Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 26 (1987), No. 1, 149--160

Persistent URL: `http://dml.cz/dmlcz/120179`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM

1987 Mathematica XXVI Vol. 88

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty University Palackého v Olomouci
Vedoucí katedry: Miroslav Laitoch, Prof., RNDr., CSc.

THE METHOD OF SOLUTION OF THE BOUNDARY VALUE PROBLEM APPLIED TO A CERTAIN FOURTH ORDER DIFFERENTIAL EQUATION

VLADIMÍR VLČEK

(Received March 24th, 1986)

Dedicated to Professor M.Laitoch on his 65th birthday

1. Hitherto knowledge of solution of the boundary value problem to 2nd order differential equation

Consider a linear differential equation of the second order

$$y''(t) + [q(t,k,m) + r(t)]y(t) = 0, \qquad (2.1)$$

where both functions $q(t)$ and $r(t)$ are continuous on the interval $\mathbb{I} = (-\infty, +\infty)$ and $k,m \in \mathbb{R}$ are parameters.
For the two-point boundary value problem

$$y(a) = y(b) = 0, \text{ where } a,b, \in \mathbb{R}, \ a < b$$

of the differential equation (2.1) there holds the following Lemma from [3] which assures the existence of the function $k = k(m)$ being continuous on the interval $\mathbb{I}$ such that the solution $y(t,m,k(m))$ of this equation vanishes at the points a and b, i.e. it is a solution of the prescribed boundary value problem.

Here and in what follows when we talk of "a solution of the differential equation" we mean nontrivial solution, only.

L e m m a 1. Consider the differential equation (2.1), where $t \in \langle a,b\rangle \subset Ⓘ$ and let $n \in Ⓝ$ be an arbitrary preassigned natural number.

A/ Let the functions $q(t,k,m)$ and $r(t)$ be continuous for all $(t,k,m) \in \langle a,b\rangle \times Ⓡ^2$, whereby $q(t,k,m)$ is an increasing function with respect to the parameter k.

B/ Let to every $m \in Ⓡ$ belong a function $\overline{k}(m)$ [or $\underline{k}(m)$] such that the solution $y(t,m,\overline{k}(m))$ [or $y(t,m,\underline{k}(m))$] of (2.1), where $q(t,\overline{k}(m),m)$ [or $q(t,\underline{k}(m),m)$], vanishing at the point a has more [or less] than n zeros on the interval $(a,b\rangle$.

Then there exists a continuous function $k(m)$, $m \in Ⓡ$, such that the solution $y(t,m,k(m))$ of (2.1) vanishing at the point a has a point b as the n-th conjugate point (on the right) to the point a.

On the basis of this Lemma may be formulated [3] an analogous theorem on solving the three-point boundary value problem

$$y(a) = y(b) = y(c) = 0, \text{ where } -\infty < a < b < c < +\infty, \qquad (3)$$

for the differential equation (2.1).

T h e o r e m : Consider the differential equation (2.1), where $t \in \langle a,c\rangle \subset Ⓡ$, $b \in (a,c)$ and $n_1, n_2 \in Ⓝ$, $n_2 > n_1$, be arbitrary preassigned natural numbers.
Let the functions $q(t,k,m)$ and $r(t)$ satisfy the assumptions A/, B/ from foregoing Lemma 1. on the interval $\langle a,b\rangle$ for $n = n_1$ and on the interval $\langle b,c\rangle$ for $n = n_2 - n_1$.

C/ Let for at least one value m^o of a parameter m exist a value k^o of the parameter k such that

$$q(t,k^o,m^o) > 0 \quad \text{for at least one } t \in \langle a,b)$$

and

$$q(t,k^o,m^o) < 0 \quad \text{for all } t \in \langle b,c\rangle$$

and let for at least one value m_o of a parameter m exist a value k_o of a parameter k such that

$$q(t,k_o,m_o) < 0 \quad \text{for at least one } t \in (b,c\rangle$$

and

$$q(t,k_o,m_o) > 0 \quad \text{for all } t \in \langle a,b\rangle$$

D/ Let the function $q(t,k,m)$ for any firmly chosen t,k,m, with $q(t,k,m) \neq 0$, satisfy the property

$$\frac{q(t,\lambda k,\lambda m)}{\operatorname{sgn}[\lambda q(t,k,m)]} \longrightarrow \infty$$

monotonically for $|\lambda| \to \infty$.

