle

$$y(t, C_1, C_2) = u^2(t)\,[C_1 u(t) + C_2 v(t)], \tag{S_2}$$

$C_i \in \mathbf{R}$, $i = 1, 2$, $C_2 \neq 0$, of a solution of (1) there corresponds the solution $Y_0(t)$ (even a whole subbundle of such solutions which differ from each other by an arbitrary nonzero multiplicative constant) relative to the threeparametric bundle

$$Y(t, C_1', C_2', C_3') = U(t)\,[C_1' U^2(t) + C_2' U(t)\,V(t) + C_3' V^2(t)], \tag{S_3}$$

$C_i' \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, of a solution of (1) such that both solutions $y_0(t)$, $Y_0(t)$ possess the same zeros whose multiplicities were, however, interchanged (with respect to the strong or weak conjugacy of the points) and thus at the same time there occurred an exchange of the weakly conjugate points for the weakly ones and vice versa. Thereby, the mutual transformation of the bundle of solutions of one type onto a bundle of solutions of the second type is realized through the exchange of the basis $[u(t), v(t)]$ of the oscillatory differential equation (2) for an appropriate basis $[U(t), V(t)]$ relative to the same differential equation. As we know, such an exchange may be realized through a regular centroaffine transformation. We show finally that the both bundles (S_2), (S_3) of solutions of (1) with the properties considered and expressed in two different bases of solutions of (2) are essentially of the same form (and with respect to the coincidence of their zeros with interchanged multiplicities, dual in some degree).

Theorem 1: Let ${}^2t_0, {}^2t_2, {}^2t_4 \in (-\infty, +\infty)$ be three consecutive strongly conjugate points of the multiplicity $\nu = 2$ relative to a twoparametric bundle (S_2) of the solutions of the oscillatory differential equation (1). Let next ${}^1t_1, {}^1t_3 \in (-\infty, +\infty)$ be two consecutive simple weakly conjugate points to the point ${}^2t_0$ [being simultaneously zeros of a solution of (2)] belonging to the same bundle (S_2) and lying respectively between the points ${}^2t_0$, ${}^2t_2$ and ${}^2t_2$, ${}^2t_4$, so that

$$ {}^2t_0 < {}^1t_1 < {}^2t_2 < {}^1t_3 < {}^2t_4 \tag{4} $$

is true.

Then there exists a certain threeparametric bundle (S_3) of solutions of the same differential equation (1) such that the points ${}^1t_1$, ${}^1t_3$ are its two consecutive simple strongly conjugate points, whereby the points ${}^2t_0$, ${}^2t_2$, ${}^2t_4$ are the three consecutive double weakly conjugate points to the point ${}^1t_1$ belonging to the same bundle (S_3). Hereby the point ${}^1t_1$ lies between the points ${}^2t_0$, ${}^2t_2$ and the point ${}^1t_3$ lies between the points ${}^2t_2$, ${}^2t_4$ i.e. (4) holds again.

Proof: From the previous considerations (used for the first time in the proof of *Lemma 1*) it becomes obvious that there exists a basis $[u(t), v(t)]$ of the oscillatory differential equation (2) satisfying the condition (P) at the point $t_0 \in (-\infty, +\infty)$. According to Statement 2. of *Lemma 1*, all solutions $y(t)$ of the oscillatory differential equation (1) vanishing together with the function $u(t)$ at the point t_0, with the multiplicity $\nu = 2$, form (up to an arbitrary nonzero multiplicative constant) a twoparametric bundle exactly of the form

$$ y(t, C_1, C_2) = u^2(t)\,[C_1 u(t) + C_2 v(t)], \tag{S_2} $$

where $C_j \in \mathbf{R}$, $j = 1, 2$, $C_2 \neq 0$, are arbitrary parameters. Since all the double zeros $t_{2k} \in (-\infty, +\infty)$, $k = 0, \pm 1, \pm 2, \ldots$, relative to this bundle, representing its only mutually strongly conjugate points are at the same time the zeros of the function $u(t)$, we see that

$$ y(t_{2k}) = u(t_{2k}) = 0 $$

(so that these equalities hold also for the three given points t_0, t_2, t_4; this condition will be written hereafter only for the point t_0). It holds for the remaining simple zeros $t_{2k+1} \in (-\infty, +\infty)$, $k = 0, \pm 1, \pm 2, \ldots$, relative to the bundle (S_2), which are altogether weakly conjugate to all foregoing points t_{2k} that they simultaneously represent the zeros of the twoparametric system of functions $y^*(t, C_1, C_2) = C_1 u(t) + C_2 v(t)$ being on $(-\infty, +\infty)$ linearly independent on the function $u(t)$. In other words

$$ y({}^1t_{2k+1}) = y^*({}^1t_{2k+1}, C_1, C_2) = 0. $$

Hereby (according to Statement 1., *Theorem 2.3* in [1]), it holds for all $k = 0, \pm 1, \pm 2, \ldots$ that

$$ {}^1t_{2k+1} \in ({}^2t_{2k}, {}^2t_{2k+2}). $$

This implies that between any two neighbouring strongly conjugate points ${}^2t_{2k}$, ${}^2t_{2k+2}$ there always lies exactly one weakly conjugate point ${}^1t_{2k+1}$ being at the same time a zero of an arbitrary function $y_0^*(t)$ resulting from the system $y^*(t, C_1, C_2)$ at a (firm) choice of both constants C_1, C_2 ($C_2 \neq 0$). Thus especially for $k = 0, 1$

