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ON A DISTRIBUTION OF ZEROS OF SOLUTIONS OF AN ITERATED DIFFERENTIAL EQUATION OF THE n-*th* ORDER

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Let us consider an ordinary linear homogeneous differential equation of the *n-th* order

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_{k+1}(t)\, y^{(k)}(t) = 0 \tag{1}$$

arising in connection with the iteration of the ordinary linear homogeneous differential equation of the *2-nd* order

$$y''(t) + q(t)\, y(t) = 0, \tag{2}$$

where the function $q(t) \in C_I^{n-2}$, $\mathbf{I} = (-\infty, +\infty)$, $n \in \mathbf{N}$, $n > 1$, is understood to be $q(t) > 0$ for all $t \in \mathbf{I}$. The differential equation (2) is assumed to be oscillatory in the sense of [2], i.e. there exist infinitely many zeros of any nontrivial solution $y(t)$ relative to this equation, lying both to the right and to the left of any arbitrary point $t \in \mathbf{I}$.

Throughout this discussion the differential equation (1), where

$$a_{k+1}(t) = a_{k+1}[q(t), \dots, q^{(n-2)}(t)],$$

$k = 0, 1, \dots, n-1$, (see [1]) will be called "the iterated differential equation of the *n-th* order", only.

If we denote the ordered pair of the oscillatory solutions $u(t)$, $v(t)$ relative to (2) and linearly independent on interval $\mathbf{I}$ as the basis of a space of all solutions relative to this equation, then the ordered n-tuple of functions

$$[u^{n-1}(t), u^{n-2}(t)\, v(t), \dots, u^{n-k-1}(t)\, v^{k}(t), \dots, u(t)\, v^{n-2}(t), v^{n-1}(t)],$$

where $k = 0, 1, \dots, n-1$, forms a basis of the space of all solutions relative to (1). Thus the system of all (nontrivial) solutions $y(t)$ relative to (1) may be written as

$$y(t) = \sum_{i=1}^{n} C_i u^{n-i}(t)\, v^{i-1}(t), \tag{3}$$

where $C_i \in \mathbf{R}$, $i = 1, \ldots, n$ ($n \in \mathbf{N}, n > 1$), are arbitrary independent constants (the parameters of the system), whereby $\sum_{i=1}^{n} C_i^2 > 0$. Since (1) is of the *n-th* order, every zero of its arbitrary (nontrivial) oscillatory solution $y(t)$ is of multiplicity $v = n - 1$ at the highest.

In all what follows, under "solution" both of (1) and (2) only nontrivial solution will be understood.

In [1] (in agreement with [2]) there were introduced the concept of the so called first conjugate point, in [3] (again in agreement with [2]) generalized to the concept of the $|k|$-*th*, $k = 0, \pm 1, \pm 2, \ldots$, conjugate point to the right or to the left to the given arbitrary chosen zero $t_0 \in \mathbf{I}$ of the solution $y(t)$ relative to (1). In Definition 1.3 [3] there were also distinguished, among the $|k|^{th}$ conjugate points $t_k \in \mathbf{I}$ to the right or to the left of t_0, the so called strongly or weakly conjugate points of the bundle $Y(t)$ of all solutions $y(t)$ relative to (1), vanishing together at t_0.

To investigate the existence and multiplicities of the strongly or weakly conjugate points of the bundle $Y(t)$ of the oscillatory solutions $y(t)$ relative to (1) we proceed as follows. Let us choose an arbitrary point $t_0 \in \mathbf{I}$ and a basis $[u(t), v(t)]$ relative to the oscillatory differential equation (2) such that, say, the function $u(t)$ from this basis would vanish at it, whereby

$$u(t_0) = v'(t_0) = 0, \tag{P}$$

(so that $v(t_0) \neq 0$, $u'(t_0) \neq 0$).

Then by (3) the bundle $Y(t)$ of all solutions $y(t)$ relative to (1) vanishing at t_0 together with the function $u(t)$ may be written as

1. $$Y(t) = \sum_{i=1}^{n-1} C_i u^{n-i}(t)\, v^{i-1}(t), \qquad C_{n-1} \neq 0$$

exactly if the point t_0 is a simple zero of all solutions $y(t)$ relative to (1) from the bundle $Y(t)$;

2. $$Y(t) = \sum_{i=1}^{n-2} C_i u^{n-i}(t)\, v^{i-1}(t), \qquad C_{n-2} \neq 0$$

exactly if the point t_0 is a double zero of all solutions $y(t)$ relative to (1) from the bundle $Y(t)$;

⋮

$n - 1$) $$Y(t) = C_1 u^{n-1}(t), \qquad C_1 \neq 0$$

exactly if the point t_0 is an $(n-1)$-fold zero of all solutions $y(t)$ relative to (1) from the bundle $Y(t)$;

Generally:

$$Y(t) = \sum_{i=1}^{n-k} C_i u^{n-i}(t) v^{i-1}(t), \qquad C_{n-k} \neq 0$$

where $k, n \in \mathbf{N}$, $n > 1$, $1 \leqq k \leqq n - 1$, exactly if the point t_0 is a k-fold zero of all solutions $y(t)$ relative to (1) from the bundle $Y(t)$ (cf Lemma 1. [1]).

Writing the bundle $Y(t)$ in an equivalent form

$$Y(t) = u^{n-k}(t)\, Y_k^*(t, C_1, \ldots, C_k), \tag{S_k}$$

where

$$Y_k^*(t, C_1, \ldots, C_k) = \sum_{i=1}^{k} C_i u^{k-i}(t) v^{i-1}(t),$$

$1 \leqq k \leqq n - 1$ ($n \in \mathbf{N}$, $n > 1$), enables us immediately to express the following assertion about the strongly conjugate points (see Definition 3.1 [3]) of the bundle $Y(t)$ of all solutions $y(t)$ relative to (1) vanishing at the ν-fold, $\nu \in \{1, \ldots, n - 1\}$, point t_0.

Statement: The only strongly conjugate points of the bundle (S_k) of all solutions $y(t)$ relative to (1) vanishing at any arbitrary, firmly chosen point $t_0 \in \mathbf{I}$ together with the function $u(t)$ from the basis $[u(t), v(t)]$ relative to the oscillatory differential equation (2) are exactly all zeros of these solutions coinciding with all zeros of the function $u^{n-k}(t)$, $k = 1, \ldots, n - 1$ ($n \in \mathbf{N}$, $n > 1$). Thereby the multiplicity $\nu \in \{1, \ldots, n - 1\}$ of the strongly conjugate points, with the given k always the same at all these points (see Theorem 1.5, [3]), is equal to the step of the $(n - k)th$ power of the function $u(t)$ acting in (S_k), i.e. $\nu = n - k$ for all $1 \leqq k \leqq n - 1$.

Besides the weakly conjugate points (see Definition 3.1 [3]) of the bundles $Y(t)$ of these solutions $y(t)$ relative to (1) may be only the zeros of the k-parametric system of the functions $Y_k^*(t, C_1, \ldots, C_k)$ from the corresponding forms of the bundle (S_k) – if, naturally, any zeros of such function system exist at all.

Let T_1 denote a neighbouring zero of the function $u(t)$ lying to the right of the point t_0, so that $T_1 > t_0$. Then simultaneously for every solution $y(t)$ relative to (1) from the bundle $Y(t)$ having the form (S_k) we have

$$y(t_0) = u(t_0) = 0, \qquad y(T_1) = u(T_1) = 0,$$

whereby for all $t \in (t_0, T_1)$ the function $u(t) \neq 0$. It is evident that also the point T_1 is a first strongly conjugate point of the bundle $Y(t)$ of all solutions $y(t)$ relative to (1), lying to the right of the (strongly conjugate) point t_0. Thus the question of the existence of zeros of the bundle $Y(t)$ of all solutions $y(t)$ relative to (1) on the open interval (t_0, T_1) reduces to the question of the existence of zeros of the k-parametric system of functions $Y_k^*(t, C_1, \ldots, C_k)$ from (S_k) on this interval.

Our object now is to look for such forms of the bundles $Y(t)$ of solutions $y(t)$ relative to (1) having on the interval (t_0, T_1) or $\langle t_0, T_1 \rangle$ the prescribed number of zeros with regard to their multiplicities. Especially we will observe such special forms of bundles $Y(t)$ having the form (S_k), when the numbers of these zeros on the interval considered are extreme. We will distinguish cases with the order n ($n \in \mathbf{N}$, $n \geqq 2$) of the differential equation (1) being even or odd.