Then there exist the values of parameters k and m such that the solution $y(t,k,m)$ vanishing at the point a has the point b as the n_1-th and the point c as the n_2-th conjugate point (on the right) to the point a.

It should be noted here that the Lemma (and so the Theorem as well) in [3] is formulated under the more generally assumption A/ with respect to the function q. It is supposed that $q(t,k,m)$ is a weakly increasing function of a parameter k, i.e. for every $m \in R$ and for every $t \in \langle a,b\rangle$ there is valid the implication: $k_1 < k_2 \Rightarrow q(t,k_1,m) \leq q(t,k_2,m)$, whereby the sharp inequality in the consequent holds on the set of points $t \in \langle a,b\rangle$ with the measure different from zero.

Instead of the weakly increasing function one may use the assumption with a weakly decreasing function.

The inequalities introduced in pairs in the assumption C/ of the cited Theorem may be interchanged for reversed, as well.

For some special forms of the function $q(t,k,m)$ occurring in (2.1) there follow the consequent theorems [3] from the Theorem above. They again establish sufficient conditions for the existence of the solution of the three-point boundary value problem formulated for the respective differential equation of the second order.

C o r o l l a r y 1. Consider the differential equation

$$y''(t) + [kq_1(t) + mq_2(t) + r(t)]y(t) = 0, \qquad (2.2)$$

where $t \in \langle a,c\rangle \subset (I)$, $b \in (a,c)$, with the functions $q_1(t)$, $q_2(t)$, $r(t)$ being continuous on the interval $\langle a,c\rangle$. Let $n_1, n_2 \in (N)$, $n_2 > n_1$, be arbitrary preassigned natural numbers.

Let $q_1(t) \geqq 0$, whereby $q_1(t) \not\equiv 0$ on the intervals $\langle a,b\rangle$, $\langle b,c\rangle$ and the zeros of the function $q_1(t)$ - except for their finite number - be coinciding with the common zeros of the functions $q_2(t)$ and $r(t)$. Let the values k^o, k_o on a parameter k exist such that there simultaneously hold the inequalities

$$k^o q_1(t) + q_2(t) > 0 \quad \text{for at least one } t \in \langle a,b)$$

$$k^o q_1(t) + q_2(t) < 0 \quad \text{for all } t \in \langle b,c\rangle$$

and

$$k_o q_1(t) + q_2(t) < 0 \quad \text{for at least one } t \in (b,c\rangle$$

$$k_o q_1(t) + q_2(t) > 0 \quad \text{for all } t \in \langle a,b\rangle .$$

The there exist the values of parameters k and m such that the solution $y(t,k,m)$ of (2.2) vanishing at the point a has the point b as the n_1-th and the point c as the n_2-th conjugate point (on the right) to the point a.

C o r o l l a r y 2. Consider the differential equation

$$y''(t) + [k + mq(t) + r(t)]\, y(t) = 0 , \qquad (2.3)$$

where $t \in \langle a,c\rangle \subset (I)$, $b \in (a,c)$, with the functions $q(t)$ and $r(t)$ being continuous on the interval $\langle a,c\rangle$. Let $n_1, n_2 \in (N)$, $n_2 > n_1$, be arbitrary preassigned natural numbers.

Let the function $q(t)$ assume its maximum [or minimum] only in the interval $\langle a,b)$ and its minimum [or maximum] only in the interval $(b,c\rangle$.

Then there exist the values of parameters k and m such that the solution $y(t,k,m)$ of (2.3) vanishing at the point a has the point b as the n_1-th and the point c as the n_2-th conjugate point (on the right) to the point a.

The requirement on the existence of the extreme values of the function $q(t)$ on the intervals $\langle a,b)$ and $(b,c\rangle$ stated in the assumption of Corollary 2. may be replaced by either of the two requirements below. Either

1/ the function $q(t)$ is continuous and monotonic on the interval $\langle a,c\rangle$

or

2/ $q(t) > 0$ for all $t \in (a,b)$ and $q(t) < 0$ for all $t \in (b,c)$ or $q(t) < 0$ for all $t \in (a,b)$ and $q(t) > 0$ for all $t \in (b,c)$.

It is shown that the requirements placed on the function $q(t)$ are to the existence to the proper values of the both parameters k and m indispensable.

