Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Svatoslav Staněk

On the intersection of the set of solutions of two Kummer's differential equations

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 23 (1984), No. 1, 25--32

Persistent URL: http://dml.cz/dmlcz/120147

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1984 VOLUMEN 79 PAG. 25–32

Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty University Palackého v Olomouci
Vedoucí katedry: Miroslav Laitoch, Prof., RNDr., CSc.

ON THE INTERSECTION OF THE SET OF SOLUTIONS OF TWO KUMMER'S DIFFERENTIAL EQUATIONS

SVATOSLAV STANĚK

(Received March 20th, 1982)

1. Introduction

We investigate the intersection of the set of solutions of two Kummer's differential equations

$$-\{X, t\} + X'^2 . p(X) = q(t), \tag{pq}$$

$$-\{X, t\} + X'^2 . P(X) = Q(t), \tag{PQ}$$

where $\{X, t\} := \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\left(\frac{X''(t)}{X'(t)}\right)^2$ is Schwarz's derivative of the function X at the point t. This intersection has been investigated with $p = q$, $P = Q$ in [3] and [4], where an algebraic approach was applied for the set of solutions of (pp) which is a three-parametric continuous group with respect to the composition of functions, providing that the equation (p) : $y'' = p(t)\,y$ is oscillatory (see [1], [2]).

2. Basic concepts and notation

Throughout the differential equations of the type

$$y'' = q(t)\,y, \qquad q \in C^0(\mathbf{R}), \tag{q}$$

are considered to be oscillatory on $\mathbf{R}$, i.e. $\pm\infty$ are cluster points of zeros of any nontrivial solution of (q).

A function $\alpha \in C^0(\mathbf{R})$ is a phase of (q) exactly if there exist independent solutions u, v of (q):

$$\operatorname{tg} \alpha(t) = u(t)/v(t) \qquad \text{for } t \in \mathbf{R} - \{t; v(t) = 0\}.$$

Every phase α of (q) has the following properties:

(i) $\alpha \in C^3(\mathbf{R})$;
(ii) $\alpha'(t) \neq 0$ for $t \in \mathbf{R}$;
(iii) $\alpha(\mathbf{R}) = \mathbf{R}$;
(iv) $-\{\alpha, t\} - \alpha'^2(t) = q(t)$ for $t \in \mathbf{R}$.

The set of phases of the equation $y'' = -y$ will be denoted by $\mathfrak{E}$. This set forms a group with respect to the composition of functions.

Let (p) and (q) be oscillatory equations with α and β being their phases, respectively. The symbols $\mathscr{L}_{pq}$, $\mathscr{L}^+_{pq}$, $\mathscr{L}^-_{pq}$ refer to the set of all solutions, to the set of all increasing solutions and to the set of all decreasing solutions of (pq), respectively. Then

$$\mathscr{L}_{pq} = \{\alpha^{-1}\varepsilon\beta;\ \varepsilon \in \mathfrak{E}\},$$
$$\mathscr{L}^+_{pq} = \{\alpha^{-1}\varepsilon\beta;\ \varepsilon \in \mathfrak{E},\ \text{sign}\,\varepsilon' = \text{sign}\,\alpha' \,.\, \text{sign}\,\beta'\},$$
$$\mathscr{L}^-_{pq} = \{\alpha^{-1}\varepsilon\beta;\ \varepsilon \in \mathfrak{E},\ \text{sign}\,\varepsilon' = -\text{sign}\,\alpha' \,.\, \text{sign}\,\beta'\},$$

and for any $X \in \mathscr{L}_{pq}$ we have $X(\mathbf{R}) = \mathbf{R}$ (see [1], [2]).

Let $\mathscr{S} \subset \mathscr{L}_{pq}$. Say that $\mathscr{S}$ is a complete set (in $\mathbf{R} \times \mathbf{R}$) if and only if there exists for each $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}$ only one element $X \in \mathscr{S}$ such that $X(t_0) = x_0$.