$$y({}^1t_i) = y_0^*({}^1t_i) = 0, \qquad i = 1, 3,$$

so that for both constants $C_j \in \mathbf{R}$, $j = 1, 2$, relative to the system $y^*(t, C_1, C_2)$, with preassigned points t_1, t_3 from the given pentand of points in the assumption of the Theorem, there must hold

$$C_1 u(t_i) + C_2 v(t_i) = 0,$$

whence – because of $u(t_i) \neq 0$, $i = 1, 3$, – we get

$$C_1 = -\frac{C_2 v(t_i)}{u(t_i)},$$

[where specially $C_1 = 0 \Leftrightarrow v(t_i) = 0$, because $C_2 \neq 0$]. Hence, all solutions $y(t)$ of (1) vanishing together with the function $u(t)$ [with respect to the condition (P) being satisfied at t_0] with a multiplicity $\nu = 2$ at the strongly conjugate points ${}^2t_{2k}$ (among which the given points t_0, t_2, t_4 belong) and with a multiplicity $\mu = 1$ at the weakly conjugate points ${}^1t_{2k+1}$, $k = 0, \pm 1, \pm 2, \ldots$, (among which also the given points t_i, $i = 1, 3$, belong), are exactly of the form

$$y(t, C_2) = \frac{C_2 u^2(t)}{u(t_i)} [-v(t_i) u(t) + u(t_i) v(t)], \tag{S$_2^*$}$$

where $C_2 \in \mathbf{R} - \{\mathbf{0}\}$ is an arbitrary constant.

Let us now look for such solutions $Y(t)$ of the same differential equation (1) that are simply vanishing at the points t_i, $i = 1, 3$, (with both these points being thereby mutually strongly conjugate) and that are at the same time doubly vanishing at the points t_0, t_2, t_4 (with these points being weakly conjugate to the points t_1, t_3).

Assume that $[U(t), V(t)]$, where

$$\begin{aligned} U(t) &= c_{11} u(t) + c_{12} v(t), \\ V(t) &= c_{21} u(t) + c_{22} v(t), \end{aligned} \tag{T}$$

$c_{ij} \in \mathbf{R}$, $i, j = 1, 2$, $c_{11}c_{22} - c_{12}c_{21} \neq 0$ [which means that the centroaffine transformation (T) is regular], is such a basis of an (oscillatory) differential equation (2), that the functions $U(t)$, $V(t)$ satisfy the condition (P) at the point t_1.
Then – according to Statement 1. of *Lemma 1* – all solutions $Y(t)$ of (1) simply vanishing at t_1 (and at the same time also at t_3) together with the function $U(t)$, i.e. for which

$$Y(t_i) = U(t_i) = 0, \qquad i = 1, 3,$$

form a thereparametric bundle (up to an arbitrary nonzero multiplicative constant) being exactly of the form

$$Y(t, C_1', C_2', C_3') = U(t) [C_1' U^2(t) + C_2' U(t) V(t) + C_3' V^2(t)], \tag{S$_3$}$$

where $C_i' \in R$, $i = 1, 2, 3$, $C_3' \neq 0$, are arbitrary parameters.

From the condition $U(t_i) = 0$, $i = 1, 3$, it can be now readily seen how to choose the function $U(t)$: in order to be vanishing at all zeros of the functions $C_1^* u(t) + C_2^* v(t)$ from the forefound form of the solution

$$y(t, C_2) = \frac{C_2 u^2(t)}{u(t_i)} [C_1^* u(t) + C_2^* v(t)]$$

of the differential equation (1), where $C_1^* = -v(t_i)$, $i = 1, 3$, $C_2^* = u(t_i)$, $i = 1, 3$, meeting the assumption of the Theorem, it must necessarily be linearly dependent with it on $(-\infty, +\infty)$, i.e. there must exist a constant $\lambda \in \mathbf{R} - \{\mathbf{0}\}$ such, that

$$U(t) = \lambda[-v(t_i)\, u(t) + u(t_i)\, v(t)].$$

As is known from the I*st* part of the proof of *Theorem 1.1* and also from *Theorems 1.3* and *2.3* in [1], a threeparametric bundle (S_3) of the solutions $Y(t)$ of (1) in order to possess double weakly conjugate points, there must between its parameters $C_j' \in \mathbf{R}$, $j = 1, 2, 3$, exactly hold

$$C_2'^2 = 4C_1'C_3'.$$

Hence the solutions $Y(t)$ searched, satisfying the conditions required must belong to (S_3) of the form

$$Y(t, C_2', C_3') = \frac{U(t)}{4C_3'} [C_2' U(t) + 2C_3' V(t)]^2, \qquad (S_3')$$

where $C_j' \in \mathbf{R}$, $j = 2, 3$, $C_3' \neq 0$, which we get from (S_3) after some modification writing $\frac{C_2'^2}{4C_3'}$ for C_1'.

Here all the double weakly conjugate points – let us write them as T_{2k}, $k = 0, \pm 1, \pm 2, \ldots$, – of such a bundle are simultaneously the zeros of a twoparametric system $Y^*(t, C_2', C_3') = C_2' U(t) + 2C_3' V(t)$ of functions on $(-\infty, +\infty)$ linearly independent of the function $U(t)$. If we write them also as ${}^2T_{2k}$, then (according to Statement 2. of *Theorem 2.3* [1]), we have

$${}^2T_{2k} \in ({}^1t_{2k-1}, {}^1t_{2k+1}),$$

$k = 0, \pm 1, \pm 2, \ldots$, whereby between two arbitrary neighbouring simple strongly conjugate points ${}^1t_{2k-1}, {}^1t_{2k+1}'$ of the boundle (S_3') there always lies its exactly one zero ${}^2T_{2k}$ $(k = 0, \pm 1, \pm 2, \ldots)$, which represents at the same time the zero of an arbitrary function from the system $Y^*(t, C_2', C_3')$. This function may be always obtained from the system above in a firm choice of constants $C_j' \in \mathbf{R}$, $j = 2, 3$.