The solution of the three-point boundary value problem for the differential equation (2.3), i.e. the differential equation (2.2) where $q_1(t) \equiv 1$ has been considered by F.M. Arscott [4] and for the differential equation (2.1) by M.Greguš [5]. The latter besides makes a connection with the solution of the boundary value problem for a 3rd order differential equation of the form $y'''(t) + q(t,k,m)\, y(t) = 0$.

2. Application to solutions of boundary value problems for the fourth order differential equation

Consider now a linear 4-th order differential equation

$$Y^{IV}(t) + 10\,[q(t)Y'(t)]' + 3\,[3q^2(t) + q''(t)]Y(t) = 0, \qquad (4)$$

where $q(t) \in C^2(-\infty,+\infty)$ on the interval $I = (-\infty,+\infty)$ arising in iterating the linear 2nd order differential equation

$$y''(t) + q(t)y(t) = 0 \qquad (2.0)$$

(hereafter the dif.equation (4) will be referred to as "iterated" equation).

If $[u(t),v(t)]$ is a basis of all solutions of (2.0), then

$$[u^3(t),\ u^2(t)v(t),\ u(t)v^2(t),\ v^3(t)]$$

is a basis of all solutions of (4). Consequently, the system of all solutions of (4) constructs a 4-parametric space of functions

$$Y(t,C_1,\dots,C_4) = \sum_{i=1}^{4} C_i u^{4-i}(t)v^{i-1}(t), \qquad (S_4)$$

where $C_i \in Ⓡ$, $i= 1,\dots,4$, are four independent arbitrarily chosen parameters of the system, whereby $\sum_{i=1}^{4} C_i^2 > 0$ (trivial solution is excluded here and below).

Let us assume the functions $u(t),v(t)$ forming the basis of (2.0) be oscillatory in the sense of [2], i.e. there lie infinitely many (simple) zeros of the functions $u(t),v(t)$ on the right and on the left from every point $t_0 \in Ⓘ$, mutually separating (in consequence of the validity of the Sturm-Liouville separation theorem). Since any two linearly independent solutions of (2.0) possess this property, equation (2.0) will be called oscillatory.

With respect to the form of the basis of the space of all solutions of the differential equation (4), every solution $Y(t)$ of this equation from the system (S_4) is oscillatory as well. Consequently, the differential equation (4) with an oscillatory basis will be also called oscillatory.

Since the order of the differential equation (4) is $n = 4$, the multiplicity of its arbitrary zero may at most be equal to 3.

<u>Oscillatory bundles of the differential equation (4)</u>.

Let $t_0 \in Ⓘ$ be an arbitrary firmly chosen point. Then all solutions $Y(t)$ of the differential equation (4) from the system (S_4) vanishing at this point contemporally with the function $u(t)$ are generating a 3-parametric system of functions

$$Y(t,C_1,C_2,C_3) = u(t) \sum_{i=1}^{3} C_i u^{3-i}(t)v^{i-1}(t), \qquad (S)$$

where $\sum_{i=1}^{3} C_i^2 > 0$.

The situation of zeros of the solutions $Y(t)$ of (4) from the individual oscillatory bundles attained in the system (S) may be summarized as follows:

1/ The bundle $Y_1(t,C_1) = C_1u^3(t)$, $C_1 \neq 0$, (S_1)
has all zeros threefold

2/ The bundle $Y_2(t,C_1,C_2) = u^2(t)[C_1u(t) + C_2v(t)]$,
$C_2 \neq 0$, (S_2)

has among all zeros simple zeros on one hand and two-fold zeros on the other which interchange

3/ The bundle $Y_3(t,C_1,C_2,C_3) = u(t)[C_1u^2(t) + C_2u(t)v(t) +$

$+ C_3v^2(t)]$, $C_3 \neq 0$, (S_3)

a/ has all zeros simple if $C_2^2 - 4C_1C_3 \neq 0$

b/ has among all zeros simple zeros on one hand and two--fold zeros on the other which interchange if $C_2^2 - 4C_1C_3 = 0$.