The main consequence of these considerations lies in possible applying the forms of the bundles $Y(t)$ of solutions $y(t)$ relative to (1) found, to solutions of special boundary value problems, such as those of Sturm–Liouville type, wherein conditions are placed only on values of solutions $y(t)$ relative to this equation at the zeros prescribed.

For completeness let us first make a detailed picture of the numbers, distribution and multiplicities of zeros in all the possible forms of bundles $Y_k(t)$, $k = 1, \ldots, n-1$ ($n \in \mathbf{N}$, $n > 1$) of solutions $y(t)$ relative to (1) on the interval $\langle t_0, T_1 \rangle$ in connection with their mutual strong or weak conjugacy.

The number, distribution and multiplicities of zeros

1. If $k = 1$, it is immediate that the one-parametric bundle $Y(t)$ of all solutions $y(t)$ relative to (1) in the form

$$Y_1(t) = C_1 u^{n-1}(t)$$

with $C_1 \neq 0$ being an arbitrary real constant, has all zeros strongly conjugate with multiplicity $\nu = n - 1$, only. These are just all zeros of the function $u^{n-1}(t)$.

On the interval $\langle t_0, T_1 \rangle$ there lie two neighbouring zeros t_0, T_1 of the function $u(t)$ and thus also of the function $u^{n-1}(t)$. Besides these two zeros of the bundle $Y_1(t)$ of the solutions $y(t)$ relative to (1), there lie no other zeros of this bundle.

2. If $k = 2$, then (up to a possible arbitrary multiplicative constant $C \in \mathbf{R} - \{0\}$), the two-parametric bundle $Y(t)$ of all solutions $y(t)$ relative to (1) is of the form

$$Y_2(t) = u^{n-2}(t) \left[C_1 u(t) + C_2 v(t) \right],$$

with $C_i \in \mathbf{R}$, $i = 1, 2$, $C_2 \neq 0$, are arbitrary constants. In consequence of the assumption $C_2 \neq 0$, every function from the two-parametric system of functions

$$Y_2^*(t, C_1, C_2) = C_1 u(t) + C_2 v(t),$$

obtained in an arbitrary choice of constants $C_i \in \mathbf{R}$ ($i = 1, 2$) is linearly independent of the function $u(t)$ – and thus also of the function $u^{n-2}(t)$ – on the interval $\mathbf{I} = (-\infty, +\infty)$. By the Sturm separation theorem of zeros of two arbitrary oscillatory solutions relative to (2) linearly independent on $\mathbf{I}$ we know that between any two neighbouring $(n-2)$-fold zeros of the function $u^{n-2}(t)$ there lies exactly one simple zero of an arbitrary function from the system of functions $Y_2^*(t)$. Thus, on the interval $\langle t_0, T_1 \rangle$ between two consecutive $(n-2)$-fold, strongly conjugate points t_0, T_1 of the bundle $Y_2(t)$ of all solutions $y(t)$ relative to (1), there lies exactly one simple weakly conjugate point t_1 of this bundle.

If we denote now (and hereafter) the sequence of the conjugate points by a subscript to the right, and the multiplicity by a superscript to the left in writing the corresponding zero, lying on the right of the point t_0, then there hold the inequalities

$$^{n-2}t_0 < {}^1t_1 < {}^{n-2}t_2,$$

where $^{n-2}t_2 = T_1$.

3. If $k = 3$, then the three-parametric bundle $Y(t)$ of all solutions $y(t)$ relative to (1) is of the form

$$Y_3(t) = u^{n-3}(t)\left[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)\right],$$

where $C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, are arbitrary constants. With respect to the existence and multiplicities of zeros of the three-parametric system of functions

$$Y_3^*(t, C_1, C_2, C_3) = C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t),$$

there may occur three possibilities:

3a) If $C_2^2 - 4C_1C_3 > 0$, then there exist four real constants $c_{ij} \in \mathbf{R}$ $(i, j = 1, 2)$ such that

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0, \qquad c_{12}c_{22} \neq 0,$$

whereby

$$C_1 = c_{11}c_{21},\ C_2 = c_{12}c_{21} + c_{11}c_{22},\ C_3 = c_{12}c_{22}$$

and

$$Y_3^*(t) = \left[c_{11}u(t) + c_{12}v(t)\right]\left[c_{21}u(t) + c_{22}v(t)\right]$$

(up to a multiplicative constant $C \in \mathbf{R} - \{0\}$) holds. Denoting

$$y_1^*(t, c_{11}, c_{12}) = c_{11}u(t) + c_{12}v(t),$$
$$y_2^*(t, c_{21}, c_{22}) = c_{21}u(t) + c_{22}v(t),$$

then for every (admissible) choice of all four constants $c_{ij} \in \mathbf{R}$ $(i, j = 1, 2)$ in both two-parametric function systems we always get any pair of functions $y_1^*(t), y_2^*(t)$. These functions are the two solutions of the differential equation (2) linearly independent on $\mathbf{I} = (-\infty, +\infty)$, whereby each of them is besides linearly independent of the solution $u(t)$ relative to (2) on an interval $\mathbf{I}$.

By the Sturm theorem all zeros of these three functions (in pairs linearly independent) mutually separate on $\mathbf{I}$; whereby between any two consecutive zeros of the function $u(t)$ there is exactly one simple zero of either function $y_1^*(t)$, $y_2^*(t)$. If we denote these simple zeros of the functions $y_1^*(t), y_2^*(t)$ on the interval (t_0, T_1) by t^*, t^{**}, respectively, then either $t_0 < t^* < t^{**} < T_1$ or $t_0 < t^{**} < t^* < T_1$. In the first case t^* and t^{**} are, respectively, the first and the second weakly conjugate point from the right to t_0. In the latter case t^{**} and t^* are, respectively, the first and the second weakly conjugate point from the right to t_0. Thus, with respect to the multiplicity of the four zeros of the bundle $Y_3(t)$ of solutions $y(t)$ relative to (1) on the interval $\langle t_0, T_1 \rangle$ we have either

$${}^{n-3}t_0 < {}^1t_1^* < {}^1t_2^{**} < {}^{n-3}t_3$$

or

$${}^{n-3}t_0 < {}^1t_1^{**} < {}^1t_2^* < {}^{n-3}t_3,$$

where ${}^{n-3}t_0$, ${}^{n-3}t_3 = T_1$ are mutually strongly conjugate points of all solutions $y(t)$ from this bundle.

3b) If $C_2^2 - 4C_1C_3 = 0$, then there exist two real constants $c_{11}, c_{12} \in \mathbf{R}$, such that $c_{12} \neq 0$, whereby

$$C_1 = c_{11}^2,\ C_2 = 2c_{11}c_{12},\ C_3 = c_{12}^2$$

and

$$Y_3^*(t) = [c_{11}u(t) + c_{12}v(t)]^2$$

(up to a multiplicative constant $C \in \mathbf{R} - \{0\}$) holds. In every admissible choice of constants $c_{11}, c_{12} \in \mathbf{R}$ in the two-parametric system of functions

$$y^*(t) = c_{11}u(t) + c_{12}v(t),$$

there always arises a function which is a solution of the differential equation (2) linearly independent of the function $u(t)$ on $\mathbf{I}$, so that all zeros of both functions mutually separate here. Thus, also all double zeros of the two-parametric system of functions $Y_3^*(t) = Cy^{*2}(t)$ with the $(n-3)$-fold zeros of the function $u^{n-3}(t)$ from the bundle $Y_3(t)$ of the solutions $y(t)$ relative to (1) mutually separate on the interval $\mathbf{I}$; whereby the first and the latter are, respectively, the weakly and the strongly conjugate points of all solutions $y(t)$ from this bundle.

Denoting by t^* the double zero of an arbitrary function from the system of functions $Y_3^*(t)$ lying on an open interval (t_0, T_1), then with respect to the multiplicity of these three points t_0, t^* and T_1 on the interval $\langle t_0, T_1 \rangle$ the inequalities

$${}^{n-3}t_0 < {}^2t_1 < {}^{n-3}t_2$$

hold, where ${}^{n-3}t_2 = T_1$.