Thus, in order that these weakly conjugate points T_{2k} of its may be exactly the points t_{2k}, $k = 0, \pm 1, \pm 2, \ldots$, the system of functions $Y^*(t, C_2', C_3')$ must necessarily be on $(-\infty, +\infty)$ linearly dependent on the function $u(t)$, by means of which [and by means of the function $v(t)$ from the original basis $[u(t), v(t)]$ of (2)] the foregoing solution $y(t)$ of (1) was expressed, i.e. there must exist a constant $\varkappa \in \mathbf{R} - \{\mathbf{0}\}$ such that

$$C_2' U(t) + 2C_3' V(t) = \varkappa u(t).$$

However, because of $u(t_{2k}) = 0$ for $t = t_{2k}$, $k = 0, \pm 1, \pm 2, \ldots$, we have

$$C_2' U(t_0) + 2C_3' V(t_0) = 0$$

for every $\varkappa$, whence we get that for both parameters $C_j' \in \mathbf{R}$, $j = 2, 3$ [in view of $U(t_0) \neq 0$]

$$C_2' = -\frac{2C_3' V(t_0)}{U(t_0)}$$

holds. It follows specially that $C_2' = 0 \Leftrightarrow V(t_0) = 0$, for $C_3' \neq 0$.

Thus, all the solutions $Y(t)$ of (1) vanishing together with the function $U(t)$ with the multiplicity $\nu = 1$ at the strongly conjugate points ${}^1t_{2k+1}$ (among which the given points t_1, t_3 belong) and with the multiplicity $\mu = 2$ at the weakly conjugate points ${}^2t_{2k}$, $k = 0, \pm 1, \pm 2, \ldots$, (among which also the given points t_0, t_2, t_4 belong) are exactly of the form

$$Y(t, C_3') = \frac{C_3' U(t)}{U^2(t_0)} [-V(t_0) U(t) + U(t_0) V(t)]^2, \qquad (S_3^*)$$

where $C_3' \in \mathbf{R} - \{0\}$ is an arbitrary constant.

It still remains to determine the coefficients c_{ij} $(i, j = 1, 2)$ of the transformation (T) used, specifying by means of both functions $u(t)$, $v(t)$ of the original basis $[u(t), v(t)]$ both functions $U(t)$, $V(t)$ in the new basis $[U(t), V(t)]$ of (2), where exactly the bundle (S_3^*) of all solutions $Y(t)$ was expressed.

Writing, however, both conditions for the coincidence of the corresponding zeros in both bundles (S_2^*) and (S_3^*)

$$U(t) = \lambda[-v(t_i) u(t) + u(t_i) v(t)]$$
$$-V(t_0) U(t) + U(t_0) V(t) = \varkappa u(t)$$

i.e.

$$U(t) = \lambda[-v(t_i) u(t) + u(t_i) v(t)]$$
$$V(t) = \frac{1}{U(t_0)} \{[\varkappa - \lambda v(t_i) V(t_0)] u(t) + \lambda u(t_i) V(t_0) v(t)\},$$

we get [in comparing with (T)] that

$$c_{11} = -\lambda v(t_i)$$
$$c_{12} = \lambda u(t_i)$$
$$c_{21} = \frac{1}{U(t_0)} [\varkappa - \lambda v(t_i) V(t_0)],$$
$$c_{22} = \frac{1}{U(t_0)} \lambda u(t_i) V(t_0),$$

[whereby – in view of $\lambda \varkappa u(t_i) U(t_0) \neq \emptyset$ – in fact

$$c_{11}c_{22} - c_{12}c_{21} = -\frac{\lambda\varkappa u(t_i)}{U(t_0)} \neq 0$$

holds], which was to be demonstrated.
Completely analogous we could prove

Theorem 2: Let ${}^1t_0, {}^1t_2, {}^1t_4 \in (-\infty, +\infty)$ be three immediately consecutive strongly conjugate points with multiplicity $\nu = 1$ of a certain threeparametric bundle (S_3) of solutions of an oscillatory differential equation (1) and let ${}^2t_1, {}^2t_3 \in (-\infty, +\infty)$ be two immediately consecutive double weakly conjugate points to ${}^1t_0$ [being simultaneously the zeros of a solution of (2)] belonging to the same bundle (S_3) lying respectively between the points ${}^1t_0, {}^1t_2$ and ${}^1t_2, {}^1t_4$, so that

$${}^1t_0 < {}^2t_1 < {}^1t_2 < {}^2t_3 < {}^1t_4. \tag{5}$$

Then there exists a certain twoparametric bundle (S_2) of solutions of the same differential equation (1) such that the points ${}^2t_1, {}^2t_3$ are its two immediately consecutive double strongly conjugate points, whereby the points ${}^1t_0, {}^1t_2, {}^1t_4$ are three immediately consecutive simple weakly conjugate points to the point ${}^2t_1$ belonging to the same bundle (S_2), with the point ${}^2t_1$ lying between the points ${}^1t_0, {}^1t_2$ and the point ${}^2t_3$ lies between the points ${}^1t_2, {}^1t_4$, i.e. (5) holds again.