In what follows we take the equations (p), (q) to be oscillatory and $P, Q \in C^0(\mathbf{R})$, which fact will be explicitly pointed out in assumptions of Theorems, only.

The symbols $\mathscr{P}^+_{pqPQ}$ and $\mathscr{P}^-_{pqPQ}$ refer to the sets $\mathscr{L}^+_{pq} \cap \mathscr{L}^+_{PQ}$ and $\mathscr{L}^-_{pq} \cap \mathscr{L}^-_{PQ}$, respectively.

3. Lemmas

Lemma 1. *Let X, Y be increasing or decreasing solutions of* (pq), *$X \neq Y$ and let $X(t_0) = Y(t_0)$ for a $t_0 \in \mathbf{R}$. Then the functions p, q are uniquely determined by the functions X, Y.*

Proof. Let $X, Y \in \mathscr{L}_{PQ}$. We prove $p = P$, $q = Q$. First we have

$$\tau \,.\, X'(t)\,(|\,P[X(t)] - p[X(t)]\,|)^{1/2} = (|\,Q(t) - q(t)\,|)^{1/2}, \tag{1}$$
$$\tau \,.\, Y'(t)\,(|\,P[Y(t)] - p[Y(t)]\,|)^{1/2} = (|\,Q(t) - q(t)\,|)^{1/2}, \qquad t \in \mathbf{R},$$

where $\tau := \text{sign}\, X'$. Herefrom for $t \in \mathbf{R}$

$$X'(t)\,(|\,P[X(t)] - p[X(t)]\,|)^{1/2} = Y'(t)\,(|\,P[Y(t)] - p[Y(t)]\,|)^{1/2} \tag{2}$$

Integrating (2) from t_0 to t in applying the substitution method gives

$$\int_{X(t)}^{Y(t)} (|\,P(s) - p(s)\,|)^{1/2}\, ds = 0, \qquad t \in R. \tag{3}$$

By the theorem on the uniqueness of solutions of (pq) (see [1]) the equality $X(t) = Y(t)$ holds on no interval. It follows from (3) that $p = P$ and then from (1) we obtain $q = Q$.

Remark 1. It follows from Lemma 1 that the sets $\mathscr{P}^{+}_{pqPQ}$, $\mathscr{P}^{-}_{pqPQ}$ may be "at most" a complete set provided that at least one of the assumptions $p \neq P$, $q \neq Q$ holds.

Lemma 2. *Let $\mathscr{P}^{+}_{pqPQ}$ (or $\mathscr{P}^{-}_{pqPQ}$) be a complete set. Then*

$$(Q(t) - q(t))(P(t) - p(t)) > 0 \qquad \textit{for } t \in \mathbf{R}. \tag{4}$$

Proof. Let $\mathscr{P}^{-}_{pqPQ}$ be a complete set. Then we have for every $X \in \mathscr{P}^{-}_{pqPQ}$

$$X'^2(t)(P[X(t)] - p[X(t)]) = Q(t) - q(t), \qquad t \in \mathbf{R}, \tag{5}$$

and from this in view of the fact that $\mathscr{P}^{-}_{pqPQ}$ is a complete set

$$(P(t) - p(t))(Q(t) - q(t)) \geqq 0, \qquad t \in \mathbf{R}. \tag{6}$$

Let $q(t_0) = Q(t_0)$, $t_1 \in \mathbf{R}$ and let $X \in \mathscr{P}^{-}_{pqPQ}$, $X(t_0) = t_1$. Then, putting $t = t_0$ in (5), gives $p(t_1) = P(t_1)$. Therefore $p = P$ and naturally also $q = Q$. Consequently $\mathscr{P}^{-}_{pqPQ}$ is a three-parametric set.

Let $p(t_0) = P(t_0)$, $t_1 \in \mathbf{R}$ and $X \in \mathscr{P}^{-}_{pqPQ}$, $X(t_1) = t_0$. Then, putting $t = t_1$ in (5), gives $q(t_1) = Q(t_1)$. Therefore $q = Q$ and naturally also $p = P$. Consequently $\mathscr{P}^{-}_{pqPQ}$ is a three-parametric set.

