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ON SOME PROPERTIES OF SOLUTIONS OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS HAVING A COMMON ONEPARAMETRIC CONTINUOUS GROUP OF DISPERSIONS

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1. INTRODUCTION

O. Borůvka described in [4] the set of coefficients of all both side oscillatory equations on $\mathbf{R}$ $(=(-\infty, \infty))$ having the form

$$y'' = r(t)\,y, \qquad r \in C^0(\mathbf{R}), \tag{r}$$

which have a common oneparametric continuous group of dispersions. If the coefficients of two different equations relating to this set are π-periodic (on $\mathbf{R}$), then all its function possess the above property. The paper below presents another characterization of the set of equations with π-periodic coefficients having a common oneparametric continuous group of dispersions. This characterization is connected with the coincidence of finite intervals of nonstability in a certain equation $y'' = (q(t) + \lambda p(t))\,y$ with π-periodic coefficients p, q; $\lambda \in \mathbf{R}$.

2. DEFINITIONS, NOTATION, BASIC PROPERTIES

Let (r) be an on both side oscillatory equation on $\mathbf{R}$ (meaning thereby that any nontrivial solution of (r) has infinitely many zeros on the right and on the left of each point t_0). Say that a function α is the (first) phase of (r) if there exist its independent solutions u, v:

$$\operatorname{tg} \alpha(t) = u(t)/v(t) \qquad \text{for } t \in \mathbf{R} - \{t \in \mathbf{R};\, v(t) = 0\}.$$

Any phase α of (r) has the following three properties:

(i) $\alpha \in C^3(\mathbf{R})$,

(ii) $\alpha'(t) \neq 0$ for $t \in \mathbf{R}$,

(iii) $\alpha(\mathbf{R}) = \mathbf{R}$.

The phase α of (r) uniquely determines the coefficient r of this equation in the sense as follows

$$r(t) = -\{\alpha, t\} - \alpha'^2(t), \qquad t \in \mathbf{R},$$

where $\{\alpha, t\} := \alpha'''(t)/2\alpha'(t) - (3/4)(\alpha''(t)/\alpha'(t))^2$ is the Schwarzian derivative of the function α at the point t.

Say that the function α is a phase function if it possesses the properties (i)–(iii). Any phase of (r) is a phase function. The set of phase functions forms the group $\mathfrak{G}$ with respect to rule of composition of functions.

Let (q), (Q) be both side oscillatory equations, $q \in C^\circ(\mathbf{R})$, $Q \in C^0(\mathbf{R})$. Any solution X, $X'(t) \neq 0$ for $t \in \mathbf{R}$, of the differential equation

$$\text{(Qq)} \qquad -\{X, t\} + X'^2 \,.\, Q(X) = q(t)$$

is called the general dispersion of (q) and (Q) (in the above order). The general dispersion X of (q) and (Q) is a phase function and possesses the following characteristic property: There is one-to-one correspondence of solutions y of (q) and Y of (q) given by

$$y(t) = YX(t)/|X'(t)|^{1/2}, \qquad t \in \mathbf{R}. \tag{1}$$

Let X be a general dispersion of (q) and (Q) and let X^{-1} denote the inverse function to the function X (on $\mathbf{R}$). Then X^{-1} is a general dispersion of (Q) and (q), consequently, it is a solution of (qQ).

Let $q = Q$. Then the solutions of (qq) are called the dispersions (of the first kind) of (q). The dispersions of (q) form a group with respect to the rule of composition, which is called the group of dispersions of (q); denotion $\mathscr{L}_q$. The set $\mathscr{L}_q^+$ of increasing dispersions of (q) is a subgroup of the group $\mathscr{L}_q$. Let α be a phase of (q). Then the function $\varphi(t) := \alpha^{-1}(\alpha(t) + \pi \,.\, \operatorname{sign} \alpha')$, $t \in \mathbf{R}$, is a dispersion of (q); $\varphi \in \mathscr{L}_q^+$. The function φ is called the basic central dispersion (of the first kind) of (q).

All the above definitions and properties are stated in [2], [3].

Conformably with [4] let us say that a group $\mathfrak{A}$, $\mathfrak{A} \subset \mathfrak{G}$, is a oneparametric continuous group if exactly one element from $\mathfrak{A}$ passes through any point $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}$. In other words, to any point (t_0, x_0) there exists one and only one function $X \in \mathfrak{A}$ such that $X(t_0) = x_0$.

Lemma 1 ([4]).