3c) If $C_2^2 - 4C_1C_3 < 0$, then there exist four real constants $c_{ij} \in \mathbf{R}$ $(i, j = 1, 2)$, such that

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0, \qquad c_{11}^2 + c_{21}^2 > 0, \qquad c_{12}^2 + c_{22}^2 > 0,$$

whereby

$$C_1 = c_{11}^2 + c_{21}^2,\ C_2 = 2(c_{11}c_{12} + c_{21}c_{22}),\ C_3 = c_{12}^2 + c_{22}^2$$

and it holds

$$Y_3^*(t) = C\{[c_{11}u(t) + c_{12}v(t)]^2 + [c_{21}u(t) + c_{22}v(t)]^2\},$$

(up to a multiplicative constant $C \in \mathbf{R} - \{0\}$).

Because of the linear independence of any pair of functions $y_1^*(t)$, $y_2^*(t)$ from the both two-parametric systems of functions

$$y_1^*(t, c_{11}, c_{12}) = c_{11}u(t) + c_{12}v(t),$$
$$y_2^*(t, c_{21}, c_{22}) = c_{21}u(t) + c_{22}v(t),$$

whose zeros mutually separate, there exists no zero of the functions

$$Y_3^*(t, C) = C[y_1^{*2}(t) + y_2^{*2}(t)],$$

on $\mathbf{I} = (-\infty, +\infty)$. Let us remark that always at least one function either from the system $y_1^*(t, c_{11}, c_{12})$ or from the system $y_2^*(t, c_{21}, c_{22})$ is linearly independent of the function $u(t)$ on $\mathbf{I}$ and this on account of the assumption $c_{12}^2 + c_{22}^2 > 0$. Thus for all $t \in \mathbf{I}$ we have

$$Y_3^*(t, C) > 0, \quad \text{if} \quad C > 0 \quad \text{or} \quad Y_3^*(t, C) < 0, \quad \text{if} \quad C < 0.$$

Since no weakly conjugate point of the bundle $Y_3(t)$ of solutions $y(t)$ relative to (1) exists on the open interval (t_0, T_1), then the only zeros of this bundle on the interval $\langle t_0, T_1 \rangle$ are exactly both the $(n-3)$-fold zeros ${}^{n-3}t_0$ and ${}^{n-3}t_1 = T_1$, only. These points are at the same time mutually strongly conjugate points of all solutions $y(t)$ from this bundle.

4. If $k = 4$, then the four-parametric bundle $Y(t)$ of all solutions $y(t)$ relative to (1) have the form

$$Y_4(t) = u^{n-4}(t)\left[C_1u^3(t) + C_2u^2(t)\,v(t) + C_3u(t)\,v^2(t) + C_4v^3(t)\right],$$

where $C_i \in \mathbf{R}$, $i = 1, \dots, 4$, $C_4 \neq 0$, are arbitraiy constants. With regard to the existence and to the multiplicities of zeros of the four-parametric system of functions

$$Y_4^*(t, C_1, \dots, C_4) = C_1u^3(t) + C_2u^2(t)\,v(t) + C_3u(t)\,v^2(t) + C_4v^3(t),$$

there may arise the following four possibilities:

4a) There exist two real constants $c_{1j} \in \mathbf{R}$ $(j = 1, 2)$, $c_{12} \neq 0$, such that

$$C_1 = c_{11}^3,\ C_2 = 3c_{11}^2c_{12},\ C_3 = 3c_{11}c_{12}^2,\ C_4 = c_{12}^3$$

and it holds

$$Y_4^*(t) = [c_{11}u(t) + c_{12}v(t)]^3,$$

(up to a multiplicative constant $C \in \mathbf{R} - \{0\}$).

Every function $y^*(t)$, obtained in an arbitrary (admissible) choice of constants c_{1j} $(j = 1, 2)$ from the two-parametric system of functions

$$y^*(t, c_{11}, c_{12}) = c_{11}u(t) + c_{12}v(t)$$

is linearly independent of the function $u(t)$ on the interval $\mathbf{I}$. Thus, all zeros of both foregoing functions mutually separate on $\mathbf{I}$.

Denoting by t^* the zero of the function $y^*(t)$ on the interval (t_0, T_1), then this point is a thiee-fold weakly conjugate point from the right to the point t_0 of the bundle

$$Y_4(t) = u^{n-4}(t)\,y_1^{*3}(t, c_{11}, c_{12})$$

of all solutions $y(t)$ relative to (1) on (t_0, T_1). So, with regard to the multiplicities of these three zeros t_0, t^* and T_1 of the solutions $y(t)$ on the interval $\langle t_0, T_1\rangle$ we have

$$^{n-4}t_0 < {}^3t_1^* < {}^{n-4}t_2,$$

where $^{n-4}t_0$, $^{n-4}t_2 = T_1$ are mutually strongly conjugate points of this bundle.

4b) There exist four real constants $c_{ij} \in \mathbf{R}$ $(i\ j = 1, 2)$, such that

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0, \qquad c_{12}c_{22} \neq 0,$$

whereby

$$C_1 = c_{11}c_{21}^2, C_2 = c_{21}(2c_{11}c_{22} + c_{12}c_{21}), C_3 = c_{22}(c_{11}c_{22} + 2c_{12}c_{21}), C_4 = c_{12}c_{22}^2$$

and it holds

$$Y_4^*(t) = [c_{11}u(t) + c_{12}v(t)]\,[c_{21}u(t) + c_{22}v(t)]^2$$

(up to a multiplicative constant $C \in \mathbf{R} - \{0\}$). Every two functions $y_1^*(t)$, $y_2^*(t)$, obtained in an arbitrary (admissible) choice of constants $c_{ij} \in \mathbf{R}$ $(i, j = 1, 2)$ in both two-parametric systems of functions

$$y_1^*(t, c_{11}, c_{12} = c_{11}u(t) + c_{12}v(t),$$
$$y_2^*(t, c_{21}, c_{22}) = c_{21}u(t) + c_{22}v(t)$$

are linearly independent on $\mathbf{I} = (-\infty, +\infty)$ and besides either of them is also linearly independent of the function $u(t)$ on the interval $\mathbf{I}$. Thus, all zeros of the three functions $u(t)$, $y_1^*(t)$ and $y_2^*(t)$ mutually separate on $\mathbf{I}$. Then between any two consecutive zeros of the function $u(t)$ there lies exactly one zero both of the function $y_1^*(t)$ and the function $y_2^*(t)$. If we denote the zeros of the functions $y_1^*(t)$, $y_2^*(t)$ on the interval (t_0, T_1) by t^* and t^{**}, respectively, then either

$$t_0 < t^* < t^{**} < T_1 \qquad \text{or} \qquad t_0 < t^{**} < t^* < T_1.$$

In the first case t^* and t^{**} are the first and the second weakly conjugate points from the right to t_0, respectively. In the latter case t^{**} and t^* are the first and the second weakly conjugate points from the right to t_0, respectively. Thus, with regard to the multiplicities of the four zeros t_0, t^*, t^{**}, T_1 of the bundle

$$Y_4(t) = u^{n-4}(t)\, y_1^*(t, c_{11}, c_{12})\, y_2^{*2}(t, c_{21}, c_{22})$$

of solutions $y(t)$ relative to (1) on the interval $\langle t_0, T_1\rangle$ we have either

$$^{n-4}t_0 < {}^1t_1^* < {}^2t_2^{**} < {}^{n-4}t_3,$$

or

$$^{n-4}t_0 < {}^2t_1^{**} < {}^1t_2^* < {}^{n-4}t_3,$$

where $^{n-4}t_0$, $^{n-4}t_3 = T_1$ are mutually strongly conjugate points of all solutions $y(t)$ from this bundle.