Remark 1. In the above mutually dual theorems there is shown as their corollary that among the weakly conjugate points whether it be of the twoparametric bundle (S_2) or of the threeparametric bundle of the special form (S_3') of the solutions of (1) there exists a relation characteristic for the strongly conjugate points (see *Definition 2.1* [1]), i.e. that all the weakly conjugate points always belong to the same bundle of solutions of (1) representing at the same time the zeros of the functions from the subsystem $y^*(t, C_1, C_2)$ [with respect to the bundle (S_2)] or $Y^*(t, C_2', C_3')$ [with respect to the bundle (S_3')] mutually differing by an arbitrary nonzero multiplicative constant $C \in \mathbf{R}$ – are mutually strongly conjugate among themselves. This property appears in passing from one basis of the differential equation (2) to an other one, whereby the transformation realizes not only between both bundles of solutions (S_2) and (S_3) mutually, but also an exchange of all the weakly conjugate points for the strongly conjugate points and vice versa. The mutual position of all these zeros is to such a transformation invariant. This fact will be of importance for determining the position of the weakly conjugate points by means of functions describing their position with respect to that of the strongly conjugate points in the corresponding bundle of solutions of the differential equation (1) which especially shines up after identifying the functions describing the position of the strongly conjugate points with the function $\varphi_\kappa(t)$ introduced in [2]. On functions of this type see [1] and [3].

Remark 2. Notice that in the main, in both foregoing theorems the formulation of a certain five-point boundary value problem was involved (with prescribed multiplicities of points) for an oscillatory differential equation (1). It was known in advance about the triple or about the couple of points alternating (along with their multiplicities) in the given pentand that the corresponding boundary value problem with three or two points (naturally with the multiplicity 1) is solvable for two linearly independent solutions of an oscillatory differential equation (2).

It was shown at the same time that this problem with the properties required and concerning the decompositions and multiplicities of the given five points is always solvable in two ways in correspondence to solutions of (1) chosen either from (S_2) or (S_3').

Unlike to the first case (in *Theorems 1* or *2*), where an exchange of the solution $y(t)$ of (1) from (S_2) for the solution $Y(t)$ of the same equation from (S_3) was involved, vanishing at the same points as the solution $y(t)$ with a simultaneous exchange of the strong conjugacy for the weak conjugacy at the same zeros and vice versa, combined with an exchange of a basis of (2) for another basis of (2), we will give one more case where the coincidence of all zeros was reached in both bundles (S_2) and (S_3) of solutions of (1) with one and the same (properly chosen) basis $[u(t), v(t)]$ of (2) whereby the properties of a strong or weak conjugacy of zeros (with distinct multiplicities) in both bundles were retained simultaneously.

We will formulate the respective Theorem for such two bundles of solutions of the differential equation (1) bearing in mind that the condition (P) has been just satisfied by both functions $u(t)$, $v(t)$ in both bundles (S_2), (S_3) simultaneously at an arbitrary firmly chosen point $t_0 \in (-\infty, +\infty)$.

Theorem 3: In two different bundles

$$y(t, C_1, C_2) = u^2(t)\,[C_1 u(t) + C_2 v(t)], \qquad (S_2)$$
$$C_i \in \mathbf{R}, \qquad i = 1, 2, \qquad C_2 \neq 0,$$

and

$$\hat{y}(t, C_1', C_2', C_3') = u(t)\,[C_1' u^2(t) + C_2' u(t)\, v(t) + C_3' v^2(t)], \qquad (S_3)$$
$$C_j \in \mathbf{R}, \qquad j = 1, 2, 3, \qquad C_3 \neq 0,$$

of solutions of one and the same oscillatory differential equation (1) expressed by means of both functions $u(t)$, $v(t)$ from the basis $[u(t), v(t)]$ of the oscillatory differential equation (2) and vanishing together with the function $u(t)$ at an arbitrary firmly chosen point $t_0 \in (-\infty, +\infty)$ there realizes a mutual coincidence of all their zeros, so that both bundles (S_2), (S_3) are vanishing

1. at the same points $t_{2k} \in (-\infty, +\infty)$, $k = 0, \pm 1, \pm 2, \ldots$, [being strongly conjugate points for both bundles and namely double for (S_2) and simple for (S_3)] and

2. at the same points $t_{2k+1} \in (-\infty, +\infty)$, $k = 0, \pm 1, \pm 2, \ldots$, [being weakly conjugate points for both bundles and namely simple for (S_2) and double for (S_3)]

exactly if there exists a constant $\lambda \in \mathbf{R} - \{0\}$ such that

$$C_1' = \frac{\lambda C_1^2}{2C_2}, \qquad C_2' = \lambda C_1, \qquad C_3' = \frac{\lambda C_2}{2}$$

simultaneously holds.

Proof: The Statement 1. of the above Theorem on a mutual coincidence of all (double) strongly conjugate points ${}^2t_{2k} \in (-\infty, +\infty)$, $k = 0, \pm 1, \pm 2, \dots$, of (S_2) with all (simple) strongly conjugate points ${}^1t_{2k} \in (-\infty, +\infty)$, $k = 0, \pm 1, \pm 2, \dots$, of (S_3) follows (according to Statements 1. and 2. of *Lemma 1*) just from the very form of both these bundles of solutions $y(t)$ and $\hat{y}(t)$ of the differential equation (1), expressed by means of the functions $u(t), v(t)$ relative to one and the same basis $[u(t), v(t)]$ of the oscillatory differential equation (2) satisfying the condition (P) at the point $t_0 \in (-\infty, +\infty)$ i.e. for $k = 0$. The Statement 2. of the above Theorem on a coincidence of all (simple) weakly conjugate points ${}^1t_{2k+1} \in ({}^2t_{2k}, {}^2t_{2k+2})$, $k = 0, \pm 1, \pm 2, \dots$, of (S_2) with all (double) weakly conjugate points ${}^2T_{2k+1} \in$ $\in ({}^1t_{2k}, {}^1t_{2k+2})$, $k = 0, \pm 1, \pm 2, \dots$, of (S_3) will be readily obtained in applying the relation $C_2'^2 = 4C_1'C_3'$ for the parameters $C_j' \in \mathbf{R}$, $j = 1, 2, 3$, $C_3' \neq 0$, necessary and sufficient (see the I*st* part of the proof to *Theorem 1.1* and the Statement 2a) of *Theorem 2.3* [1]) for the existence of the double weakly conjugate points ${}^2T_{2k+1}$, $k = 0, \pm 1, \pm 2, \dots$, of (S_3) assuming thus the form