Let (q_1), (q_2) *be both side oscillatory equations,* $q_1 \in C^0(\mathbf{R})$, $q_2 \in C^0(\mathbf{R})$, $q_1 - q_2 \in C^2(\mathbf{R})$, $q_1(t) \neq q_2(t)$, $t \in \mathbf{R}$. *Then* $\mathrm{P}^+_{q_1 q_2} := \mathscr{L}^+_{q_1} \cap \mathscr{L}^+_{q_2}$ *is a oneparametric continuous group (we say that* (q_1) *and* (q_2) *have a common oneparametric continuous group of dispersions) just if there exist a phase function* X *and positive numbers* $k_1, k_2, k_1 \neq k_2$, *such that*

$$q_i(t) = -\{X, t\} - k_i \,.\, X'^2(t), \qquad t \in \mathbf{R}, \qquad i = 1, 2.$$

Let X be a phase function and put $[\mathbf{X}] := \{q(t);\ q(t) = -\{X, t\} - k \,.\, X'^2(t),\ k \in \mathbf{R}^+,\ t \in \mathbf{R}\}$, where $\mathbf{R}^+ := (0, \infty)$. It follows from Lemma 1 that $[\mathbf{X}]$ is the set of coefficients q of exactly those both side oscillatory equations (q), that $\bigcap_{q \in [X]} \mathscr{L}_q^+$ is a oneparametric continuous group. It is easy to verify that $[\mathbf{X}] = [\mathbf{Y}]$ holds for the phase functions X, Y exactly if $X = aY + b$, where $a \neq 0$, b are constants. The equality $[\mathbf{X}] = [\mathbf{Y}]$ is to be taken as an equality of two sets.

Assume that r is a continuous and a π-periodic function. Let $\lambda \in \mathbf{R}$ and u, v be solutions of the equation $y'' = (r(t) + \lambda)\, y$ satisfying the initial conditions $u(0) = v'(0) = 0$, $u'(0) = v(0) = 1$. Let us put $\Delta(\lambda) := v(\pi) + u'(\pi)$. We know from the Floquet theory ([6]–[8], [10]) that there exist sequences $\{\lambda_i\}_{i=0}^{\infty}$, $\{\lambda'_i\}_{i=1}^{\infty}$,

$$\lambda_0 > \lambda'_1 \geqq \lambda'_2 > \lambda_1 \geqq \lambda_2 > \lambda'_3 \geqq \lambda'_4 > \dots$$

such that $\Delta(\lambda) = 2$ exactly for $\lambda = \lambda_i$ $(i = 0, 1, 2, \dots)$ and $\Delta(\lambda) = -2$ exactly for $\lambda = \lambda'_i$ $(i = 1, 2, 3, \dots)$. The intervals $[\lambda_{2n-1}, \lambda_{2n}]$, $[\lambda'_{2n-1}, \lambda'_{2n}]$ $(n = 1, 2, 3, \dots)$ are called the finite intervals of nonstability of the equation (r + λ). The above mentioned intervals degenerate to one point, i.e. $\lambda'_1 = \lambda'_2$, $\lambda_1 = \lambda_2$, $\lambda'_3 = \lambda'_4, \dots$ exactly if $r(t) =$ a constant. That is, the equation (r + λ) is for any $\lambda < \lambda_0$ stable on $\mathbf{R}$ (all solutions of the equation (r + λ) are bounded on $\mathbf{R}$) exactly if $r(t) =$ a constant (cf. [1], [6], [10]). The equation (r + λ) is stable exactly if there exists a phase α of this equation such that

$$\alpha(t + \pi) = \alpha(t) + a, \qquad t \in \mathbf{R},$$

where $a \neq 0$ is a constant (cf. [3]).

It should be noted here that the equation (r) with a π-periodic coefficient r is either both side oscillatory or disconjugate (i.e. any nontrivial solution of (r) has one zero on $\mathbf{R}$ at most).

3. MAIN RESULTS

Theorem 1.

Let p, q be π-periodic functions, $p \in C^2(\mathbf{R})$, $q \in C^0(\mathbf{R})$, $p(t) > 0$ for $t \in \mathbf{R}$. Let the equation

(q + λp) $$y'' = (q(t) + \lambda p(t))\, y, \qquad \lambda \in \mathbf{R},$$

be both side oscillatory for $\lambda < \lambda_0 > 0$ and disconjugate for $\lambda \geqq \lambda_0$. The equation (q + λp) is then stable on $\mathbf{R}$ for any $\lambda < \lambda_0$ exactly if there exists a phase function X such that

$$[\mathbf{X}] = \{q + \lambda p;\, \lambda < \lambda_0\}.$$

Remark 1.