4c) There exist six real constants $c_{ij} \in \mathbf{R}$ ($i = 1, 2, 3$, $j = 1, 2$), such that

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0, \quad \begin{vmatrix} c_{11} & c_{12} \\ c_{31} & c_{32} \end{vmatrix} \neq 0, \quad \begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix} \neq 0, \quad c_{12}c_{22}c_{32} \neq 0,$$

whereby

$$C_1 = c_{11}c_{21}c_{31},\ C_2 = c_{12}c_{21}c_{31} + c_{11}c_{22}c_{31} + c_{11}c_{21}c_{32},$$
$$C_3 = c_{12}c_{22}c_{31} + c_{12}c_{21}c_{32} + c_{11}c_{22}c_{32},\ C_4 = c_{12}c_{22}c_{32}$$

and it holds

$$Y_4^*(t) = [c_{11}u(t) + c_{12}v(t)]\,[c_{21}u(t) + c_{22}v(t)]\,[c_{31}u(t) + c_{32}v(t)],$$

(up to a multiplicative constant $C \in \mathbf{R} - \{0\}$).

Every two from the three functions $y_1^*(t)$, $y_2^*(t)$, $y_3^*(t)$, obtained in an arbitrary (admissible) choice of constants $c_{ij} \in \mathbf{R}$ ($i = 1, 2, 3$, $j = 1, 2$) in the three two-parametric systems of functions

$$y_1^*(t, c_{11}, c_{12}) = c_{11}u(t) + c_{12}(t),$$
$$y_2^*(t, c_{21}, c_{22}) = c_{21}u(t) + c_{22}v(t),$$
$$y_3^*(t, c_{31}, c_{32}) = c_{31}u(t) + c_{32}v(t),$$

are linearly independent on $\mathbf{I} = (-\infty, +\infty)$ and besides, each of them is also linearly independent of the function $u(t)$ on $\mathbf{I}$. Thus, all zeros of the four functions $u(t)$, $y_1^*(t)$, $y_2^*(t)$ and $y_3^*(t)$ mutually separate of $\mathbf{I}$. Then between any two consecutive zeros of the function $u(t)$ there always lies exactly one zero both of the function $y_1^*(t)$ and the function $y_2^*(t)$ and also of the function $y_3^*(t)$.

If we denote the zeros of the three functions $y_1^*(t)$, $y_2^*(t)$, $y_3^*(t)$ on the interval (t_0, T_1) by t^*, t^{**}, t^{***}, respectively, then either $t_0 < t^* < t^{**} < t^{***} < T_1$ or $t_0 < t^* < t^{***} < t^{**} < T_1$ or $t_0 < t^{**} < t^* < t^{***} < T_1$ or $t_0 < t^{**} < t^{***} < < t^* < T_1$ or $t_0 < t^{***} < t^* < t^{**} < T_1$ or $t_0 < t^{***} < t^{**} < t^* < T_1$. In all the above cases the points t^*, t^{**}, t^{***} are weakly conjugate points from the right to the point t_0 and this in a successive order.

Thus, with regard to the multiplicities of the five zeros $t_0, t^*, t^{**}, t^{***}, T_1$ of the bundle

$$Y_4(t) = u^{n-4}(t)\, y_1^*(t, c_{11}, c_{12})\, y_2^*(t, c_{21}, c_{22})\, y_3^*(t, c_{31}, c_{32})$$

of the solutions $y(t)$ relative to (1) on the interval $\langle t_0, T_1 \rangle$ we have either

$${}^{n-4}t_0 < {}^{1}t_1^* < {}^{1}t_2^{**} < {}^{1}t_3^{***} < {}^{n-4}t_4,$$

or

$${}^{n-4}t_0 < {}^{1}t_1^* < {}^{1}t_2^{***} < {}^{1}t_3^{**} < {}^{n-4}t_4,$$

or

$${}^{n-4}t_0 < {}^{1}t_1^{**} < {}^{1}t_2^* < {}^{1}t_3^{***} < {}^{n-4}t_4,$$

or

$${}^{n-4}t_0 < {}^{1}t_1^{**} < {}^{1}t_2^{***} < {}^{1}t_3^* < {}^{n-4}t_4,$$

or

$$ {}^{n-4}t_0 < {}^{1}t_1^{***} < {}^{1}t_2^{*} < {}^{1}t_3^{**} < {}^{n-4}t_4, $$

or

$$ {}^{n-4}t_0 < {}^{1}t_1^{***} < {}^{1}t_2^{**} < {}^{1}t_3^{*} < {}^{n-4}t_4, $$

where ${}^{n-4}t_0$, ${}^{n-4}t_4 = T_1$ are mutually strongly conjugate points of all solutions $y(t)$ from this bundle.

4d) There exist six real constants $c_{ij} \in \mathbf{R}$ ($i = 1, 2, 3, j = 1, 2$), such that

$$ \begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix} \neq 0, $$

$$ c_{12} \neq 0, c_{21}^2 + c_{31}^2 > 0, c_{22}^2 + c_{32}^2 > 0, $$

whereby

$$ C_1 = c_{11}(c_{21}^2 + c_{31}^2), C_2 = c_{12}(c_{21}^2 + c_{31}^2) + 2c_{11}(c_{21}c_{22} + c_{31}c_{32}), $$
$$ C_3 = c_{11}(c_{22}^2 + c_{32}^2) + 2c_{12}(c_{21}c_{22} + c_{31}c_{32}), C_4 = c_{21}(c_{22}^2 + c_{32}^2) $$

and it holds

$$ Y_4^*(t) = [c_{11}u(t) + c_{12}v(t)]\{[c_{21}u(t) + c_{22}v(t)]^2 + [c_{31}u(t) + c_{32}v(t)]^2\}, $$

(up to a multiplicative constant $C \in \mathbf{R} - \{0\}$).

Any two functions $y_2^*(t)$, $y_3^*(t)$ obtained in an arbitrary (admissible) choice of constants $c_{ij} \in \mathbf{R}$ ($i = 2, 3, j = 1, 2$) from the two-parametric systems of functions

$$ y_2^*(t, c_{21}, c_{22}) = c_{21}u(t) + c_{22}v(t), $$
$$ y_3^*(t, c_{31}, c_{32}) = c_{31}u(t) + c_{32}v(t), $$

are linearly independent on the interval $\mathbf{I} = (-\infty, +\infty)$. For any (admissible) choice of constants $c_{ij} \in \mathbf{R}$ ($i = 2, 3, j = 1, 2$) in a four-parametric system of functions in the form

$$ \bar{Y}^*(t, c_{21}, c_{22}, c_{31}, c_{32}) = y_2^{*2}(t, c_{21}, c_{22}) + y_3^{*2}(t, c_{31}, c_{32}) $$

we have

$$ \bar{Y}^*(t, c_{21}, c_{22}, c_{31}, c_{32}) > 0 $$

on $\mathbf{I}$. Consequently, the arbitrary function $\bar{y}^*(t) = y_2^{*2}(t) + y_3^{*2}(t)$ from this system has no zero on $\mathbf{I}$.

So, the only zeros of the functions from the system (up to a multiplicative constant $C \in \mathbf{R} - \{0\}$) in the form

$$ Y_4^*(t) = y_1^*(t, c_{11}, c_{12})\ \bar{Y}^*(t, c_{21}, c_{22}, c_{31}, c_{32}) $$

on the interval $\mathbf{I} = (-\infty, +\infty)$ are the simple zeros of every function $y_1^*(t)$ obtained in an (admissible) choice of constants $c_{1j} \in \mathbf{R}$ ($j = 1, 2$) from the two-parametric system of functions

$$ y_1^*(t, c_{11}, c_{12}) = c_{11}u(t) + c_{12}v(t). $$

Since every function $y_1^*(t)$ from this two-parametric system of functions $y_1^*(t, c_{11}, c_{12})$ is — with respect to the assumption $c_{12} \neq 0$ — linearly independent of the function $u(t)$ on **I**, all zeros of both functions mutually separate on the interval **I**.

If we denote by t^* the zero of such a function $y_1^*(t)$ on the open interval (t_0, T_1), then with regard to the multiplicity of all three zeros t_0, t^* and T_1 of the bundle

$$Y_4(t) = u^{n-4}(t)\, y_1^*(t, c_{11}, c_{12})\, \bar{Y}^*(t, c_{21}, c_{22}, c_{31}, c_{32})$$

of solutions $y(t)$ relative to (1) on the interval $\langle t_0, T_1 \rangle$ we have

$$^{n-4}t_0 < {}^1t_1^* < {}^{n-4}t_2,$$

where $^1t_1^*$ is weakly conjugate and $^{n-4}t_0, {}^{n-4}t_2 = T_1$ are mutually strongly conjugate points of all solutions $y(t)$ from this bundle.