On that account (4) is true. Similarly for $\mathscr{P}^{+}_{pqPQ}$.

Lemma 3. *Let $X, Y \in \mathscr{P}^{+}_{pqPQ}$ (or $X, Y \in \mathscr{P}^{-}_{pqPQ}$), $X \neq Y$, and either $p \neq P$ or $q \neq Q$.* Let us put

$$r(t) := \int_{t_0}^{t} (|P(s) - p(s)|)^{1/2}\,\mathrm{d}s,\; s(t) := \int_{t_0}^{t} (|Q(s) - q(s)|)^{1/2}\,\mathrm{d}s, \qquad t \in \mathbf{R}. \tag{7}$$

Then $r(\mathbf{R}) = s(\mathbf{R}) = \mathbf{R}$.

Proof. Let us put $\tau := \operatorname{sign} X'$ $(= \operatorname{sign} Y')$. Then, integrating (1) from t_0 to t and with reference to definition (7) of the functions r and s, we obtain

$$\begin{aligned} \tau\,.\,r[X(t)] &= s(t) + a_1, \\ \tau\,.\,r[Y(t)] &= s(t) + a_2, \qquad t \in \mathbf{R}, \end{aligned} \tag{8}$$

where $a_1 = \tau\,.\,r[X(t_0)]$, $a_2 = \tau\,.\,r[Y(t_0)]$. If $a_1 = a_2$, then we get

$$\int_{X(t)}^{Y(t)} (|P(s) - p(s)|)^{1/2}\,\mathrm{d}s = 0 \qquad \text{for } t \in \mathbf{R}$$

from (8), which however holds exactly if $p = P$ and $q = Q$. Consequently $a_1 \neq a_2$.

To prove the above assertion is suffices to show that $s(\mathbf{R}) = \mathbf{R}$. The functions r, s are monotone on $\mathbf{R}$. If $\lim_{t \to -\infty} s(t) = b > -\infty$, then from (8) we get

$$\tau \lim_{t \to -\tau\infty} r(t) = \tau \lim_{t \to -\infty} r[X(t)] = b + a_1,$$

$$\tau \lim_{t \to -\tau\infty} r(t) = \tau \lim_{t \to -\infty} r[Y(t)] = b + a_2,$$

which, however, conflicts with $a_1 \neq a_2$. Hence $b = -\infty$. If $\lim_{t\to\infty} s(t) = c < \infty$, then it follows from (8)

$$\tau \lim_{t\to\tau\infty} r(t) = \tau \lim_{t\to\infty} r[X(t)] = c + a_1,$$

$$\tau \lim_{t\to\tau\infty} r(t) = \tau \lim_{t\to\infty} r[Y(t)] = c + a_2,$$

which again conflicts with $a_1 \neq a_2$. Thus $c = \infty$.

Lemma 4. Let $\mathscr{P}^+_{pqPQ}$ (or $\mathscr{P}^-_{pqPQ}$) be a complete set. Then

$$p - P \in C^2(\mathbf{R}), \qquad q - Q \in C^2(\mathbf{R}).$$

Proof. Let $\mathscr{P}^-_{pqPQ}$ be a complete set. Then necessarily $p \neq P$, $q \neq Q$. Let the functions r, s be defined by (7). Then it holds for every $X \in \mathscr{P}^-_{pqPQ}$ that

$$-r[X(t)] = s(t) + a, \qquad t \in \mathbf{R}, \tag{9}$$

with $a := -r[X(t_0)]$. It follows from Lemma 3 that $r(\mathbf{R}) = s(\mathbf{R}) = \mathbf{R}$.