$$\hat{y}(t, C_2', C_3') = \frac{u(t)}{4C_3'}[C_2'u(t) + 2C_3'v(t)]^2 \qquad (S_3')$$

and besides from the condition of the mutual coincidence of all zeros t_{2k+1} and T_{2k+1}, $k = 0, \pm 1, \pm 2, \dots$, of both twoparametric systems of functions, namely

$$y^*(t, C_1, C_2) = C_1u(t) + C_2v(t),$$

[appearing in the bundle (S_2)] and

$$\hat{y}^*(t, C_2', C_3') = C_2'u(t) + 2C_3'v(t),$$

[appearing in the bundle (S_3)],
with respect to the assumptions $C_2 \neq 0$ and $C_3' \neq 0$ linearly independent of the function $u(t)$ on $(-\infty, +\infty)$.

The last condition which at the seme time expresses the coincidence of all weakly conjugate points t_{2k+1}, T_{2k+1}, $k = 0, \pm 1, \pm 2, \dots$, (of distinct multiplicities) of both bundles (S_2), (S_3'), since

$$y({}^1t_{2k+1}, C_1, C_2) = y^*({}^1t_{2k+1}, C_1, C_2)$$

and

$$\hat{y}({}^2T_{2k+1}, C_2, C_3) = \hat{y}^{*2}({}^2T_{k+1}, C_2, C_3),$$

for all $k = 0, \pm 1, \pm 2, \dots$, is exactly the linear dependence of these both systems of functions on the interval $(-\infty, +\infty)$, i.e. the existence of the constant $\lambda \in \mathbf{R} - \{\mathbf{0}\}$ such that

$$\hat{y}^*(t, C_2', C_3') = \lambda y^*(t, C_1, C_2),$$

which means that at the same time

$$C_2' = \lambda C_1 \qquad \text{and} \qquad C_3' = \frac{\lambda C_2}{2},$$

$\left(\text{so that } C_1' = \dfrac{\lambda C_1^2}{2C_2}\right)$ and therefore the second of both bundles, i.e. the bundle (S_3') of the solutions $y(t)$ of (1) then is of the form

$$\hat{y}(t, C_1, C_2) = Cu(t)\left[C_1 u(t) + C_2 v(t)\right]^2,$$

where $C\left(= \dfrac{\lambda}{2C_2}\right) \in \mathbf{R} - \{\mathbf{0}\}$ is an arbitrary constant.

Specially: if $C_1 = 0$ holds in (S_2), then $C_1' = C_2' = 0$ must hold in (S_3) and the mutual coincidence considered of all corresponding zeros (with distinct multiplicities) occurs now in oneparametric bundles (S_2), (S_3') in the form

$$y(t, C_2) = C_2 u^2(t)\, v(t),$$
$$\hat{y}(t, C_3') = C_3' u(t)\, v^2(t),$$

where $C_2, C_3' \in \mathbf{R} - \{\mathbf{0}\}$ are arbitrary parameters.

It is worth mentioning that at (S_2), (S_3) with equally denoted initial both parameters, i.e. at the bundles

$$y(t, C_1, C_2) = u^2(t)\left[C_1 u(t) + C_2 v(t)\right],$$

$C_i \in \mathbf{R}$, $i = 1, 2$, $C_2 \neq 0$, and

$$\hat{y}(t, C_1, C_2, C_3) = u(t)\left[C_1 u^2(t) + C_2 u(t)\, v(t) + C_3 v^2(t)\right],$$

$C_3 \in \mathbf{R} - \{\mathbf{0}\}$,
of solutions of the same differential equation (1), the mutual coincidence considered, of their all corresponding zeros (i.e. including the weakly conjugate points) – with respect to the condition on nontriviality of solutions of (1) – cannot occur [since the nonlinear system of the three algebraic equations for the parameters $C_i \in \mathbf{R}$, i = 1, 2, 3:

$$\begin{aligned} C_1(\lambda C_1 - 2C_2) &= 0, \\ \lambda C_1 - C_2 &= 0, \\ \lambda C_2 - 3C_3 &= 0, \end{aligned}$$

meeting the necessary and sufficient condition of such a complete coincidence given in the statement of the theorem just proved, possesses only one trivial solution $C_1 = C_2 = C_3 = 0$ at every $\lambda \in \mathbf{R}$].