⋮

n − 1) If $k = n - 1$, then the $(n-1)$-parametric bundle $Y(t, C_1, \dots, C_{n-1})$ of all solutions $y(t)$ relative to (1) is of the form

$$Y_{n-1}(t, C_1, \dots, C_{n-1}) = u(t) \sum_{i=1}^{n-1} C_i u^{n-i-1}(t)\, v^{i-1}(t),$$

where $C_i \in \mathbf{R}$, $i = 1, \dots, n-1$; $C_{n-1} \neq 0$, are arbitrary constants. With respect to the existence of zeros of the $(n-1)$-parametric system of functions

$$Y_{n-1}^*(t, C_1, \dots, C_{n-1}) = \sum_{i=1}^{n-1} C_i u^{n-i-1}(t)\, v^{i-1}(t),$$

on the interval $\mathbf{I} = (-\infty, +\infty)$, let us distinguish the step $n-1$ of this homogeneous functional polynomial being odd or even. If $n-1$ is an odd number, then there always exists between any two consecutive zeros of the function $u(t)$ at least one zero of any of the functions $y_{n-1}^*(t)$, obtained in an arbitrary (admisible) choice of the constants $C_i \in \mathbf{R}$ $(i = 1, \dots, n-1)$ from the system $Y_{n-1}^*(t, C_1, \dots, C_{n-1})$. But if $n-1$ is an even number, then there need not exist any zero of this system on **I**, so that either $Y_{n-1}^*(t, C_1, \dots, C_{n-1}) > 0$ or $Y_{n-1}^*(t, C_1, \dots, C_{n-1}) < 0$ holds on this interval.

From here on we will direct our attention to the study of the existence and to the multiplicities of zeros of the system of functions $Y_{n-1}^*(t)$ on an open interval (t_0, T_1), because the situation concerning the existence and the multiplicities of zeros of the system $Y_{n-1}^*(t)$, between any other two consecutive zeros of the function $u(t)$ from the bundle $Y_{n-1}(t)$, is analogous.

According to the fundamental theorem of algebra generalized to the functional polynomials there exist $2(n-2)$ — generally complex — constants c_{ij} $(i = 1, \dots, n-2; j = 1, 2)$ such that the $(n-1)$-parametric system of functions $Y_{n-1}^*(t, C_1, \dots,$

$C_{n-1})$ may be written in an equivalent form as

$$Y^*_{n-1}(t, C_1, \ldots, C_{n-1}) = \prod_{i=1}^{n-2} [c_{i1}u(t) + c_{i2}v(t)], \qquad (Y^*)$$

whereby

$$C_1 = \prod_{i=1}^{n-2} c_{i1} \in \mathbf{R}, \ldots, C_{n-1} = \prod_{i=1}^{n-2} c_{i2} \in \mathbf{R},$$

where $\prod_{i=1}^{n-2} c_{i2} \neq 0$.

Especially if $n - 1$ is an odd number, then (Y^*) may be written in the form

$$Y^*_{n-1}(t, C_1, \ldots, C_{n-1}) = [c_{11}u(t) + c_{12}v(t)] \prod_{i=2}^{n-2} [c_{i1}u(t) + c_{i2}v(t)],$$

where both constants c_{1j} $(j = 1, 2)$, $c_{12} \neq 0$, are real. Thus the zeros of this functional system $Y^*_{n-1}(t)$ are surely the zeros of the functions $y^*_1(t)$ obtained in an arbitrary (admissible) choice of the constants c_{1j} $(j = 1, 2)$ from the two-parametric subsystem

$$y^*_1(t, c_{11}, c_{12}) = c_{11}u(t) + c_{12}v(t).$$

If for further $i = 2, \ldots, n - 2$ the corresponding pair of constants c_{ij} $(j = 1, 2)$ is no more real, then also the further zeros of the system $Y^*_{n-1}(t)$ – besides the cited zeros of the function $y^*_1(t)$ – no more exist.

In studying the existence and the multiplicities of zeros of the functional system $Y^*_{n-1}(t)$ in the form (Y^*) – and thus also the bundle $Y_{n-1}(t)$ of the solutions $y(t)$ relative to (1) – we distinguish the following four significant cases:

1. Let in (Y^*) exist exactly $n - 2$ always two and two linearly independent pairs of real constants c_{ij}, $c_{i2} \neq 0$ $(i = 1, \ldots, n - 2;\ j = 1, 2)$. [Remark: two ordered pairs of real or complex numbers (c_{11}, c_{12}), (c_{21}, c_{22}) are called linearly independent exactly if

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0.]$$

Then there lie on the interval (t_0, T_1) exactly $n - 2$ simple zeros of the $2(n - 2)$-parametric system of functions

$$Y^*_{n-1}(t) = \prod_{i=1}^{n-2} [c_{i1}u(t) + c_{i2}v(t)], \qquad (Y^*_1)$$

each of them always belongs to one function $y^*_i(t)$ $(i = 1, \ldots, n - 2)$ obtained in an arbitrary (admissible) choice of the constants $c_{ij} \in \mathbf{R}$ $(i = 1, \ldots, n - 2; j = 1, 2)$ in a corresponding two-parametric system of functions $y^*_i(t, c_{i1}, c_{i2})$ in (Y^*_1). Then all these points are weakly conjugate points with respect to both simple mutually strongly conjugate points ${}^1t_0$, ${}^1t_n = T_1$ of all solutions $y(t)$ relative to (1) from the bundle $Y_{n-1}(t, C_1, \ldots, C_{n-1})$.

2. Let in (Y^*) exist m ($m \in \mathbf{N}$, $m \leqq n-2$) always two and two linearly independent pairs of real constants c_{ij}, $c_{i2} \neq 0$ ($i = 1, \dots, m$; $j = 1, 2$) such that

$$Y_{n-1}^*(t) = \prod_{i=1}^{m} [c_{i1}u(t) + c_{i2}v(t)]^{\nu_m}, \qquad (Y_2^*)$$

where $\nu_i \in N$ ($i = 1, \dots, m$), $\sum_{i=1}^{m} \nu_i = n - 2$.

Then there lie on the interval (t_0, T_1) exactly m zeros of the functional system (Y_2^*) with the multiplicities ν_i ($i = 1, \dots, m$), each of which belongs to the function $[y_i^*(t)]^{\nu_i}$, obtained in an arbitrary (admissible) choice of constants $c_{ij} \in \mathbf{R}$ from the corresponding two-parametric system of functions $[y_i^*(t, c_{i1}, c_{i2})]^{\nu_i}$. Here all these points are weakly conjugate points with respect to both simple, mutually strongly conjugate points ${}^1t_0$, ${}^1t_{m+1} = T_1$ of all solutions $y(t)$ relative to (1) from the bundle $Y_{n-1}(t, C_1, \dots, C_{n-1})$.

Let us remark that in case of $\nu_1 = \nu_2 = \dots = \nu_m = \nu$, $\nu \in \{1, \dots, n-2\}$, the multiplicities of all these m zeros of the functions $[y_i^*(t)]^\nu$ ($i = 1, \dots, m$) from the $2m$-parametric system

$$Y_{n-1}^*(t) = \prod_{i=1}^{m} [y_i^*(t, c_{i1}, c_{i2})]^\nu$$

are the same on the interval (t_0, T_1).

Especially, we get the case 1) exactly for $m = n - 2$, when $\nu_1 = \nu_2 = \dots = \nu_{n-2} = 1$.

3. Let in (Y^*) exist two pairs of complex linearly independent constants $\tilde{c}_{ij}$ ($i, j = 1, 2$) such that $\tilde{c}_{11}\tilde{c}_{21}$, $\tilde{c}_{12}\tilde{c}_{21} + \tilde{c}_{11}\tilde{c}_{22}$, $\tilde{c}_{12}\tilde{c}_{22} \in \mathbf{R}$, where $\tilde{c}_{12}\tilde{c}_{22} \neq 0$.