From the above enumerated bundles may those under 2/ and 3/ be more briefly written as follows:

2/ $Y_2(t) = u^2(t)y_1(t,c_{11},c_{12})$, $c_{12} \neq 0$ (S_2)

3/ a) $Y_{31}(t) = u(t)y_1(t,c_{11},c_{12})y_2(t,c_{21},c_{22})$,
$c_{i2} \neq 0, i=1,2$ (S_{31})

b) $Y_{32}(t) = u(t)[y_1^2(t,c_{11},c_{12}) + y_2^2(t,c_{21},c_{22})]$,
$c_{i2} \neq 0, i=1,2$ (S_{32})

c) $Y_{33}(t) = u(t)\ y_1^2(t,c_{11},c_{12})$, $c_{12} \neq 0$ (S_{33})

where $y_i(t,c_{i1},c_{i2}) = c_{i1}u(t) + c_{i2}v(t)$, $c_{i2} \neq 0$, $i = 1,2$, denotes the system of all solutions of the differential equation (2,0) linearly independent with the function $u(t)$ on the interval I [in case of the bundles 3/a), 3/b) then $u(t)$, $y_1(t)$, $y_2(t)$ are always in pairs linearly independent]. Hereby the zeros of the solutions from the bundles $Y_2(t)$, $Y_{31}(t)$ and $Y_{33}(t)$ belonging to the systems of the functions $y_i(t)$ are arbitrarily moving always between the two neighbouring zeros of the function $u(t)$. Their location is then fixed by an admissible choice of values of parameters $c_{i1}, c_{i2} \in$ Ⓡ, $i = 1,2$, in these functional systems. It is just this movable character of zeros in the systems $y_i(t)$, $i = 1,2$, of the solutions of the differential equation (2.0) which essentially helps to solve the boundary value problems for the differential equation (4).

A solution of a simple n-point ($n < 4$) boundary value problem for the differential equation (4) under the oscillatority of the dif.equation (2.0) has been treated in [1].

Suppose now the function $q(t)$ in (2.0) is $q = q(t,k,m)$, where $k,m \in$ Ⓡ are parameters, satisfying the assumptions of the Theorem on the existence of the solution of the three-point boundary value problem from the introductory chapter of this article. The there will be appropriately modified the functional coefficients in the differential equation (4) arisen on iterating the differential equations (2.1) or (2.2) or (2.3). There will however remain unchanged the structures of the individual bundles (S_1), (S_2), (S_3) of solutions $Y(t)$ of the differential equation (4) vanishing together with the function $u(t)$ always at the point $a \in$ Ⓘ [we continue to use the notation $u(t), v(t)$ for the bases also in the case of dif. equations (2.i), i=1,2,3].

I. On account of the fact that for every solution $Y(t)$ of the dif.equation (4) from the bundle (S_1)

$$Y(a) = Y'(a) = 0$$

holds, the three-point boundary value problem (3) for the dif. equation (4) is of the form

$$Y(a) = Y'(a) = Y(b) = Y(c) = 0 \tag{5}$$

whose solution is the function $Y(t) = u^3(t)$ [up to an arbitrary multiplicative constant $C_1 \in$ Ⓡ, $C_1 \neq 0$]. Then the points

$a,b,c \in (I)$, $-\infty < a < b < c < +\infty$ are threefold zeros of the solution $Y(t)$; for the 1st derivative $Y'(t)$ is the point a a twofold zero. The three-point boundary value problem (3) for the dif.equation (4) naturally satisfies any solution $Y(t)$ from all bundles (S_i), $i = 1,2,3$, if the solution of this problem is the function $u(t)$ - the solution of the dif.equation (2.i), $i = 1,2,3$. Especially then the solution $Y(t)$ of the dif.equation (4) of the form

$$Y(t) = u(t) \sum_{i=1}^{2} y_i^2(t)$$

from the bundle (S_{32}) wherein $C_2^2 - 4C_1C_3 < 0$ [in this case all zeros of the solution $Y(t)$ of (4) coincide with the zeros of the function $u(t)$].

II. The function $Y(t)$ from the bundle (S_2) is the solution of the four-point boundary value problem

$$Y(a) = Y(b) = Y(c) = Y(d) = 0\,, \tag{6}$$

where $a,b,c,d \in (I)$, $-\infty < a < b < c < d < +\infty$, for the dif.equation (4), if the function $u(t)$ is a solution of the three-point boundary value problem (3)

$$u(a) = u(b) = u(d) = 0$$

or

$$u(a) = u(c) = u(d) = 0$$

for the dif.equation (2.i), $i = 1,2,3$, on the interval $\langle a,d \rangle$ and it simultaneously holds for the function $y_1(t)$ that

$$y_1(c) = 0$$

or

$$y_1(b) = 0$$

[this may be always achieved by properly chosen coefficients $c_{11}, c_{12} \neq 0$ in the systems of functions $y_1(t,c_{11},c_{12})$ - being linearly independent with the function $u(t)$].