Let α and β be increasing phases of (p) and (q), respectively. Then there exists to every $X \in \mathscr{P}^-_{pqPQ}$ (exactly one) $\varepsilon \in \mathfrak{E}$, sign $\varepsilon' = -1$: $X = \alpha^{-1}\varepsilon\beta$. From this and from (9) we find that $-r\alpha^{-1}\varepsilon = s\beta^{-1} + a$. Putting $R(t) := r[\alpha^{-1}(t)]$, $S(t) := s[\beta^{-1}(t)]$, $t \in \mathbf{R}$, yields

$$-R[\varepsilon(t)] = S(t) + a, \qquad t \in \mathbf{R}, \tag{10}$$

whereby to every $b \in \mathbf{R}$ there exists (exactly one) $\varepsilon_1 \in \mathfrak{E}$, such that $-R[\varepsilon_1(t)] = S(t) + b$. The set of such ε_1 will be denoted by $\mathfrak{E}_1$ ($\subset \mathfrak{E}$). Let now $\varepsilon_2 \in \mathfrak{E}_1$ be such that $-R[\varepsilon_2(t)] = S(t)$. Then $R\{\varepsilon_1[\varepsilon_2^{-1}(t)]\} = R(t) - b$ and therefore $\mathfrak{E}_1\varepsilon_2^{-1} = \{R^{-1}[R(t) + R(a)]; a \in \mathbf{R}\}$. The set $\mathfrak{E}_1\varepsilon_2^{-1}$ is a complete set forming a group with respect to the composition of functions. We put $\tau(t, a) := R^{-1}[R(t) + R(a)]$, $(t, a) \in \mathbf{R}\times\mathbf{R}$. Since $\tau(t, a) = \tau(a, t)$, both $\tau(., a)$ and $\tau(t, .)$ are phases of $y'' = -y$. Consequently, $\tau(t, a)$ has continuous partial derivatives of all orders on $\mathbf{R}\times\mathbf{R}$. Differentiating with respect to the variable a in $R[\tau(t, a)] = R(t) + R(a)$, we find that $R'[\tau(t, a)] \cdot \frac{\partial \tau}{\partial a}(t, a) = R'(a)$, and specially for $a = 0$: $R'[\tau(t, 0)] \cdot \frac{\partial \tau}{\partial a}(t, 0) = R'(0)$. Putting $\varepsilon_3(t) := \tau(t, 0)$, $v(t) := \frac{\partial \tau}{\partial a}(t, 0)$ $(\neq 0)$ for $t \in \mathbf{R}$, gives $R'[\varepsilon_3(t)] \times v(t) = R'(0)$, whence $R'(t) = \frac{R'(0)}{v[\varepsilon_3^{-1}(t)]}$, $t \in \mathbf{R}$, and, since ε_3 and v have derivatives of all orders on $\mathbf{R}$, it is easily seen that the function R possesses this property, too. Because of $r = R\alpha$ and $\alpha \in C^3(\mathbf{R})$ we have $r \in C^3(\mathbf{R})$ and from (9) we have $s \in C^3(\mathbf{R})$. Further, on taking account of Lemma 2 we obtain the assertion of the Lemma.

The proof proceeds similarly for $\mathscr{P}^+_{pqPQ}$ being a complete set.

Lemma 5. *$\mathscr{P}^+_{pqPQ}$ is a complete set exactly if*

$$P(t) = p(t) + k \,.\, \alpha'^2(t), \qquad Q(t) = q(t) + k \,.\, \beta'^2(t), \qquad t \in \mathbf{R}, \tag{11}$$

where $k \neq 0$ is a constant and α and β are appropriate or arbitrary phases of (p) *and* (q), *respectively.*