Then there exist two pairs of linearly independent real constants c_{ij} ($i, j = 1, 2$) such that

$$\begin{aligned} \tilde{c}_{11}\tilde{c}_{21} &= c_{11}^2 + c_{21}^2 \\ \tilde{c}_{12}\tilde{c}_{21} + \tilde{c}_{11}\tilde{c}_{22} &= 2(c_{11}c_{12} + c_{21}c_{22}) \\ \tilde{c}_{12}\tilde{c}_{22} &= c_{12}^2 + c_{22}^2, \end{aligned}$$

where $c_{12}^2 + c_{22}^2 > 0$, and the $2(n-2)$-parametric system of the functions (Y^*) may be written (up to a multiplicative constant $C \in \mathbf{R} - \{0\}$) in the form

$$Y_{n-1}^*(t) = \{[c_{11}u(t) + c_{12}v(t)]^2 + [c_{21}u(t) + c_{22}v(t)]^2\} \prod_{i=3}^{n-4} [c_{i1}u(t) + c_{i2}v(t)]. \quad (Y_3^*)$$

Since it holds

$$\tilde{Y}_2^*(t) = [c_{11}u(t) + c_{12}v(t)]^2 + [c_{21}u(t) + c_{22}v(t)]^2 > 0$$

on the interval $\mathbf{I} = (-\infty, +\infty)$ for all (admissible) choices of the constants $c_{ij} \in \mathbf{R}$ ($i, j = 1, 2$), then the system of functions (Y_3^*) may have at most $n - 4$ zeros (including their multiplicities) on the interval (t_0, T_1).

The functional system (Y_3^*) has thereby exactly $n - 4$ simple zeros on (t_0, T_1) if and only if all the other ordered pairs of real constants $c_{ij} \in \mathbf{R}$ $(i = 3, \dots, n-3;$ $j = 1, 2)$ appearing in the two-parametric systems of functions

$$y_i^*(t, c_{i1}, c_{i2}) = c_{i1}u(t) + c_{i2}v(t)$$

contained in (Y_3^*) are always two and two linearly independent.

All such zeros are weakly conjugate points with respect to both simple, mutually strongly conjugate points ${}^1t_0, {}^1T_1$ of all solutions $y(t)$ relative to (1) from the bundle $Y_{n-1}(t, C_1, \dots, C_{n-1})$.

4. Let in (Y^*) exist $2m$ $(m \in \mathbf{N}, 2m \leqq n-2, n > 2)$ always two and two linearly independent pairs of complex constants $\tilde{c}_{ij}$ $(i = 1, \dots, 2m; j = 1, 2)$ such that

$$\begin{aligned}
&\tilde{c}_{11}\tilde{c}_{21}, \tilde{c}_{12}\tilde{c}_{21} + \tilde{c}_{11}\tilde{c}_{22}, \tilde{c}_{12}\tilde{c}_{22} \in \mathbf{R}\\
&\vdots\\
&\tilde{c}_{2m-1,1}\tilde{c}_{2m,1}, \tilde{c}_{2m-1,2}\tilde{c}_{2m,1} + \tilde{c}_{2m-1,1}\tilde{c}_{2m,2}, \tilde{c}_{2m-1,2}\tilde{c}_{2m,2} \in \mathbf{R},
\end{aligned}$$

whereby $\tilde{c}_{12}\tilde{c}_{22} \neq 0, \dots, \tilde{c}_{2m-1,2}\tilde{c}_{2m,2} \neq 0$.

Then there exist $2m$ always two and two linearly independent pairs of real constants c_{ij} $(i = 1, \dots, 2m; j = 1, 2)$, such that

$$\begin{aligned}
\tilde{c}_{11}\tilde{c}_{21} &= \tilde{c}_{11}^2 + \tilde{c}_{21}^2\\
\tilde{c}_{12}\tilde{c}_{21} + \tilde{c}_{11}\tilde{c}_{22} &= 2(c_{11}c_{12} + c_{21}c_{22})\\
\tilde{c}_{12}\tilde{c}_{22} &= c_{12}^2 + c_{22}^2\\
&\vdots\\
\tilde{c}_{2m-1,1}\tilde{c}_{2m,1} &= c_{2m-1,1}^2 + c_{2m,1}^2\\
\tilde{c}_{2m-1,2}\tilde{c}_{2m,1} + \tilde{c}_{2m-1,1}\tilde{c}_{2m,2} &= 2(c_{2m-1,1}c_{2m-1,2} + c_{2m,1}c_{2m,2})\\
\tilde{c}_{2m-1,2}\tilde{c}_{2m,2} &= c_{2m-1,2}^2 + c_{2m,2}^2,
\end{aligned}$$

where $c_{11}^2 + c_{21}^2 > 0, \dots, c_{2m-1,2}^2 + c_{2m,2}^2 > 0$, and the $2(n-1)$-parametric system of functions (Y^*) of the form

$$Y_{n-1}^*(t) = \prod_{i=1}^{m} [\tilde{c}_{i1}u(t) + \tilde{c}_{i2}v(t)]^{v_m} \prod_{i=m+1}^{n-2} [c_{i1}u(t) + c_{i2}v(t)],$$

where $v_i \in \mathbf{N}$ $(i = 1, \dots, m)$, $\sum_{i=1}^{n} v_i = M \leqq n-2$, may be written in an equivalent form

$$Y_{n-1}^*(t) = \prod_{i=1}^{2m-1} \{[c_{i1}u(t) + c_{i2}v(t)]^2 + [c_{i+1,1}u(t) + c_{i+1,2}v(t)]^2\}^{v_m} \times$$
$$\times \prod_{i=2m+1}^{n-2} [c_{i1}u(t) + c_{i2}v(t)] \qquad (\mathrm{Y}_4^*)$$

(up to a multiplicative constant $C \in \mathbf{R} - \{0\}$).

Since it holds

$$\tilde{Y}_m^*(t) = \prod_{i=1}^{2m-1} \{\sum_{k=i}^{i+1} [c_{k1}u(t) + c_{k2}v(t)]^2\}^{v_m} > 0$$

on the interval $\mathbf{I} = (-\infty, +\infty)$ for all (admissible) choices of constants $c_{ij} \in \mathbf{R}$ ($i = 1, \ldots, 2m$; $j = 1, 2$), then the $2(n-2)$-parametric system of functions (Y_4^*) may have at most $n - 2 - M$ zeros (including their multiplicities) on the interval (t_0, T_1). These zeros – if any at all exist – may be only those zeros of the $2(n - 2m - 3)$-parametric subsystem of the functions

$$\hat{Y}^*_{n-m-1}(t) = \prod_{i=2m+1}^{n-2} [c_{i1}u(t) + c_{i2}v(t)].$$

All such zeros are weakly conjugate points with respect to both simple, mutually strongly conjugate points ${}^1t_0$, ${}^1T_1$ of all solutions $y(t)$ relative to (1) from the bundle $Y_{n-1}(t, C_1, \ldots, C_{n-1})$.

Thus in case of a general $k = n - 1$, we may summarize that exactly so many zeros of any solution $y(t)$ relative to (1) from the bundle $Y_{n-1}(t)$ will lie on the open interval (t_0, T_1) as many – always two by two linearly independent – pairs of real constants $c_{ij} \in \mathbf{R}$ ($i = 1, \ldots, m$; $m \leqq n - 2$; $j = 1, 2$) exist in the system of functions

$$Y^*_{n-1}(t) = \prod_{i-1}^{m} [c_{i1}u(t) + c_{i2}v(t)]^{\nu_i},$$

where $c_{i2} \neq 0$ and where $\nu_i \in \mathbf{N}$ ($i = 1, \ldots, m$), $\sum_{i=1}^{m} \nu_i = M \leqq n - 2$, denote the multiplicities of these zeros. All these points are the zeros (always two and two linearly independent) of the functions $y_i^*(t)$, obtained in an arbitrary (admissible) choice of constants $c_{ij} \in \mathbf{R}$ in the corresponding two-parametric subsystems of functions

$$y_i^*(t, c_{i1}, c_{i2}) = c_{i1}u(t) + c_{i2}v(t),$$

$i = 1, \ldots, m$, contained in the function system $Y^*_{n-1}(t)$. All such points with the multiplicities ν_i, $i \in \{1, \ldots, m\}$ will be the weakly conjugate points of the bundle $Y_{n-1}(t)$ of solutions $y(t)$ relative to (1) with respect to both simple, mutually strongly conjugate points ${}^1t_0$, ${}^1T_1$ from this bundle.

