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A NOTE TO A CERTAIN NONLINEAR DIFFERENTIAL EQUATION OF THE THIRD-ORDER

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Consider a nonlinear third order differential equation

$$x'''(t) + ax''(t) + bx'(t) + h[x(t)] = p(t), \qquad (1)$$

where $a,b \in R$, $a > 0$, $b > 0$ are given constants and the functions $h[x(t)]$, $p(t)$ with continuous first derivatives are oscillatory on the interval $I = (-\infty, +\infty)$ possessing simple zeros t_k, $k = 0, \pm1, \pm2, \ldots$ [with respect to the function $p(t)$] and $x_m(t)$, $m = 0, \pm1, \pm2, \ldots$ [with respect to the function $h[x(t)]$]. All roots $x_m(t)$ of the function $h[x(t)]$ are isolated here.

The boundedness of solutions $x(t)$ related to equation (1) has been investigated in [1] on condition of the inequality $a^2 > 4b$ being valid. It is the purpose of this paper to show that on the same assumptions to both functions $h[x(t)]$ and $p(t)$ as introduced in [1], the assumption of positive real constants a and b may be extended to both remaining cases, where $a^2 = 4b$ or $a^2 < 4b$.

Suppose, there exist such constants $H > 0$ and $P > 0$ that the inequalities

$$\left| h[x(t)] \right| \leqq H \tag{2}$$

$$\left| p(t) \right| \leqq P \tag{3}$$

hold for all functions $x(t)$, $x \in (-\infty, +\infty)$ and for all $t \in I_1 = \langle 0, +\infty)$ [completely analogous we would proceed in case of $t \in (-\infty, 0\rangle$]

Now we will prove that from the boundedness of functions $h[x(t)]$ and $p(t)$ on the interval I_1 there follows the existence of such a constant $D_1 > 0$ that in both above mentioned cases the inequality

$$\limsup_{t \to \infty} |x'(t)| \leqq D_1$$

holds. Hereby $D_1 = \frac{P + H}{b}$.

I. Let $a^2 = 4b$ hold in equation (1). Substituting $x'(t) = y(t)$, we obtain from (1) the differential equation

$$y''(t) + ay'(t) + by(t) = p(t) - h[x(t)] \; , \tag{4}$$

where $x(t) = \int y(t)dt$.

Applying the well-known Lagrange's method of variation of constants $C_j \in R$ $(j = 1,2)$ in a general solution $\overline{y}(t) = C_1 y_1(t) + C_2 y_2(t)$ of the differential equation

$$\overline{y}''(t) + a\overline{y}'(t) + b\overline{y}(t) = 0, \tag{5}$$

where $y_1(t) = e^{-\frac{a}{2}t}$, $y_2(t) = te^{-\frac{a}{2}t}$, $w[y_1(t), y_2(t)] = e^{-at}$, yields

$$C_1(t) = -\int te^{\frac{a}{2}t}\left[p(t) - h[x(t)]\right] dt + C_1 \; ,$$

$$C_2(t) = \int e^{\frac{a}{2}t}\left[p(t) - h[x(t)]\right] dt + C_2 \; ,$$

so that the solution y(t) of (4) on the interval $I_1 = \langle 0, +\infty)$ is of the form $y(t) = \overline{y}(t) + y_p(t)$, where

$$\overline{y}(t) = (C_1 + C_2 t)\, e^{-\frac{a}{2}t},$$

$$y_p(t) = e^{-\frac{a}{2}t}\left\{t \int e^{\frac{a}{2}t}\left[p(t) - h[x(t)]\right] dt - \right.$$

$$\left. - \int t e^{\frac{a}{2}t}\left[p(t) - h[x(t)]\right] dt\right\} =$$

$$= \int_0^t e^{-\frac{a}{2}(t-\tau)}(t-\tau)\left[p(\tau) - h[x(\tau)]\right] d\tau .$$