Proof. ($\Rightarrow$) Let α_1 and β_1 be some phases of (p) and (q), respectively, with r, s being defined by (7). By Lemma 4 $r, s \in C^3(\mathbf{R})$. Let us put $g := \alpha_1 r^{-1}$, $h := \beta_1 s^{-1}$. Then $g(\mathbf{R}) = h(\mathbf{R}) = \mathbf{R}$ and $g, h \in C^3(\mathbf{R})$. Let g and h be phases of (u) and (v), respectively. Then is $u(t) = -\{g, t\} - g'^2(t)$, $v(t) = -\{h, t\} - h'^2(t)$, $t \in \mathbf{R}$. There exists exactly one $\varepsilon \in \mathfrak{E}$ to every $a \in \mathbf{R}$, which can be proved in analogy with the proof of Lemma 4, such that $g^{-1}\{\varepsilon[h(t)]\} = t + a$, $t \in \mathbf{R}$. For every $a \in \mathbf{R}$ is thus the function $t + a$ a solution of the equation $-\{X, t\} + X'^2 \,.\, u(X) = v(t)$. Then, of couse, $u(t + a) = u(t)$, $t \in \mathbf{R}$, for every $a \in \mathbf{R}$ and $u(t) = v(t) =$ a constant $(:= m)$. On account of the fact that (u) is an oscillatory equation, we have $m < 0$ and so, there exist $\varepsilon_1, \varepsilon_2 \in \mathfrak{E}$:

$$\alpha_1[r^{-1}(t)] = \varepsilon_1(\sqrt{-m}t), \qquad \beta_1[s^{-1}(t)] = \varepsilon_2(\sqrt{-m}t), \qquad t \in R. \tag{12}$$

Putting $\alpha := \varepsilon_1^{-1}\alpha_1$, $\beta := \varepsilon_2^{-1}\beta_1$, then α and β are phases of (p) and (q), respectively, and we obtain further from (12): $r(t) = \frac{1}{\sqrt{-m}}\alpha(t)$, $s(t) = \frac{1}{\sqrt{-m}}\beta(t)$, that is $|P(t) - p(t)| = -\frac{1}{m}\alpha'^2(t)$, $|Q(t) - q(t)| = -\frac{1}{m}\beta'^2(t)$. In putting $k := := -\frac{1}{m}\operatorname{sign}(P - p)\left(= -\frac{1}{m}\operatorname{sign}(Q - q) \text{ by Lemma 2}\right)$, then relation (11) is satisfied, where $k \neq 0$.

($\Leftarrow$) Let $k \neq 0$ be a constant with α and β being phases of (p) and (q), respectively. Let (11) be valid. It may be assumed without any loss of generality that $\operatorname{sign}\alpha' = = \operatorname{sign}\beta' = 1$. Then $X \in \mathscr{P}^+_{pqPQ}$ exactly if

$$-\{X, t\} + X'^2 \,.\, p(X) = q(t),$$
$$-\{X, t\} + X'^2 \,.\, (P(X) + k \,.\, \alpha'^2(X)) = q(t) + k \,.\, \beta'^2(t),$$

hence $X \in \mathscr{P}^+_{pqPQ}$ exactly if $X \in \mathscr{L}^+_{pq}$ and $X'^2 \,.\, \alpha'^2(X) = \beta'^2$. Then $\alpha[X(t)] = = \beta(t) + a$, $a \in \mathbf{R}$, and if we put $\mathscr{S} := \{\alpha^{-1}[\beta(t) + a]; a \in \mathbf{R}\}$ is $\mathscr{S}$ $(\subset \mathscr{L}^+_{pq})$ a complete set and $\mathscr{S} = \mathscr{P}^+_{pqPQ}$.

4. Main results

Theorem 1. *Let* (p), (q) *be oscillatory equations, $P, Q \in C^0(\mathbf{R})$. Let α and β be phases of* (p) *and* (q), *respectively, with r, s being defined by* (7). *Then P^+_{pqPQ} and necessarily also $\mathscr{P}^-_{pqPQ}$ are complete sets exactly if the functions αr^{-1}, βs^{-1} are phases of* (m), *where $m < 0$ is a constant.*

Proof. ($\Rightarrow$) Let $\mathscr{P}^+_{pqPQ}$ be a complete set. Then it follows from the proof ($\Rightarrow$) of Lemma 5 that the functions αr^{-1}, βs^{-1} are the phases of (m), where $m < 0$ is a constant and it is clear from the proof ($\Leftarrow$) of Lemma 5 that $\mathscr{P}^-_{pqPQ} = \{\alpha^{-1}[-\beta(t) + a]; a \in \mathbf{R}\}$ is a complete set.