The remaining $n - 2 - m$ pairs of constants $\tilde{c}_{ij}$ ($i = m + 1, \ldots, n - 2$; $m \leqq \leqq n - 3$; $j = 1, 2$) in the system $Y^*_{n-1}(t)$, for which it holds that the corresponding four-parametric subsystems of functions

$$\tilde{Y}_i^*(t) = [\tilde{c}_{i1}u(t) + \tilde{c}_{i2}v(t)] [\tilde{c}_{i+1,1}u(t) + \tilde{c}_{i+1,2}v(t)],$$

where $\tilde{c}_{i2} \neq 0$, $\tilde{c}_{i+1,2} \neq 0$, have no zero on the open interval (t_0, T_1) must, and namely in an even number, be complex conjugate.

[Remark: Two ordered pairs of complex constants $(\tilde{c}_{11}, \tilde{c}_{12})$, $(\tilde{c}_{21}, \tilde{c}_{22})$ are conjugate if there exist two ordered pairs of real constants (c_{11}, c_{12}), (c_{21}, c_{22}), such that simultaneously

$$\tilde{c}_{11}\tilde{c}_{21} = c_{11}^2 + c_{21}^2$$

$$\tilde{c}_{12}\tilde{c}_{21} + \tilde{c}_{11}\tilde{c}_{22} = 2(c_{11}c_{12} + c_{21}c_{22})$$
$$\tilde{c}_{12}\tilde{c}_{22} = c_{12}^2 + c_{22}^2$$

is true].

The density of a distribution of zeros

The end summarizing of our considerations on the existence, the number and the multiplicities of zeros of the functional system $Y_{n-1}^*(t)$ – and thus of all solutions $y(t)$ relative to (1) from the corresponding bundle $Y_{n-1}(t)$ – on the open interval (t_0, T_1), carried out in case of $k = n - 1$, may analogous be performed in all the foregoing cases for $k \in \{1, \dots, n-2\}$. Instead of the summary form of the bundle

$$Y_k(t, C_1, \dots, C_k) = u^{n-k}(t) \sum_{i=1}^{n} C_i u^{k-i}(t)\, v^{i-1}(t),$$

with the parameters $C_i \in \mathbf{R}$, $i = 1, \dots, k$; $C_k \neq 0$, it will be useful to apply the equivalent product form

$$Y_k(t, c_{i1}, c_{i2}) = u^{n-k}(t)\; Y_k^*(t, c_{i1}, c_{i2}), \qquad (\mathrm{S_k})$$

where (up to a multiplicative constant $C \in \mathbf{R} - \{\mathbf{0}\}$)

$$Y_k^*(t, c_{i1}, c_{i2}) = \prod_{i=1}^{k-1} [c_{i1}u(t) + c_{i2}v(t)] \qquad (\mathrm{Y_k^*})$$

with – generally complex – constants c_{ij} $(i = 1, \dots, k-1; j = 1, 2)$, for which

$$C_1 = \prod_{i=1}^{k-1} c_{i1} \in \mathbf{R}, \dots, C_k = \prod_{i=1}^{k-1} c_{i2} \in \mathbf{R},$$

whereby $\prod_{i=1}^{k-1} c_{i2} \neq 0$ for all $i = 1, \dots, k-1$.

This enables us to express several theorems on the prescribed number of zeros of solutions $y(t)$ relative to (1) from the bundles $(\mathrm{S_k})$ on the interval (t_0, T_1) or $\langle t_0, T_1 \rangle$ and especially to decide for which types of the bundles $(\mathrm{S_k})$ the number of zeros belonging to $y(t)$ of (1) with respect to its order n on the interval considered, will be extremal.

On the basis of the analysis made for all types of the bundles $(\mathrm{S_k})$, $k \in \{1, \dots, n-1\}$, with respect to the increasing multiplicity $v = n - k$ of an arbitrary firmly chosen point $t_0 \in \mathbf{I} = (-\infty, +\infty)$, at which the bundles $(\mathrm{S_k})$ are vanishing together with the function $u(t)$ from the basis $[u(t), v(t)]$ relative to differential equation (2), we can immediately express the evident following

Statement: The higher is the multiplicity $v \in \{1, \dots, n-1\}$, $n \in \mathbf{N}$, $n > 1$, of the point $t_0 \in \mathbf{I}$ at which all solutions $y(t)$ relative to (1) from the oscillatory bundle $(\mathrm{S_k})$ are vanishing, the less number of weakly conjugate zeros of this bundle may lie on the open interval $({}^{\nu}t_0, {}^{\nu}T_1)$, where ${}^{\nu}T_1$ is the first strongly conjugate point from the right to the point ${}^{\nu}t_0$.

Especially: if $\nu = n - 1$, then there lies no zero of the bundle

$$(S_k),\ k = 1, \qquad \text{on the interval } ({}^{n-1}t_0, {}^{n-1}T_1).$$

However, the bundle (S_1) is not the only bundle of solutions $y(t)$ relative to (1) for which it holds that it has no zero on the interval (t_0, T_1). All bundles (S_k), $k \in \{1, \dots, n - 1\}$, with this property are treated in the following

Theorem 1.: If $n = 2m$ [or $n = 2m - 1$], then there lies no zero of the bundle (S_k) on the open interval (t_0, T_1) exactly if $\nu \in \{1, 3, \dots, 2m - 1\}$ [or $\nu \in \{2, 4, \dots, 2(m - 1)\}$], whereby all the ordered pairs of constants c_{ij} $(i = 1, 2, \dots, k - 1; j = 1, 2)$ in (Y_k^*), being of even number, are the two and two corresponding pairs complex conjugate. Then the only zeros of the bundle (S_k) of all solutions $y(t)$ relative to (1) on the closed interval $\langle t_0, T_1 \rangle$ are exactly both the boundary $(n - k)$-tuple points ${}^{n-k}t_0$, ${}^{n-k}T_1$. In this case all solutions $y(t)$ relative to (1) on the interval $\mathbf{I} = (-\infty, +\infty)$ have nothing but $(n - k)$-tuple strongly conjugate points, being simultaneously the zeros of the function $u(t)$.

Remark: The conditions stated in the foregoing theorem are at the same time the necessary and sufficient conditions for the thinnest distribution of zeros ever possible for the oscillatory solution $y(t)$ relative to (1) on the interval $\mathbf{I} = (-\infty, +\infty)$.

Especially it holds: If the distribution of all zeros of the function $u(t)$ from the basis $[u(t), v(t)]$ of the differential equation (2) is equidistant with the step $\delta = T_1 - t_0$ [where $t_0, T_1, T_1 > t_0$ are two consecutive zeros of the function $u(t)$] on the interval $\mathbf{I}$, then in all cases of the bundles (S_k) from the above Theorem, the distribution of the ν-tuple zeros, $\nu \in \{1, \dots, n - 1\}$, of all solutions $y(t)$ relative to (1) from the corresponding bundles (S_k) on $\mathbf{I}$, are also equidistant and namely with the same step δ.

The following theorem gives the forms of all bundles (S_k) of such solutions $y(t)$ relative to (1) having on the interval (t_0, T_1) exactly one zero, weakly conjugate with respect to both mutually strongly conjugate points ${}^{\nu}t_0, {}^{\nu}T_1$, $\nu \in \{1, \dots, n - 2\}$.

Theorem 2.: If $n = 2m - 1$ [or $n = 2m$], then on the open interval (t_0, T_1) there lies exactly one zero t^* of the bundle (S_k) of solutions $y(t)$ relative to (1) and namely of multiplicity $\mu = p$, exactly if it holds for the functional system (Y_k^*) in the bundle (S_k)

$$Y_k^*(t) = [c_{11}u(t) + c_{12}v(t)]^p \prod_{i=1}^{k-p-1} [\tilde{c}_{i1}u(t) + \tilde{c}_{i2}v(t)],$$

where

$$p \in \{1, 3, \dots, 2m - 3\} \qquad \text{for } k = 2q - 1, \qquad q = 1, 2, \dots, m - 1$$

and

$$p \in \{2, 4, \dots, 2(m-2)\} \qquad \text{for } k = 2q, \qquad q = 1, 2, \dots, m-2$$

[or

$$p \in \{2, 4, \dots, 2(m-2)\} \qquad \text{for } k = 2q-1, \qquad q = 1, 2, \dots, m-1$$

and

$$p \in \{1, 3, \dots, 2m-3\} \qquad \text{for } k = 2q, \qquad q = 1, 2, \dots, m-2],$$

whereby $c_{1j} \in \mathbf{R}$ $(j = 1, 2)$, $c_{12} \neq 0$, and all the remaining ordered pairs of the complex constants $(\tilde{c}_{i1}, \tilde{c}_{i2})$, $\tilde{c}_{i2} \neq 0$ $(i = 1, 2, \dots, k-p-1)$ are by twos conjugate.