It is implied from Theorem 1 that for $\lambda < \lambda_0$ both side oscillatory equations (q + λp) have a common oneparametric continuous group of dispersions exactly if the equation (q + λp) is stable for any $\lambda < \lambda_0$.

To prove Theorem 1 we use the following lemmas:

Lemma 2.

Let X be a phase function. Then there exists exactly one function $q \in [\mathbf{X}]$ *such that X is a phase of* (q).

Proof. Let X be a phase function. Putting $q(t) := -\{X, t\} - X'^2(t)$, $t \in \mathbf{R}$, we have $q \in [\mathbf{X}]$ and X is a phase of (q) (see § 2).

Lemma 3.

Let X be a phase function. Then two different π*-periodic functions lie in the set* $[\mathbf{X}]$ *if and only if all functions in* $[\mathbf{X}]$ *are* π*-periodic.*

Proof. The proof in one direction is obvious. Let now q_1, q_2 be π-periodic function in $[\mathbf{X}]$, $q_1 \neq q_2$. It follows from the definition of the set $[\mathbf{X}]$ the existence of numbers $k_1, k_2 \in \mathbf{R}^+, k_1 \neq k_2 : q_i(t) = -\{X, t\} - k_i \,.\, X'^2(t)$. Since $q_2(t) - q_1(t)$ is a π-periodic function and $q_2(t) - q_1(t) = (k_1 - k_2)\, X'^2(t)$, then $X'(t)$ is a π-periodic function. Hence, also the function $-\{X, t\} - k \,.\, X'^2(t)$ is π-periodic for any $k \in \mathbf{R}^+$.

From the proof of Lemma 3 follows:

Corollary 1.

Let X be a phase function. Then $[\mathbf{X}]$ *contains two different* π*-periodic functions exactly if* X' *is a* π*-periodic function.*

Lemma 4.

Let X be a phase function and X' *a* π*-periodic function. Let* $q \in [\mathbf{X}]$. *Then* (q) *is stable on* $\mathbf{R}$.

Proof. Let the assumptions of Lemma 4 be fulfilled. According to Corollary 1 is q a π-periodic function. From the assumptions now follows the existence of numbers $a \neq 0$ and $k \in \mathbf{R}^+$ such that $X(t + \pi) = X(t) + a$, $q(t) = -\{X, t\} - k \,.\, X'^2(t)$. Putting $\alpha(t) := k^{1/2} \,.\, X(t)$, $t \in \mathbf{R}$, then α is a phase of (q) and it follows from $\alpha(t + \pi) = k^{1/2} \,.\, X(t + \pi) = k^{1/2} \,.\, X(t) + ak^{1/2} = \alpha(t) + ak^{1/2}$ that (q) is stable on $\mathbf{R}$.

Corollary 2.

Let X be a phase function and let two different π*-periodic functions exist in* $[\mathbf{X}]$. *Then the equation* (q) *is stable on* $\mathbf{R}$ *for any* $q \in [\mathbf{X}]$.

Proof. It follows from Corollary 1 that the function X' is π-periodic and by Lemma 4 the equation (q) are stable on $\mathbf{R}$ for any $q \in [\mathbf{X}]$.

Lemma 5.

Let p, q be continuous π*-periodic functions,* $p(t) > 0$ *for* $t \in \mathbf{R}$. *Then there exists a number* λ_0 *such that the equation* $(q + \lambda p)$ *is both side oscillatory for* $\lambda < \lambda_0$, *and disconjugate for* $\lambda \geqq \lambda_0$.

Proof. Let the assumptions of Lemma 5 be fulfilled. It is evident that the equation $(q + \lambda p)$ is not for all λ either both side oscillatory or disconjugate. We know ([5]) that for a fixed t is the basic central dispersion $\varphi(t, \lambda)$ of $(q + \lambda p)$ a continuous function of λ whenever it is defined. Herefrom and from the Sturm comparison theorem then follows the existence of the number λ_0 having the properties given in Lemma 5.

Lemma 6.

Let p, q be π-periodic function, $p \in C^2(\mathbf{R})$, $q \in C^0(\mathbf{R})$, $p(t) > 0$ for $t \in \mathbf{R}$. Let $\lambda_0 > 0$ be a number with the property stated in Lemma 5. Then the equation (q + λp) *is stable for all $\lambda < \lambda_0$ if and only if $\lambda_0^{1/2} \int_0^t p^{1/2}(\tau)\, d\tau$ is a phase of* (q).