($\Leftarrow$) Letting αr^{-1}, βs^{-1} be the phases of (m), where $m < 0$ is a constant, yields $r, s \in C^3(\mathbf{R})$, $r(\mathbf{R}) = s(\mathbf{R}) = \mathbf{R}$ and there exist $\varepsilon_1, \varepsilon_2 \in \mathfrak{E}$: $\alpha(t) = \varepsilon_1(\sqrt{-m}\,.\,r(t))$, $\beta(t) = \varepsilon_2(\sqrt{-m}\,.\,s(t))$, $t \in \mathbf{R}$, whence $r(t) = \dfrac{\alpha_1(t)}{\sqrt{-m}}$, $s(t) = \dfrac{\beta_1(t)}{\sqrt{-m}}$, where $\alpha_1 := \varepsilon_1^{-1}\alpha$ is a phase of (p) and $\beta_1 := \varepsilon_2^{-1}\beta$ is a phase of (q). Then, of course, $P = p - \dfrac{\tau}{m}\alpha_1'^2$, $Q = q - \dfrac{\tau}{m}\beta_1'^2$, where $\tau = \pm 1$. By Lemma 5, $\mathscr{P}^+_{pqPQ}$ and thus also $\mathscr{P}^-_{pqPQ}$ are complete sets.

Theorem 2. *Let* (p), (q) *be oscillatory equations,* $P, Q \in C^0(\mathbf{R})$, $p(t) \neq P(t)$, $q(t) \neq Q(t)$ *for* $t \in \mathbf{R}$, $p - P$, $q - Q \in C^2(\mathbf{R})$. *Let* α and β *be phases of* (p) *and* (q), *respectively, with* r, s *defined by* (7), $r(\mathbf{R}) = s(\mathbf{R}) = \mathbf{R}$. *Let* αr^{-1} *be a phase of* (u) *and* βs^{-1} *be a phase of* (v). *Then*

(i) $\mathscr{P}^+_{pqPQ}$ *and* $\mathscr{P}^-_{pqPQ}$ *are countable sets exactly if there exist* $a > 0$, b, c *such that* u, v *are inconstant* a*-periodic functions,* $u(t + b) = v(t)$, $u(-t + c) = v(t)$ *for* $t \in \mathbf{R}$;

(ii) $\mathscr{P}^+_{pqPQ}$ *is a countable set and* $\mathscr{P}^-_{pqPQ}$ *is the empty set exactly if there exist* $a > 0$, b *such that* u, v *are* a*-periodic functions,* $u(t + b) = v(t)$ *for* $t \in \mathbf{R}$ *and no number* c *exists such that* $u(-t + c) = v(t)$ *for* $t \in \mathbf{R}$;

(iii) $\mathscr{P}^+_{pqPQ}$ *is the empty set and* $\mathscr{P}^-_{pqPQ}$ *is a countable set exactly if there exist* $a > 0$, c *such that* u, v *are* a*-periodic functions,* $u(-t + c) = v(t)$ *for* $t \in \mathbf{R}$ *and no number* b *exists such that* $u(t + b) = v(t)$ *for* $t \in \mathbf{R}$;

(iv) $\mathscr{P}^+_{pqPQ}$ *and* $\mathscr{P}^-_{pqPQ}$ *are one-element sets exactly if* u, v *are not periodic functions and if there exist numbers* b, c: $u(t + b) = v(t)$, $u(-t + c) = v(t)$ *for* $t \in \mathbf{R}$;

(v) $\mathscr{P}^+_{pqPQ}$ *is a one-element set and* $\mathscr{P}^-_{pqPQ}$ *is the empty set exactly if the functions* u, v *are not periodic, there exists a number* b: $u(t + b) = v(t)$ *for* $t \in \mathbf{R}$ *and there exists no number* c *such that* $u(-t + c) = v(t)$ *for* $t \in \mathbf{R}$;