Thereby

a) in case of $n = 2m - 1$ it holds: if the multiplicity $\nu = n - k$ of the point $t_0 \in \mathbf{I}$ is odd [even], then the multiplicity μ of the weakly conjugate point t^* is also odd [even],

b) in case of $n = 2m$ it holds: if the multiplicity $\nu = n - k$ of the point $t_0 \in \mathbf{I}$ is odd [even], then the multiplicity μ of the weakly conjugate point t^* is even [odd].

Remark: The conditions expressed in the Theorem above are at the same time the necessary and sufficient conditions for the forms of the bundles (S_k) of all such solutions $y(t)$ relative to (1) whose strongly conjugate points alternate with the weakly conjugate points [i.e. in which the strongly and weakly conjugate zeros mutually separate].

The question when on the open interval (t_0, T_1) there exist weakly conjugate points of the bundle (S_k) of solutions $y(t)$ relative to (1), whereby the multiplicities of all zeros of such solutions $y(t)$ on the closed interval $\langle t_0, T_1 \rangle$ are the same, discusses the following

Theorem 3.: If $n = 2m - 1$ [or $n = 2m$], then on the interval $\langle {}^{\nu}t_0, {}^{\nu}T_1 \rangle$ there exist weakly conjugate points of the bundle (S_k) of solutions $y(t)$ relative to (1), having throughout the same multiplicity $\mu = p = \nu$ exactly if $m \leqq k \leqq 2(m-1)$ [or $m + 1 \leqq k \leqq 2m - 1$] and for the functional system (Y_k^*) in the bundle (S_k) we have

$$Y_k^*(t) = \prod_{i=1}^{s} [c_{i1}u(t) + c_{i2}v(t)]^p \prod_{i=s+1}^{k-s-1} [\tilde{c}_{i1}u(t) + \tilde{c}_{i2}v(t)],$$

where

$$p \in \{1, 3, \dots, m-1\},\ s \in \{1, \dots, 2m-3\} \qquad \text{for } k \text{ odd}, \ p + k = 2m - 1$$

and

$$p \in \{2, 4, \dots, m-2\},\ s \in \{1, \dots, 2(m-2)\} \qquad \text{for } k \text{ even}, \ p + k = 2m - 1,$$

[or

$$p \in \{2, 4, \dots, m-2\},\ s \in \{1, \dots, 2(m-2)\} \qquad \text{for } k \text{ odd}, \ p + k = 2m$$

and

$$p \in \{1, 3, \dots, m-1\},\ s \in \{1, \dots, 2m-3\} \qquad \text{for } k \text{ even}, \ p + k = 2m]$$

whereby the ordered pairs (c_{i1}, c_{i2}) of real constants $c_{ij} \in \mathbf{R}$, $c_{i2} \neq 0$ $(i = 1, \ldots, s;$ $j = 1,2)$ are always two and two linearly independent and all the remaining ordered pairs $(\tilde{c}_{i1}, \tilde{c}_{i2})$ of complex constants $\tilde{c}_{ij}, \tilde{c}_{i2} \neq 0$ $(i = 1, \ldots, k - s - 1)$ are in (corresponding) pairs conjugate.

Remark: The conditions expressed in the above Theorem are at the same time necessary and sufficient for the existence of the bundles $S(_k)$ of all such solutions $y(t)$ relative to (1), having both strongly and weakly conjugate points of the same multiplicity on the whole interval $\mathbf{I} = (-\infty, +\infty)$.

The following theorem discussing the form of the bundle (S_k) of solutions $y(t)$ relative to (1) with the maximal number of zeros is a special case of the above Theorem.

Theorem 4.: There exists exactly one bundle (S_k) of solutions $y(t)$ relative to (1) having maximal number of weakly conjugate zeros on the open interval (t_0, T_1). The bundle is of the form

$$Y_1(t) = u(t)\, Y_1^*(t, c_{i1}, c_{i2}),$$

where for the corresponding $2(n-2)$-parametric system of functions (Y_1^*) it holds: there exist exactly $n - 2$ always two and two linearly independent ordered pairs of real constants $c_{ij} \in \mathbf{R}$, $c_{i2} \neq 0$, $(i = 1, \ldots, n-2; j = 1, 2)$, such that

$$Y_1^*(t, c_{i1}, c_{i2}) = \prod_{i=1}^{n-2} [c_{i1}u(t) + c_{i2}v(t)].$$

Each from the $n - 2$ zeros on the interval (t_0, T_1) belong always only to one of the $n - 2$ functions $y_i^*(t)$ obtained in an arbitrary (admissible) choice of constants c_{ij} from the corresponding two-parametric subsystem

$$y_i^*(t, c_{i1}, c_{i2}) = c_{i1}u(t) + c_{i2}v(t)$$

being always two and two linearly independent on the interval $\mathbf{I} = (-\infty, +\infty)$.

All these simple zeros are weakly conjugate with respect to both simple, mutually strongly conjugate points ${}^1t_0, {}^1T_1$.

Remark: Theorem 4 expresses the statement on the existence of exactly one bundle (S_k) of solutions $y(t)$ relative to (1) with the maximal density of zeros ever possible in a solution $y(t)$ of the considerated differential equation of the n-th order on the interval (t_0, T_1) – and thus also on the whole interval $\mathbf{I} = (-\infty, +\infty)$. It appears thereby that all zeros of anyhow solution $y(t)$ from this bundle – both the weakly and the strongly conjugate points – have the same lowest possible multiplicity, i.e. equal to 1.

REFERENCES

[1] Vlček, V.: *The first conjugate point of solution of the N-th order iterated differential equation* Acta Univ. Palackianae Olomucensis F. R. N., 1982, Tom 73.

[2] Borůvka, O.: *Lineare Differentialtransformationen 2. Ordnung.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1967.

[3] Vlček, V.: *Conjugate points of solutions of an iterated differential equation of the N-th order.* Acta Univ. Palackianae Olomucensis F. R. N., 1983, Tom 76.

Souhrn

ROZLOŽENÍ NULOVÝCH BODŮ ŘEŠENÍ ITEROVANÉ DIFERENCIÁLNÍ ROVNICE *n*-TÉHO ŘÁDU

VLADIMÍR VLČEK

V práci je vyšetřováno rozložení nulových bodů oscilatorických svazků řešení diferenciální rovnice *n*-tého řádu jistého speciálního typu. K jejich popisu je využito pojmů silně resp. slabě konjugovaných bodů řešení, zavedených v předchozích autorových pracích. Přitom se existence, počet popříp. uspořádání nulových bodů vyšetřuje na zvoleném intervalu mezi libovolnými dvěma navzájem silně konjugovanými body. Současně se řeší vždy otázka jejich násobnosti.

V příslušných větách jsou ukázány takové tvary svazků řešení, která mají na uvažovaném intervalu nejmenší resp. největší hustotu nulových bodů popříp. kdy na tomto intervalu jich leží předepsaný počet.

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Резюме

РАСПОЛОЖЕНИЕ НУЛЕВЫХ ТОЧЕК РЕШЕНИЙ ИТЕРИРОВАННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ N-ГО ПОРЯДКА

ВЛАДИМИР ВЛЧЕК

В работе изучается расположение нулевых точек колеблющихся пучков решений дифференциального уравнения N-го порядка наверно специального типа. К их описыванию использованы понятия так называемых сильно или

слабо сопряженных точек решений, внесенных во внимание автором в его предыдущих работах. При этом существование, номер или упорядочение нулевых точек изучается на выбранном интервале между любими двумя соседними сильно сопряженными точками решений. Современно решается совсем и вопрос об их насобностьях.

В надлежащих теоремах показаны такие формы пучков решений у которых на учитыванном интервале наименьшая или наибольшая плотность нулевых точек или их вопред данный номер на таком интервале.