Proof. Let the assumptions of Lemma 6 be fulfilled. We prove first that (q + λp) may be transformed onto an equation of the type

$$(\mathrm{Q} + \mu) \qquad y'' = (Q(t) + \mu) . y, \qquad \mu \in \mathbf{R},$$

where Q is a continuous π-periodic function (cf. [9]). Putting $s := \pi / \int_0^\pi p^{1/2}(\tau)\, d\tau (> 0)$, $X(t) := s \int_0^\pi p^{1/2}(\tau)\, d\tau$, $t \in \mathbf{R}$, then $X \in C^3(\mathbf{R})$, $X'(t) > 0$, $X(t + \pi) = s \int_0^\pi p^{1/2}(\tau)\, d\tau = X(t) + s \int_t^{t+\pi} p^{1/2}(\tau)\, d\tau = X(t) + \pi$ for $t \in \mathbf{R}$ and $X(\mathbf{R}) = \mathbf{R}$. Let X^{-1} be the inverse function to X on $\mathbf{R}$. Obviously $X^{-1} \in C^3(\mathbf{R})$, $X^{-1\prime}(t) > 0$ and $X^{-1}(t + \pi) = X^{-1}(t) + \pi$ for $t \in \mathbf{R}$. Putting $Q(t) := -\{X^{-1}, t\} + X^{-1\prime 2}(t) . q(X^{-1}(t))$, $t \in \mathbf{R}$, then $Q \in C^0(\mathbf{R})$ is a π-periodic function and it holds

$$(2) \qquad -\{X, t\} + X'^2(t) . Q(X(t)) = q(t), \qquad t \in \mathbf{R},$$

whence

$$-\{X, t\} + X'^2(t) . (Q(X(t)) + \lambda s^{-2}) = q(t) + \lambda p(t).$$

It follows from the theory of dispersions that there is one-to-one correspondence of solutions y of (q + λp) and Y of (Q + λs^{-2}) given by (1).

According to the assumption, the equation (q + λp) is both side oscillatory for $\lambda < \lambda_0$ and disconjugate for $\lambda \geqq \lambda_0$. Consequently also (Q + λs^{-2}) is both side oscillatory for $\lambda < \lambda_0$ and disconjugate for $\lambda \geqq \lambda_0$. We know from the Floquet theory that (Q + λs^{-2}) is nonstable for $\lambda \geqq \lambda_0$. Hence (q + λp) is also nonstable for these λ. Let (q + λp) be stable for any $\lambda < \lambda_0$ and thus (Q + λs^{-2}) is also stable for this λ. This is possible exactly if the finite intervals of nonstability of (Q + λs^{-2}) coincide which occurs exactly if $Q(t) =$ a constant $(:= k)$. Since (k + λs^{-2}) is stable exactly if $k + \lambda s^{-2} < 0$, there is necessarily $k = -\lambda_0 s^{-2}$. Then $-\{X, t\} - \lambda_0 s^{-2} . X'^2(t) = q(t)$ follows from (2), which implies that $\lambda_0^{1/2} s^{-1} . X(t) = \lambda^{1/2} \int_0^t p^{1/2}(\tau)\, d\tau$ is a phase of (q).

Let $X(t) := \lambda_0^{1/2} \int_0^t p^{1/2}(\tau)\, d\tau$, $t \in \mathbf{R}$, be a phase of (q). Then $X(t + \pi) = X(t) + a$, where $a := \lambda_0^{1/2} \int_0^\pi p^{1/2}(\tau)\, d\tau$ and therefore (q) is stable. Let $\mu \in \mathbf{R}^+$ and put $Y_\mu(t) := \mu . X(t)$, $t \in \mathbf{R}$. Then Y_μ is a phase of (r_μ), where $r_\mu(t) = -\{Y_\mu, t\} - Y_\mu'^2(t) = -\{X, t\} - \mu^2 . X'^2(t) = -\{X, t\} - X'^2(t) + (1 - \mu^2) X'^2(t) = q(t) + (1 - \mu^2) \lambda_0$.

$. p(t) = q(t) + \lambda p(t);\ \lambda := (1 - \mu^2)\,\lambda_0$. Since $Y_\mu(t + \pi) = \mu\,.\,(X(t) + a) = Y_\mu(t) + \mu a$, the equation (r_μ) is stable. Consequently $(q + \lambda p)$ is stable for $\lambda < \lambda_0$, which we were to prove.