R e m a r k 3. Let us consider besides the bundle (S_2) of solutions $y(t)$ of the differential equation (1) written by means of the functions $u(t)$, $v(t)$ from the basis $[u(t), v(t)]$ of the differential equation (2) satisfying the condition (P) at a point $t_0 \in (-\infty, +\infty)$ the bundle (S_3) of solutions $Y(t)$ of the same differential equation (1), written by means of the functions $U(t)$, $V(t)$ from another basis $[U(t), V(t)]$ of (2) satisfying the condition (P) at a point $T_0 \in (-\infty, +\infty)$. Thus – according to Statement 2. or Statement 1. of *Lemma 1* – the twoparametric bundle (S_2) or the threeparametric bundle (S_3) of solutions $y(t)$ or of $Y(t)$ of (1) vanishing together with the function $u(t)$ or $U(t)$ at t_0 with the multiplicity $\nu = 2$ (i.e. ${}^2t_0$) or at T_0 with the multiplicity $\nu = 1$ (i.e. ${}^1T_0$), is exactly of the form

$$y(t, C_1, C_2) = u^2(t)\,[C_1 u(t) + C_2 v(t)], \tag{S_2}$$

$C_i \in \mathbf{R}$, $i = 1, 2$, $C_2 \neq 0$, or

$$Y(t, C_1', C_2', C_3') = U(t)\,[C_1' U^2(t) + C_2' U(t)\, V(t) + C_3' V^2(t)], \tag{S_3}$$

$C_j' \in \mathbf{R}$, $j = 1, 2, 3$, $C_3' \neq 0$.

Let us try to find the coefficients $c_{ij} \in \mathbf{R}$, $i, j = 1, 2$, of the centroaffine transformation

$$\begin{aligned} U(t) &= c_{11} u(t) + c_{12} v(t), \\ V(t) &= c_{21} u(t) + c_{22} v(t), \end{aligned} \tag{T}$$

where $c_{11}c_{22} - c_{12}c_{21} \neq 0$ such that the mutual coincidence of both the strongly conjugate points ${}^2t_{2k}$, ${}^1T_{2k}$, $k = 0, \pm 1, \pm 2, \dots$, and the weakly conjugate points ${}^1t_{2k+1}$, ${}^2T_{2k+1}$, $k = 0, \pm 1, \pm 2, \dots$, of ($S_2$) and ($S_3$) occurs on the interval $(-\infty, +\infty)$.

Since we assume at (S_3) the existence of the double weakly conjugate points for which ${}^2T_{2k+1} \in ({}^1T_{2k}, {}^1T_{2k+2})$, $k = 0, \pm 1, \pm 2, \dots$, there must hold for its parameters $C_j' \in \mathbf{R}$, $j = 1, 2, 3$, $C_3' \neq 0$, the condition $C_2'^2 - 4C_1'C_3' = 0$, hence it must be of the form

$$Y(t, C_2', C_3') = \frac{U(t)}{4C_3'}\,[C_2' U(t) + 2C_3' V(t)]^2. \tag{S_3^*}$$

It becomes apparent now that for both bundles (S_2), (S_3') to be vanishing (with distinct multiplicities) at the same mutual strongly conjugate points, there must exist a constant $\varkappa \in \mathbf{R} - \{\mathbf{0}\}$ such that

$$U(t) = \varkappa u(t),$$

which means that both functions $U(t)$ and $u(t)$ are linearly dependent on the interval $(-\infty, +\infty)$ so that $T_{2k} = t_{2k}$, $k = 0, \pm 1, \pm 2, \dots$, (thus specially for $k = 0$: $T_0 = t_0$) whence it follows that both pairs of the functions $u(t)$, $v(t)$ and $U(t)$, $V(t)$ meet the condition (P) at the same point t_0, where simultaneously $u(t_0) = U(t_0) = 0$, while $v(t_0) \neq 0$, $V(t_0) \neq 0$.

For both bundles (S_2), (S_3') to be vanishing (with distinct multiplicities) at the same weakly conjugate points, i.e. for to hold $T_{2k+1} = t_{2k+1}$, $k = 0, \pm 1, \pm 2, \ldots$, there must exist a constant $\lambda \in \mathbf{R} - \{0\}$ such that

$$C_2' U(t) + 2C_3' V(t) = \lambda [C_1 u(t) + C_2 v(t)],$$

wherefrom we get for $t = t_0$ that

$$2C_3' = \frac{\lambda C_2 v(t_0)}{V(t_0)}.$$

Combining these, we obtain the following result

$$U(t) = \varkappa u(t),$$

$$V(t) = \frac{(\lambda C_1 - \varkappa C_2') V(t_0)}{C_2 v(t_0)} u(t) + \frac{V(t_0)}{v(t_0)} v(t),$$

so that [in comparing with (T)]

$$c_{11} = \varkappa,$$

$$c_{12} = 0,$$

$$c_{21} = \frac{(\lambda C_1 - \varkappa C_2') V(t_0)}{C_2 v(t_0)},$$

$$c_{22} = \frac{V(t_0)}{v(t_0)},$$

where $\dfrac{V(t_0)}{v(t_0)} = \dfrac{\lambda C_2}{2C_3'}$; in fact $c_{11}c_{22} - c_{12}c_{21} = \dfrac{\varkappa V(t_0)}{v(t_0)} \neq 0.$

In this way we obtain following

Statement: The bundles (S_2) and (S_3') of solutions $y(t)$ and $Y(t)$ of the differential equation (1) in the forms and with conditions given in *Remark 3* are vanishing at the same strongly conjugate [for (S_2) double, for (S_3') simple] points $t_{2k} \in (-\infty, +\infty)$, $k = 0, \pm 1, \pm 2, \ldots$, and simultaneously at the same weakly conjugate [for (S_2) simple, for (S_3) double] points t_{2k+1} such that $t_{2k+1} \in (t_{2k}, t_{2k+2})$, $k = 0, \pm 1, \pm 2, \ldots$, if between the bases $[u(t), v(t)]$ and $[U(t), V(t)]$ of the oscillatory differential equation (2) there holds (T) with the just found expression of coefficients $c_{ij} \in \mathbf{R}$, $i, j = 1, 2$, of this transformation.