(vi) $\mathscr{P}^+_{pqPQ}$ *is the empty set and* $\mathscr{P}^-_{pqPQ}$ *is a one-element set exactly if the functions* u, v *are not periodic, there exists a number* c: $u(-t + c) = v(t)$ *for* $t \in \mathbf{R}$ *and there exists no number* b *such that* $u(t + b) = v(t)$ *for* $t \in \mathbf{R}$;

(vii) $\mathscr{P}^+_{pqPQ}$ *and* $\mathscr{P}^-_{pqPQ}$ *are the empty sets exactly if there exist no numbers* b, c *such that* $u(t + b) = v(t)$ *and* $u(-t + c) = v(t)$ *for* $t \in \mathbf{R}$.

Proof. Let $\mathscr{P}^+_{pqPQ}$ be a countable set, $X_1, X_2 \in \mathscr{P}^+_{pqPQ}$, $X_1 \neq X_2$ such that $a_i := r[X_i(t_0)] \neq 0$, $i = 1, 2$. Let us put $g := \alpha r^{-1}$, $h := \beta s^{-1}$ and $X_i = \alpha^{-1}\varepsilon_i\beta$, where $\varepsilon_i \in \mathfrak{E}$. Then it follows from the necessary part of the proof of Lemma 5 that $g^{-1}\{\varepsilon_i[h(t)]\} = t + a_i$. Consequently $t + a_i$ are solutions of

$$-\{X, t\} + X'^2\,.\,u(X) = v(t), \tag{13}$$

hence $u(t + a_i) = v(t)$ for $t \in \mathbf{R}$, $i = 1, 2$. Thence $u(t + a_1 - a_2) = u(t)$, and because $a_1 \neq a_2$, u is a inconstant periodic function with respect to Theorem 1. Let $a > 0$ be a period of the function u. If we put $b := a_1$, then u, v are a-periodic functions and $u(t + b) = v(t)$ for $t \in \mathbf{R}$.

Let $\mathscr{P}^-_{pqPQ}$ be a countable set, $Y_1, Y_2 \in \mathscr{P}^-_{pqPQ}$, $Y_1 \neq Y_2$ such that $d_i := r[Y_i(t_0)] \neq 0$, $i = 1, 2$. Let $Y_i = \alpha^{-1}\varepsilon_i\beta$, where $\varepsilon_i \in \mathfrak{E}$. Evidently $d_1 \neq d_2$ and it follows that (8) yields $g^{-1}\{\varepsilon_i[h(t)]\} = -t + d_i$. Consequently $-t + d_i$ are solutions of (13). Hence $u(-t + d_i) = v(t)$. Then $u(t + d_1 - d_2) = u(t)$. Thus, u is a periodic function, inconstant, with respect to Theorem 1. Let $d > 0$ be a period of the function u. If we put $c := d_1$, then u, v are d-periodic functions and $u(-t + c) = v(t)$ for $t \in \mathbf{R}$.

Let the functions u, v be a-periodic and let $u(t + b) = v(t)$ for $t \in \mathbf{R}$, where $b \in \mathbf{R}$. Let the function u be inconstant. For every integer k then $t + b + ka$ are all solutions of (13), which are of the form $t + d$, where d is a constant. To every k there exists an $\varepsilon_k \in \mathfrak{E}$: $g^{-1}\{\varepsilon_k[h(t)]\} = t + b + ka$. If we put $X_k := \alpha^{-1}\varepsilon_k\beta$, where k is an integer, then $X_k \in \mathscr{L}^+_{pq}$ and

$$r[X_k(t)] = s(t) + b + ka, \qquad t \in \mathbf{R}, \tag{14}$$

whence it follows that $\mathscr{P}^+_{pqPQ}$ is a countable set and $\mathscr{P}^+_{pqPQ} = \{X_k;\ k$ being an integer$\}$.