Then in case 1, where simultaneously $u(a) = u(b) = u(d) = 0$, $y_1(c) = 0$, the points a,b,d are twofold zeros, the point c is a simple zero of the solution $Y(t)$ of the dif.equation (4) from the bundle (S_2).

In case 2, where simultaneously $u(a) = u(c) = u(d) = 0$, $y_1(b) = = 0$, the points a,c,d are twofold zeros, the point b is a simple zero of the solution $Y(t)$ of the dif.equation (4) from the bundle (S_2).

Similarly the function $Y(t)$ from the bundle (S_3) c), with $C_2^2 - 4C_1C_3 = 0$, may be used to the solution of the boundary value problem (6). On account of the fact that the bundles (S_2), (S_{33}) are dual to each other, now the zeros a,b,d are simple and the point c is a twofold zero [or the points a,c,d are simple and the point b is a twofold zero] of the solution $Y(t)$ of the dif.equation (4) from this bundle (S_{33}).

III. The function $Y(t)$ from the bundle (S_{31}) is the solution of the five-point boundary value problem

$$Y(a) = Y(b) = Y(c) = Y(d) = Y(e) = 0 , \tag{7}$$

where $a,b,c,d,e \in (I)$, $-\infty < a < b < c < d < e < +\infty$ for the dif.equation (4), if the function $u(t)$ is a solution of the three-point boundary value problem (3)

$$u(a) = u(b) = u(e) = 0$$

or

$$u(a) = u(c) = u(e) = 0$$

or

$$u(a) = u(d) = u(e) = 0$$

for the dif.equation (2.i), i=1,2,3, on the interval $\langle a,e\rangle$ and it simultaneously holds for the functions $y_i(t)$, i=1,2, that

$$y_1(c) = 0 \ , \ y_2(d) = 0 \quad \text{(or conversely)}$$

or

$$y_1(b) = 0 \ , \ y_2(d) = 0 \quad \text{(or conversely)}$$

or

$$y_1(b) = 0 \ , \ y_2(c) = 0 \quad \text{(or conversely)}$$

[this may be always achieved by properly chosen coefficients c_{i1}, $c_{i2} \neq 0$, $i = 1,2$, in the systems of functions $y_i = y_i(t, c_{i1}, c_{i2})$ with regard to the fact that any two of the three functions $u(t)$, $y_1(t)$, $y_2(t)$ are linearly independent].

Hereby all zeros $a,b,c,d,e \in (I)$ both of the function $u(t)$ and of the two functions $y_i(t)$, $i = 1,2$, from the bundle (S_{31}) 3a) of the solution $Y(t)$ of the dif.equation (4) with $C_2^2 - 4C_1C_3 > 0$, are altogether simple.

REFERENCES

[1] V l č e k, V.: On a certain boundary value problem for a fourth-order iterated differential equation; Acta UP Olom., F.R.N., Tom 69, 1981

[2] B o r ů v k a, O.: Lineare Differentialtransformationen 2.Ordnung; VEB Deutscher Verlag der Wissenschaften, Berlin, 1967

[3] G r e g u š, M., N e u m a n, F., A r s c o t t, F.M.: Three-point boundary value problems in differential equations; J.London Math.Soc. (2), 3(1971), 429-436

[4] A r s c o t t, F.M.: Two-parameter eigenvalue problems in differential equations; Proc.London Math.Soc. (3), 14(1964), 459-470

[5] G r e g u š, M.: Application of the theory of dispersions to the boundary problem of the second order; Mat.-fyzik. čas.SAV, Bratislava, 1963

SOUHRN

Aplikace metody řešení okrajové úlohy pro jistou diferenciální rovnici 4. řádu

V l a d i m í r   V l č e k

V práci je uvažováno řešení několikabodových okrajových úloh formulovaných pro jistou lineární diferenciální rovnici 4.řádu s použitím výsledků, dosažených při řešení tříbodové okrajové úlohy pro lineární diferenciální rovnici 2.řádu.

РЕЗЮМЕ

Приложение метода решения краевой задачи для дифференциального уравнения 4-го порядка определенного типа

В л а д и м и р   В л ч е к

В работе предлагается решение нескольких краевых задач сформулированных для определенного линейного дифференциального уравнения 4-го порядка из точки зрения результатов, достигнутых для решения трехточечной краевой задачи для линейного дифференциального уравнения 2-го порядка.