Specially: At $U(t) = u(t)$ [so that $\varkappa = 1$] and $V(t) = v(t)$ we get a necessary and sufficient condition for the mutual coincidence of the strongly conjugate points and simultaneously of the weakly conjugate points relative to both bundles (S_2), (S_3') of solutions $y(t)$, $Y(t)$ of (1) according to Statements 1. and 2. of the foregoing Theorem.

Now we will show two different types of bundles of solutions of (1). Namely a oneparametric bundle

$$y(t, C_1) = C_1 u^3(t), \qquad C_1 \in \mathbf{R} - \{0\}, \tag{S_1}$$

and a certain form of a threeparametric bundle

$$Y(t, C_1', C_2', C_3') = U(t)\left[C_1'U^2(t) + C_2'U(t)\,V(t) + C_3'V^2(t)\right], \tag{S$_3$}$$

$C_i' \in \mathbf{R}$, $i = 1, 2, 3$, $C_3' \neq 0$,

whose all zeros mutually coincide. They however differ from one another by their multiplicities: while all zeros of the bundle (S_1) have the multiplicity $\nu = 3$, all those of (S_3) have the multiplicity $\nu = 1$.

Theorem 4: There exist to every solution $y_0(t)$ from (S_1) of the solutions of (1), whose all zeros are triple, a subbundle of the solutions $Y_0(t)$ (differing from one another by an arbitrary nonzero multiplicative constant) from the bundle (S_3) of the solutions of the same differential equation (1) such that all zeros of $Y_0(t)$ are simple and coincide with all zeros of $y_0(t)$. All zeros of $y_0(t)$ and $Y_0(t)$ from both bundles (S_1), (S_3) are thereby mutually strongly conjugate.

Proof: We require from the functions $u(t)$, $v(t)$ from the basis $[u(t), v(t)]$ of the oscillatory differential equation (2) to satisfy the condition (P) at an arbitrary firmly chosen point $t_0 \in (-\infty, +\infty)$ again. Thus, all the zeros $t_k \in (-\infty, +\infty)$, $k = 0, \pm 1, \pm 2, \ldots$, of every solution $y_0(t)$ of the oscillatory differential equation (1) from a oneparametric bundle (S_1) are simultaneously the zeros of the function $u(t)$; they are altogether triple and mutually strongly conjugate (see Statement 3. of *Lemma 1* and Statement 1. of *Theorem 2.1* [1]).

The existence of solution $Y_0(t)$ [or of the subbundle of all such solutions differing from one another by an arbitrary nonzero multiplicative constant] from (S_3) of the solutions of (1), such that all its zeros coincide with all zeros of the solution $y_0(t)$ from the bundle (S_1) of the solutions of the same differential equation (1) having thereby altogether the multiplicity $\nu = 1$, will be proved by constructing such a solution.

Let us choose the function $U(t)$ from the basis $[U(t), V(t)]$ of the same differential equation (2) so that $U(t) = \lambda u(t)$, where $\lambda \in \mathbf{R} - \{0\}$, such that both functions $U(t)$ and $u(t)$ linearly dependent on the interval $(-\infty, +\infty)$ are vanishing at the same points t_k, $k = 0, \pm 1, \pm 2, \ldots$; hence, they both meet the condition (P) at the same point t_0. The condition $C_3' \neq 0$ in (S_3) means (according to Statement 1. of *Lemma 1*) that all solutions $Y(t)$ of (1) from (S_3) are simply vanishing at all zeros of the function $U(t)$ [and thus simultaneously of the function $u(t)$]. In other words, the points t_k, $k = 0, \pm 1, \pm 2, \ldots$, are also strongly conjugate (according to Statement 3. of *Theorem 2.1* [1] – altogether simple) points of (S_3) of the solutions of (1).

Now it is necessary to determine the condition for the constants $C_i' \in \mathbf{R}$, $i = 1, 2, 3$, $C_3' \neq 0$, so that the threeparametric system of all functions of the form

$$Y^*(t, C_1', C_2', C_3') = C_1'U^2(t) + C_2'U(t)\,V(t) + C_3'V^2(t),$$

from the bundle (S_3) doesnot possess any zeros on the interval $(-\infty, +\infty)$ [and thus, the bundle (S_3) of solutions $Y(t)$ of (1) doesnot possess any weakly conjugate points]. This condition exactly means (according to the I*st* part of the proof of *Theorem 1.1* [1]) that the inequality $C_2'^2 - 4C_1'C_3' < 0$ holds, whence it specially follows that there must $\operatorname{sgn} C_1' = \operatorname{sgn} C_3' \neq 0$; then

$$Y(t, C_1', C_2', C_3') =$$
$$= \frac{U(t)}{4C_3'} \{C_2'^2U^2(t) + 4C_3'[C_2'U(t)V(t) + C_3'V^2(t) + C_1'U^2(t)] - C_2'^2U^2(t)\} =$$
$$= \frac{U(t)}{4C_3'} \{[C_2'U(t) + 2C_3'V(t)]^2 + (4C_1'C_3' - C_2'^2)\, U^2(t)\} =$$
$$= \frac{U(t)}{4C_3'} [F^2(t) + G^2(t)], \qquad (S_3'')$$

where the functions $F(t) = \sqrt{4C_1'C_3' - C_2'^2}U(t)$ and $G(t) = C_2'U(t) + 3C_3'V(t)$ form a pair of linearly independent solutions of (2), hence they do **not possess any common** zeros on the interval $(-\infty, +\infty)$.