Since

$$|y_p(t)| = \left|\int_0^t e^{-\frac{a}{2}(t-\tau)}(t-\tau)\left[p(\tau) - h[x(\tau)]\right] d\tau\right| \leqq$$

$$\leqq (P + H)\int_0^t \left| e^{-\frac{a}{2}(t-\tau)}(t-\tau)\right| d\tau =$$

$$= \frac{2(P+H)}{a}\left| t e^{-\frac{a}{2}t} + \frac{2}{a}\left(e^{-\frac{a}{2}t} - 1\right)\right| ,$$

then it holds for $t \to +\infty$ that

$$(C_1 + C_2 t)\, e^{-\frac{a}{2}t} \to 0 \quad \text{for all } C_j \in R (j = 1,2)$$

and

$$\frac{2(P+H)}{a}\left| t e^{-\frac{a}{2}t} + \frac{2}{a}\left(e^{-\frac{a}{2}t} - 1\right)\right| \to \frac{4(P+H)}{a^2} = \frac{P+H}{b} .$$

Consequently

$$\limsup_{t \to \infty} |x'(t)| \leqq \frac{P+H}{b} .$$

II. Let $a^2 < 4b$ hold in (1) and denote

$$\alpha = -\frac{a}{2}, \qquad \beta = \frac{\sqrt{4b - a^2}}{2}. \tag{6}$$

Proceeding analogous to I., where the basis of all solutions $\bar{y}(t)$ of (5) is now constituted by a pair of functions $y_1(t) = e^{\alpha t}\cos\beta t$, $y_2(t) = e^{\alpha t}\sin\beta t$ with the Wronskian $w[y_1(t), y_2(t)] = \beta e^{2\alpha t}$, gives

$$C_1(t) = -\frac{1}{\beta}\int e^{-\alpha t}\sin\beta t\left[p(t) - h[x(t)]\right] dt + C_1 ,$$

$$C_2(t) = \frac{1}{\beta}\int e^{-\alpha t}\cos\beta t\left[p(t) - h[x(t)]\right] dt + C_2 .$$

Consequently, the solution $y(t)$ of (4) on $I_1 = \langle 0, +\infty)$ is of the form $y(t) = \bar{y}(t) + y_p(t)$, where

$$\bar{y}(t) = e^{\alpha t}(C_1\cos\beta t + C_2\sin\beta t) = e^{\alpha t}\sqrt{C_1^2 + C_2^2}\cos(\beta t - \gamma),$$

putting here $\dfrac{C_1}{\sqrt{C_1^2 + C_2^2}} = \cos\gamma$, $\dfrac{C_2}{\sqrt{C_1^2 + C_2^2}} = \sin\gamma$ for arbitrary constants $C_j \in R$ $(j = 1,2)$, $C_1^2 + C_2^2 > 0$
and

$$y_p(t) = \frac{1}{\beta} e^{\alpha t}\left\{\sin\beta t\int e^{-\alpha t}\cos\beta t\left[p(t) - h[x(t)]\right] dt - \right.$$

$$\left. - \cos\beta t\int e^{-\alpha t}\sin\beta t\left[p(t) - h[x(t)]\right] dt\right\} =$$

$$= \frac{1}{\beta}\int_0^t e^{\alpha(t-\tau)}\sin\beta(t-\tau)\left[p(\tau) - h[x(\tau)]\right] d\tau .$$

Since

$$\left|y_p(t)\right| = \frac{1}{\beta}\left|\int_0^t e^{\alpha(t-\tau)}\sin\beta(t-\tau)\left[p(\tau) - h[x(\tau)]\right]d\tau\right| \leqq$$

$$\leqq \frac{P+H}{\beta}\int_0^t\left|e^{\alpha(t-\tau)}\sin\beta(t-\tau)\right|d\tau =$$

$$= \frac{P+H}{\beta(\alpha^2+\beta^2)}\left|\left[e^{\alpha(t-\tau)}\left[\alpha\sin\beta(t-\tau) - \beta\cos\beta(t-\tau)\right]\right]_0^t\right| =$$

$$= \frac{P+H}{\beta(\alpha^2+\beta^2)}\left|e^{\alpha t}(\alpha\sin\beta t - \beta\cos\beta t) + \beta\right| =$$

$$= \frac{P+H}{\beta(\alpha^2+\beta^2)}\left|\sqrt{\alpha^2+\beta^2}\, e^{\alpha t}\sin(\beta t - \delta) + \beta\right| ,$$

where we set $\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}} = \cos\delta$, $\dfrac{\beta}{\sqrt{\alpha^2+\beta^2}} = \sin\delta$,

then for $t \to +\infty$ it holds

$$e^{\alpha t}\sqrt{C_1^2 + C_2^2}\cos(\beta t - \gamma) \to 0 \text{ for all } C_j \in R \ (j = 1,2)$$

