Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Svatoslav Staněk

A note on the oscillation of solutions of the differential equation $y' + \lambda q(t)y = 0$ with an almost periodic coefficient

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 24 (1985), No. 1, 67--70

Persistent URL: http://dml.cz/dmlcz/120166

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1985 MATHEMATICA XXIV Vol. 82

Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Prof. RNDr. Miroslav Laitoch, CSc.

A NOTE ON THE OSCILLATION OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y'' + \lambda q(t)y = 0$ WITH AN ALMOST PERIODIC COEFFICIENT

SVATOSLAV STANĚK

(Received October 16, 1983)

Let q be a (real) almost periodic function ([1], [2]), $q \neq 0$, and λ a (real) parameter. Consider the differential equation

$$y'' + \lambda q(t)\, y = 0. \qquad (\lambda q)$$

It is well known (see [3]) that (λq) is either disconjugate, i.e. every nontrivial solution of (λq) has one zero (on $\mathbf{R}$) at most, or it is oscillatory, i.e. $\pm\infty$ are the cluster points of zeros relative to every nontrivial solution of this equation. If $\lambda = 0$, then (λq) is disconjugate.

Our object now is to obtain such a necessary and sufficient condition laid on the function q for the equation (λq) to be oscilla ory for every λ, $\lambda \in \mathbf{R} - \{0\}$. A necessary and sufficient condition for the special case with q being periodic is given in [6] and follows directly from

Theorem 1. *Let q be an almost periodic function, $q \not\equiv 0$. Then (λq) is oscillatory for every λ, $\lambda \in \mathbf{R} - \{0\}$, exactly if $M\{q\} = 0$, where $M\{q\} := \lim_{t\to\infty} \frac{1}{t}\int_0^t q(s)\,ds$ means the mean value of the function q.*

Prior to proving the above Theorem we will prove

Lemma 1. *Let f be an almost periodic function and $M\{f\} = 0$. Then there exists to every number $\tau > 0$ an almost periodic function p, $|p(t)| < \tau$ for $t \in \mathbf{R}$, such that the function $\alpha(t) := \int_0^t (f(s) + p(s))\,ds$, $t \in \mathbf{R}$, is bounded on $\mathbf{R}$.*

Proof. If the function $\int_0^t f(s)\,ds$ is bounded on $\mathbf{R}$, we put $p = 0$. Let the function $\int_0^t f(s)\,ds$ be unbounded on $\mathbf{R}$ and $\tau > 0$ be an arbitrary number. Then there exists a trigonometric polynom $P(t) = a_0 + \sum_{k=1}^{N} (a_k \cos \nu_k t + b_k \sin \nu_k t)$ ($\nu_k \neq 0$ for $k = 1, 2, \ldots, N$) (see e.g. [2] p. 62), such that

$$|f(t) - P(t)| < \frac{\tau}{2} \qquad \text{for } t \in \mathbf{R}. \tag{1}$$

If $M\{P\} = a_0$, then on our assumptions $M\{f\} = 0$ and (1) we get inequality

$$|a_0| < \frac{\tau}{2}. \tag{2}$$

Putting $p(t) := P(t) - f(t) - a_0$, $t \in \mathbf{R}$, then p is almost periodic function and from (1) and (2) we get $|p(t)| < \tau$ for $t \in \mathbf{R}$. Since $\alpha(t) = \int_0^t (f(s) + p(s))\,ds = \int_0^t \sum_{k=1}^{N} (a_k \cos \nu_k x + b_k \sin \nu_k x)\,dx$, it follows that $|\alpha(t)| \leqq \sum_{k=1}^{N} \left(\left|\frac{a_k}{\nu_t}\right| + 2\left|\frac{b_k}{\nu_k}\right| \right)$ for $t \in \mathbf{R}$, so that the function α is bounded on $\mathbf{R}$.

Proof of Theorem 1. ($\Leftarrow$) If $M\{q\} = 0$, then it follows from Theorem 2 ([3]) that (λq) is oscillatory for every λ, $\lambda \in \mathbf{R} - \{0\}$.

($\Rightarrow$) Suppose (λq) to be oscillatory for every λ, $\lambda \in \mathbf{R} - \{0\}$, and $M\{q\} = c\,(\neq 0)$. For certainly let $c > 0$ (for $c < 0$ the function $-q$ instead of q is considered). Let us put $q_1(t) := q(t) - c$, $t \in \mathbf{R}$. Then q_1 is an almost periodic function and $M\{q_1\} = 0$. By Lemma 1 there exists an almost periodic function p, $|p(t)| < \frac{c}{2}$ for $t \in \mathbf{R}$, such that the function $\int_0^t (q_1(s) + p(s))\,ds$ is bounded on $\mathbf{R}$. Let $|\int_0^t (q_1(s) + p(s))\,ds| \leqq K$ for $t \in \mathbf{R}$, where K is a constant. Then, of course,

$$\left|\int_x^t (q_1(s) + p(s))\,ds\right| \leqq 2K \qquad \text{for } 0 \leqq x \leqq t. \tag{3}$$