Let u, v be inconstant a-periodic functions and $u(-t + c) = v(t)$ for $t \in \mathbf{R}$, where $c \in \mathbf{R}$. For every integer k then $-t + c + ka$ are all solutions of (13), which are of the form $-t + e$, where e is a constant. To every integer k there exists an $\varepsilon_k \in \mathfrak{E}$: $g^{-1}\{\varepsilon_k[h(t)]\} = -t + c + ka$. If we put $Y_k := \alpha^{-1}\varepsilon_k\beta$, where k is an integer, then $Y_k \in \mathscr{L}^-_{pq}$ and $-r[X_k(t)] = s(t) - c - ka$, whence it follows that $\mathscr{P}^-_{pqPQ}$ is a countable set and $\mathscr{P}^-_{pqPQ} = \{Y_k;\ k$ being an integer$\}$.

It becomes evident from our consideration that $X \in \mathscr{P}^+_{pqPQ}$ exactly if there exists $b \in \mathbf{R}$: $u(t + b) = v(t)$ for $t \in \mathbf{R}$ and $X \in \mathscr{P}^-_{pqPQ}$ exactly if there exists $c \in \mathbf{R}$ $u(-t + c) = v(t)$ for $t \in \mathbf{R}$.

Theorem. 3. *Let* (p), (q) *be oscillatory equations,* $p, Q \in C^0(\mathbf{R})$ *with at least one of the following assumptions* $p \neq P$, $q \neq Q$ *being true and* r, s *defined by* (7) *satisfy* $r(\mathbf{R}) \neq \mathbf{R} \neq s(\mathbf{R})$. *Then* $\mathscr{P}^+_{pqPQ}$ *and* $\mathscr{P}^-_{pqPQ}$ *are atmost one-element sets.*

The proof follows from Lemma 1 and Lemma 3.

REFERENCES

[1] Borůvka, O.: *Linear Differential Transformations of the Second Order.* The English Univ. Press, London 1971.

[2] Борувка, О.: *Теория глобальных свойств обыкновенных линейных дифференциальных уравнений второго порядка.* Дифференциальные уравнения, № 8, т. 12, 1976, 1347—1383.

[3] **Borůvka, O.**: *Lectures at the seminar of the Institute of Mathematics of the Czechoslovak Academy of Science in Brno.*
[4] **Staněk, S.**: *On a structure of the intersection of the set of dispersions of two second-order linear differential equations.* Acta Univ. Palackianae Olomucensis, F. R. N., T. 73, 1982, 79—85.

Souhrn

STRUKTURA PRŮNIKU ŘEŠENÍ DVOU DIFERENCIÁLNÍCH ROVNIC KUMMEROVA TYPU

SVATOSLAV STANĚK

Nechť $p, q \in C^2(\mathbf{R})$, $P, Q \in C^0(\mathbf{R})$, $p - q \in C^2(\mathbf{R})$, $P - Q \in C^2(\mathbf{R})$ a nechť $y'' = q(t)\,y$, $y'' = p(t)\,y$ jsou oscilatorické rovnice. Za uvedených předpokladů je popsána struktura průniku řešení dvou diferenciálních rovnic

$$-\{X, t\} + X'^2 \,.\, p(X) = q(t),$$
$$-\{X, t\} + X'^2 \,.\, P(X) = Q(t),$$

kde $\{X, t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\left(\frac{X''(t)}{X'(t)}\right)^2$.

Резюме

СТРУКТУРА ПЕРЕСЕЧЕНИЯ РЕШЕНИЙ ДВУХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТИПА КУММЕРА

СВАТОСЛАВ СТАНЕК

Пусть $p, q \in C^2(\mathbf{R})$, $P, Q \in C^0(\mathbf{R})$, $p - q \in C^2(\mathbf{R})$, $P - Q \in C^2(\mathbf{R})$. Пусть $y'' = p(t)y$, $y'' = q(t)y$ колеблющиеся уравнения. При этих предложениях приводится описание структуры пересечения решений двух дифферециальных уравнений

$$-\{X, t\} + X'^2 \,.\, p(X) = q(t),$$
$$-\{X, t\} + X'^2 \,.\, \mathrm{P}(X) = Q(t),$$

где $\{X, t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\left(\frac{X''(t)}{X'(t)}\right)^2$.