and

$$\frac{P+H}{\beta(\alpha^2+\beta^2)}\left|\sqrt{\alpha^2+\beta^2}\, e^{\alpha t}\sin(\beta t - \delta) + \beta\right| \to \frac{P+H}{\sqrt{\alpha^2+\beta^2}} = \frac{P+H}{b}$$

because $\alpha < 0$ with respect to (6).
Thus again

$$\limsup_{t\to\infty} |x'(t)| \leqq \frac{P+H}{b} .$$

Both in I. and II. we have proved not only the boundedness of the first derivative $x'(t)$ of each solution $x(t)$ of the differential equation (1) but besides there was shown that

$\limsup_{t\to\infty} |x'(t)|$ may be bounded by the same constant $D_1 = \frac{P+H}{b}$.

This fact enables us to further proceeding in I. and II. as did the author in [1]. Therefore we only briefly summarize the results achieved.

First, it may be shown that with respect to assumptions (2) and (3) about the functions $h[x(t)]$ and $p(t)$ there is also bounded the second derivative $x''(t)$ of the solution $x(t)$ of the differential equation (1). Substituting $z(t) = y'(t)$ $[= x''(t)]$ in (4) yields the differential equation

$$z'(t) + az(t) = p(t) - h[x(t)] - by(t) , \tag{7}$$

where $y(t) = x'(t)$, $x(t) = \int y(t)\, dt$. Since a general solution of the differential equation

$$\bar{z}'(t) + a\bar{z}(t) = 0$$

is $\bar{z}(t) = Ce^{-at}$, where $C \in R$ is an arbitrary constant, the general solution $z(t)$ of (7) on $I_1 = \langle 0, +\infty)$ may be written as $z(t) = \bar{z}(t) + z_p(t)$, where

$$z_p(t) = e^{-at}\int e^{at}[p(t) - h[x(t)] - by(t)]\, dt =$$

$$= \int_{T_x}^{t} e^{-a(t-\tau)}[p(\tau) - h[x(\tau)] - bx'(\tau)]\, d\tau ,$$

whereby $T_x \in I_1$, $T_x \leqq t$, is a suitable number [generally dependent on the function $x(t)$].

In applying the result for $|x'(t)|$ on the interval $\langle T_x, +\infty)$ in I. and II. we find that

$$|z_p(t)| = \left| \int_{T_x}^{t} e^{-a(t-\tau)} \left[p(\tau) - h[x(\tau)] - bx'(\tau)\right] d\tau \right| \leqq$$

$$\leqq 2(P + H + |M_{T_x}|) \int_{T_x}^{t} e^{-a(t-\tau)} d\tau \leqq$$

$$\leqq \frac{2}{a} (P + H + |M_{T_x}|) \left| 1 - e^{-a(t-T_x)} \right| ,$$

where $M_{T_x} \longrightarrow 0$ for $t \rightarrow +\infty$.

Hence in I. and II. also

$$\limsup_{t \to \infty} |x''(t)| \leqq \frac{2(P + H)}{a} .$$

Resulting statements.

A. If in addition to the assumption saying

1) there exist such constants $H > 0$ and $P > 0$ that for all $x = x(t) \in (-\infty, +\infty)$ and all $t \in I_1 = \langle 0, +\infty)$ the inequalities

$$|h[x(t)]| \leqq H , \quad |p(t)| \leqq P$$

moreover the assumption saying

2) there exists such a constant $H_1 > 0$ that for all $x(t)$, $x \in (-\infty, +\infty)$

$$|h'(x)| \leqq H_1 \quad \text{whereby} \quad \left| \int_0^\infty p(t)\, dt \right| < +\infty$$

holds, then for every bounded solution $x(t)$ of the differential equation (1) either

$\lim_{t\to\infty} x(t) = \bar{x}$, $\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0$,

where $h(\bar{x}) = 0$ or there exists such a root $\bar{x}$ of the function $h[x(t)]$, that $x(t) - \bar{x}$ becomes oscillated.

B. If there holds besides the above cited assumptions 1) and 2) sub A. that

3) there exists such a constant $P_1 > 0$ that for all $t \in I_1 = \langle 0, +\infty)$ the inequality

$$|p'(t)| \leqq P_1 \quad \text{whereby} \quad \limsup_{t\to\infty} |p(t)| > 0$$

is valid, then there exists for every bounded solution $x(t)$ of the differential equation (1) such a root $\bar{x}$ of the function $h[x(t)]$, that $x(t) - \bar{x}$ becomes oscillated.