Consequently $F^2(t) + G^2(t) > 0$ for all $t \in (-\infty, +\infty)$ and thus the only zeros of every solution $Y(t)$ of the differential equation (1) from the bundle (S_3'') that we constructed from the bundle (S_3) are exactly the simple zeros ${}^1t_k$, $k = 0, \pm 1, \pm 2, \ldots,$ of the function $U(t)$ [i.e. also of the function $u(t)$], coinciding with all (triple) zeros ${}^3t_k$, $k = 0, \pm 1, \pm 2, \ldots,$ of every solution $y_0(t)$ from the bundle (S_1) and they are thereby in both bundles (S_1) and (S_3'') mutually strongly conjugate.

Remark 4. It is generally possible to construct a form of (S_3'') of the bundle (S_3) [in which the relation $4C_1'C_3' - C_2'^2 > 0$ holds between the parameters $C_i' \in \mathbf{R}$, $i = 1, 2, 3$, $C_3' \neq 0$] with required properties of the coincidence of all zeros relative to any solution $Y(t)$ with all zeros of $y(t)$ from the bundle (S_1) of solutions of the same differential equation (1) so that between both these bases $[u(t), v(t)]$ and $[U(t), V(t)]$ of the oscillatory differential equation (2) we choose a regular centro-affine transformation in the form

$$U(t) = c_{11}u(t),$$
$$V(t) = c_{21}u(t) + c_{22}v(t),$$

where $c_{ij} \in \mathbf{R}$, $i, j = 1,2$, $c_{12} = 0$ and where $c_{11}c_{22} \neq 0$; at this transformation all zeros of both functions $U(t)$ and $u(t)$ coincide (are immovable).

Remark 5. Similarly to the foregoing *Theorems 1–3*, also in the last case the Statement of *Theorem 4* is closely connected with some boundary value problems formulated for the oscillatory differential equation (1).

As a consequence of this theorem we get the following

Statement: If a solution $y_0(t)$ of an oscillatory differential equation (2) vanishes on a given set of points $\mathbf{M} = \{t_k\}$, $t_k \in (-\infty, +\infty)$, $k = 0, \pm 1, \pm 2, \dots$, $\lim_{k \to -\infty} t_k = -\infty$, $\lim_{k \to +\infty} t_k = +\infty$, so that $y_0(t_k) = 0$ and thereby $y_0(t) \neq 0$ for all $t \in (t_k, t_{k+1})$, $t_k < t_{k+1}$, $k = 0, \pm 1, \pm 2, \dots$, then all solutions $Y(t)$ of the oscillatory differential equation (1) are vanishing as well on this set $\mathbf{M}$ and namely either of the form $Y_1(t) = Cy_0^3(t)$ [with the multiplicity $v = 3$] or of the form $Y_2(t) = Cy_0(t)\,[y_1^2(t) + y_2^2(t)]$ (with the multiplicity $v = 1$), where $C \in \mathbf{R} - \{0\}$ is an arbitrary constant and where $y_1(t), y_2(t)$ are two arbitrary on the interval $(-\infty, +\infty)$ linearly independent solutions of the differential equation (2).

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Souhrn

O TRANSFORMACÍCH SVAZKŮ ŘEŠENÍ ITEROVANÉ LINEÁRNÍ DIFERENCIÁLNÍ ROVNICE 4. ŘÁDU

VLADIMÍR VLČEK

V práci jsou vyšetřovány vzájemné transformace dvou- a tříparametrických svazků řešení iterované lineární diferenciální rovnice 4. řádu, při nichž je u obou svazků dosaženo koincidence všech jejich nulových bodů za současné výměny slabě konjugovaných bodů za silně konjugované body a naopak. Tato výměna je realizována pomocí výměny jedné báze oscilatorické dif. rovnice (z níž uvažovaná dif. rovnice 4. řádu vznikne iterací) za jinou bázi téže rovnice 2. řádu užitím vhodné centroafinní transformace.

Dále je — při jedné a téže bázi dif. rovnice 2. řádu — ukázána transformace mezi uvedenými svazky řešení dif. rovnice 4. řádu, při níž se dosahuje koincidence všech nulových bodů svazků při ponechání vlastnosti jejich silné resp. slabé konjugovanosti, avšak kdy dochází k vzájemné výměně jejich násobností.

Nakonec je uvažována taková transformace mezi jednoparametrickým svazkem a jistým typem tříparametrického svazku řešení dif. rovnice 4. řádu, při níž všechny nulové body obou svazků — jakožto silně konjugované body s různými násobnostmi — koincidují.

Резюме

О ТРАНСФОРМАЦИЯХ ПУЧКОВ РЕШЕНИЙ ИТЕРИРОВАННОГО ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 4-ГО ПОРЯДКА

ВЛАДИМИР ВЛЧЕК

В работе изучаются трансформации между двумя типа пучков колеблющихся решений итерированного линейного дифференциального уравнения 4-го порядка. При первых двух (взаимно дуальных) отображениях двух- и трехпараметрических пучков на себя, осуществленных при помощи центроаффинных трансформаций пара базисов одного и того же линейного однородного дифференциального уравнения 2-го порядка (из которого соответствующее дифференциальное уравнение 4-го порядка возникло после итерации), достигается совпадения всех нулей этих пучков решений, но обмениваются слабо сопряженные точки за сильно сопряженные точки и наоборот.

Далее — при том же самом базисе дифференциального уравнения 2-го порядка — показана такая трансформация введенных пучков решений дифференциального уравнения 4-го порядка, при которой взаимного совпадения всех нулей обеих пучков достигается при взаимной замене их насобностей.

При последней трансформации одно— и трехпараметрических пучков решений достигается совпадение всех их нулей как сильно сопряженных точек (при взаимно отличающихся насобностях).