Let

$$0 < \lambda < \frac{c}{8K^2}. \tag{4}$$

We will prove that

$$y'' - \lambda\left(q_1(t) + p(t) + \frac{c}{2}\right) y = 0 \tag{5}$$

is nonoscillatory on $[0, \infty)$ for all λ satisfying (4). Consequently, equation (5) is for all such λ necessarily disconjuge (on $\mathbf{R}$). Suppose that $\lambda > 0$. By substitution of $y = \exp\left(\sqrt{\frac{\lambda c}{2}}\,t\right) z$ equation (5) is transformed onto equation

$$(r(t)\,z')' + Q(t)\,z = 0, \tag{6}$$

where $r(t) := \exp(\sqrt{2\lambda c}t)$, $Q(t) := -\lambda(q_1(t) + p(t))\exp(\sqrt{2\lambda c}t)$, $t \in \mathbf{R}$. Then for every λ satisfying (4) we get from (3) for every $0 \leqq x \leqq t$

$$\left|\int_x^t (Q(s)\int_s^\infty (1/r(u))\,\mathrm{d}u\,\mathrm{d}s\right| = \lambda\left|\int_x^\infty \exp(\sqrt{2\lambda c}s)(q_1(s) + p(s))\int_s^\infty \exp(-\sqrt{2\lambda c}u)\,\mathrm{d}u\,\mathrm{d}s\right| =$$

$$= \frac{\lambda}{\sqrt{2\lambda c}}\left|\int_x^t (q_1(s) + p(s))\,\sigma s\right| \leqq \sqrt{\frac{2\lambda}{c}}K < \frac{1}{2}.$$

It now follows from the above and from the Moore criterion for the nonoscillation of (6) (see [4] or [5] p. 196) that equation (6) is nonoscillatory on $[0, \infty)$ for every λ satisfying (4). It then naturally follows from the inequality $\lambda\left(q_1(t) + p(t) + \frac{c}{2}\right) <$ $< \lambda(q_1(t) + c) = \lambda q(t)$ and from the Sturm comparison theorem that (λq) is disconjugate for every λ satisfying (4) which, however, is contrary to our assumption of the Theorem.

Remark 1. If the function $\int_0^t q(s)\,\mathrm{d}s$ is bounded on $\mathbf{R}$, then the assertion of Theorem 1 may be proved directly using Theorems 2 and 6 in [4].

REFERENCES

[1] Харасахал, В. Х.: *Почти-периодические решения обыкновенных дифференциальных уравнений.* Издательство „Наука", Алма-Ата, 1970.

[2] Левитан, Б. М.: *Почти-периодические функции.* Москва 1953.

[3] Magnus, L. and Moore, R.: *Oscillation and disconjugacy for linear differential equations with almost periodic coefficients.* Acta Math., 96, 1956, 99–123.

[4] Moore, R.: *The behaviour of solutions of a linear differential equation second order.* Pac. J. Math., 5, 1955, 125–145.

[5] Reid, W. T.: *Sturmian Theory for Ordinary Differential Equations.* Springer-Verlag New York, 1980.

[6] Staněk, S.: *A note on the oscillation for solutions of the differential equation $y'' = \lambda q(t)\,y$ with a periodic coefficient.* Czech. Math. J., 29 (109), 1979, 318–323.

OSCILACE ŘEŠENÍ DIFERENCIÁLNÍ ROVNICE $y'' = \lambda q(t)\,y$ SE SKOROPERIODICKÝM KOEFICIENTEM

Souhrn

V práci je dokázána věta: Nechť q je skoroperiodická funkce, $q \not\equiv 0$. Pak rovnice

$$y'' = \lambda q(t)\,y$$

je oscilatorické pro každé λ, $\lambda \in \mathbf{R} - \{0\}$, právě když $M\{q\} = 0$, kde $M\{q\} := \lim_{T\to\infty}\frac{1}{T}\int_0^T q(t)\,\mathrm{d}t$ značí střední hodnotu funkce q.

О КОЛЕБЛЮЩИХСЯ РЕШЕНИЯХ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y'' = \lambda q(t)\, y$ С ПОЧТИ-ПЕРИОДИЧЕСКИМ КОЭФФИЦИЕНТОМ

Резюме

В работе приводится теорема: Пусть q почти-периодическая функция, $q \not\equiv 0$. Для того, чтобы дифференциальное уравнение $y'' = \lambda q(t) y$ было колеблющимся для $\lambda \in \mathbf{R} - \{0\}$, необходимо и достаточно, чтобы $M\{q\} = 0$, где $M\{q\} := \lim_{T \to \infty} \frac{1}{T} \int_0^T q(t) \mathrm{d}t$ — среднее значение функции q.