C. If there exist such constants $H > 0$, $P > 0$, $H_1 > 0$, $P_1 > 0$, $P_0 > 0$ and $R > 0$ that for all $|x(t)| > R$ on the interval $I_1 = \langle 0, +\infty)$ the inequalities

1) $$|h[x(t)]| \leqq H, \quad |h'[x(t)]| \leqq H_1$$

2) $$|p(t)| \leqq P, \quad |p'(t)| \leqq P_1,$$

$$\left|\int_0^t p(\tau)\, d\tau\right| \leqq P_0, \quad \limsup_{t\to\infty} |p(t)| > 0$$

3) $$\min\left[\rho(\bar{x}_m, \bar{x}_{m+1}), \rho(\bar{x}_{m-1}, \bar{x}_m)\right] > \frac{2(P+H)}{b}\left(\frac{2}{a} + \frac{a}{b}\right) + \frac{P_0}{b}$$

are valid, where $\bar{x}_{m-1}$, $\bar{x}_m$, $\bar{x}_{m+1}$ ($m = 0, \pm 2, \pm 4, \dots$) are the three consecutive roots of the function $h(x)$, $h'(\bar{x}_m) > 0$, whereby $\rho(\bar{x}_m, \bar{x}_{m+1})$ or $\rho(\bar{x}_{m-1}, \bar{x}_m)$ means the distance among the roots $\bar{x}_m$, $\bar{x}_{m+1}$ or $\bar{x}_{m-1}$, $\bar{x}_m$,

then all solutions $x(t)$ of the differential equation (1) are bounded and to each of them there exists such a root $\bar{x}$ of the function $h[x(t)]$ that $x(t) - \bar{x}$ becomes oscillated.

Finally, we would like to remark that the author's considerations in [1] may be carried out even in higher order differential equations of analogous type (see [2]).

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POZNÁMKA K JISTÉ NELINEÁRNÍ DIFERENCIÁLNÍ ROVNICI TŘETÍHO ŘÁDU

Souhrn

Autor se ve svém příspěvku zabývá případem, kdy v nelineární diferenciální rovnici 3.řádu tvaru

$$x'''(t) + ax''(t) + bx'(t) + h[x(t)] = p(t)$$

s oscilatorickými funkcemi $h[x(t)]$ a $p(t)$, majícími spojitou 1.derivaci na int. $(-\infty, +\infty)$ a s reálnými konstantami $a > 0$, $b > 0$, platí - krom případu $a^2 > 4b$ vyšetřovaného J.Andresem - že $a^2 = 4b$ resp. $a^2 < 4b$.

Za obou těchto předpokladů se ukazuje, že 1. i 2. derivace všech řešení $x(t)$ studované diferenciální rovnice jsou ohraničené a to týmiž konstantami jako v případě již uvažovaném. Práce tak umožňuje řešit otázku ohraničenosti a oscilatoričnosti řešení dané dif. rovnice ve všech případech vztahů mezi kladnými konstantami a a b.

ПРИМЕЧАНИЕ К НЕЛИНЕЙНОМУ ДИФФЕРЕНЦИАЛЬНОМУ УРАВНЕНИЮ 3-ГО ПОРЯДКА ОПРЕДЕЛЕННОГО ТИПА

Резюме

Автор занимается случаем, когда в нелинейном дифференциальном уравнении 3-го порядка типа

$$x'''(t) + ax''(t) + bx'(t) + h[x(t)] = p(t)$$

с колеблющимися функциями $h[x(t)]$ и $p(t)$, у которых 1. производная непрерывна на интервале $(-\infty, +\infty)$ и с вещественными постоянными $a > 0$, $b > 0$ - возле случая $a^2 > 4b$ изучаемого уже Я. Андресом - присоединяются остаточные отношения $a^2 = 4b$ или $a^2 < 4b$.

В силу этих последних предположений показывается, что 1. и 2. производные всех решений $x(t)$ дифференциального уравнения ограничены теми же самыми постоянными как в случае уже совершенном. Таким образом работа предоставляет возможность решить вопрос об ограниченности и колебании решений этого дифференциального уравнения во всех возможных случаях отношений между положительными постоянными a и b.

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