Proof of Theorem 1. Let the assumptions of Theorem 1 be fulfilled and $(q + \lambda p)$ be stable for any $\lambda < \lambda_0$. By Lemma 6 is then the function $X(t) := \lambda_0^{1/2} \int_0^t p^{1/2}(\tau)\,d\tau$, $t \in \mathbf{R}$, a phase of (q) and we get from the second part of the proof to Lemma 6 that $[\mathbf{X}] = \{q + \lambda p;\ \lambda < \lambda_0\}$.

Let $[\mathbf{X}] = \{q + \lambda p;\ \lambda < \lambda_0\}$. According to Corollary 1, X' is then a π-periodic function which means by Corollary 2 that $(q + \lambda p)$ are stable for $\lambda < \lambda_0$.

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SOUHRN

O NĚKTERÝCH VLASTNOSTECH ŘEŠENÍ LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 2. ŘÁDU S PERIODICKÝMI KOEFICIENTY, KTERÉ MAJÍ SPOLEČNOU JEDNOPARAMETRICKOU SPOJITOU GRUPU DISPERSÍ

SVATOSLAV STANĚK

Nechť $r \in C^0(\mathbf{R})$ a nechť rovnice (r): $y'' = r(t)\,y$ je oscilatorická na $\mathbf{R}$. Řekneme, že funkce X, $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ pro $t \in \mathbf{R}$, je disperse rovnice (r), když je řešením rovnice

$$-X'''/2X' + (3/4)\,(X''/X')^2 + X'^2\,.\,r(X) = r(t).$$

Podgrupa $\mathfrak{A}$ grupy $\mathscr{L}_r$ dispersí rovnice (r) se nazývá jednoparametrická spojitá grupa dispersí, jestliže každým bodem $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}$ prochází právě jedna funkce z $\mathfrak{A}$. V práci je dokázána věta: Nechť p, q jsou π-periodické funkce, $p \in C^2(\mathbf{R})$, $q \in C^0(\mathbf{R})$, $p(t) > 0$ pro $t \in \mathbf{R}$. Nechť rovnice

(q + λp) $$y'' = (q(t) + \lambda p(t))\, y$$

je oscilatorická pro $\lambda < \lambda_0 > 0$ a diskonjugovaná pro $\lambda \geqq \lambda_0$. Pak pro každé $\lambda < \lambda_0$ je rovnice (q + λp) stabilní na $\mathbf{R}$ právě když pro každé $\lambda < \lambda_0$ mají rovnice (q + λp) stejnou jednoparametrickou spojitou grupu dispersí.

РЕЗЮМЕ

ОБ НЕКОТОРЫХ СВОЙСТВАХ РЕШЕНИЙ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ 2-ОГО ПОРЯДКА С ПЕРИОДИЧЕСКИМИ КОЭФФИЦИЕНТАМИ КОТОРЫЕ ИМЕЮТ СОВМЕСТНУЮ ОДНОПАРАМЕТРИЧЕСКУЮ НЕПРЕРЫВНУЮ ГРУППУ ДИСПЕРСИЙ

СВАТОСЛАВ СТАНЕК

Пусть $r \in C^0(\mathbf{R})$ и уравнение (r) : $y'' = r(t)\, y$ — осциллирующее на $\mathbf{R}$. Функция X, $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ для $t \in \mathbf{R}$, называется дисперсией уравнения (r), если является решением уравнения

$$-X'''/2X' + (3/4)(X''/X')^2 + X'^2 \,.\, r(X) = r(t).$$

Подгруппа $\mathfrak{A}$ группы $\mathscr{L}_r$ дисперсий уравнения (r) называется однопараметрическая непрерывная группа, если каждой точкой $(t_0, x_0) \in \mathbf{R} + \mathbf{R}$ проходит только одна функция из $\mathfrak{A}$. В работе приводится теорема: Пусть p, q — периодические функции, $p \in C^2(\mathbf{R})$, $q \in C^0(\mathbf{R})$, $p(t) > 0$, $t \in \mathbf{R}$. Пусть

(q + λp) $$y'' = (q(t) + \lambda p(t))\, y$$

является осциллирующим уравнением для $\lambda < \lambda_0 > 0$ и уравнением без сопряженных точек для $\lambda \geqq \lambda_0$. Тогда уравнение (q + λp) устойчиво для $\lambda < \lambda_0$ тогда и только тогда, когда уравнения (q + λp) имеют для $\lambda < \lambda_0$ совместную однопараметрическую нептерывную группу дисперсий.