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Ján Andres; Vladimír Vlček

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ON THE EXISTENCE OF SQUARE INTEGRABLE SOLUTIONS AND THEIR DERIVATIVES TO FOURTH AND FIFTH ORDER DIFFERENTIAL EQUATIONS

JAN ANDRES, VLADIMÍR VLČEK

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I. Consider the fourth-order nonlinear differential equation

$$x^{IV}(t) + ax'''(t) + bx''(t) + cx'(t) + h[x(t)] = p(t), \qquad (1)$$

where $a,b,c \in R^+$ are the constants with $ab > c, h(x) \in C^1(-\infty,+\infty)$, $h'(x) < 0$, $h(0) = 0$. Let positive constants H and P exist for the functions $h(x)$ and $p(t) \in C(-\infty,+\infty)$ such that the inequalities

$$|h(x)| \leqq H, \quad |p(t)| \leqq P \qquad (2)$$

hold for all $t,x \in I = (-\infty, +\infty)$ with

$$\liminf_{|x|\to\infty} |h(x)| > P.$$

Using the method introduced in [1] and Yoshizawa's converse theorem [2, p.107], it follows immediately (see the final remark in [1] and cf. [3]) from the results of [4] and [5] concerning our equation (1) that the existence of a solution x(t) to (1) is guaranteed such that

$$\limsup_{t\to\infty} |x(t)| < \infty \quad \text{and} \quad \limsup_{t\to\infty} |x^{(j)}(t)| \leqq D_j \quad (j=1,2,3), \qquad (3)$$

where $D_1 = \frac{H + P}{c}$, $D_2 = \frac{2(H + P)}{b}$, $D_3 = \frac{3(H + P)}{a}$.

Moreover, it can be proved quite analogically to [6] that this (bounded) solution x(t) either oscillates or

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x^{(j)}(t) = 0, \quad j = 1,2,3. \qquad (4)$$

We will prove (in the analogy to [7]) that under

$$\int_0^t |p(\tau)|d\tau \leqq P_0 \quad \text{for all } t \geqq 0 \qquad (5)$$

such a bounded solution x(t) of (1) jointly with its derivatives $x^{(j)}(t)$, j=1,2,3, satisfy

$$x^{(j)}(t) \in L_2 \langle 0,+\infty), \quad j=0,1,2,3. \qquad (6)$$

This is the aim of the first section.

Hence, multiplying (1) by x(t) and $x^{(j)}(t)$, j=1,2,3, successively and integrating the obtained identities by parts from a suitable $T_x \geqq 0$ to $t \geqq T_x$ (cf.(3)), we receive

$$\Big[x'''(\tau)x(\tau) - x''(\tau)x'(\tau) + ax''(\tau)x(\tau) + bx'(\tau)x(\tau) - \frac{a}{2}x'^2(\tau) + \frac{c}{2}x^2(\tau)\Big]\Big|_{T_x}^{t} + \int_{T_x}^{t} x''^2(\tau)d\tau - b\int_{T_x}^{t} x'^2(\tau)d\tau =$$

$$= \int_{T_x}^{t} \big[p(\tau) - h[x(\tau)]\big]\, x(\tau)d\tau ,$$

$$\left[x'''(\tau)x'(\tau) - \tfrac{1}{2}x''^2(\tau) + ax''(\tau)x'(\tau) + \tfrac{b}{2}x'^2(\tau)\right]\Big|_{T_x}^{t} -$$

$$- a\int_{T_x}^{t} x''^2(\tau)d\tau + c\int_{T_x}^{t} x'^2(\tau)d\tau = \int_{T_x}^{t}\left[p(\tau) - h[x(\tau)]\right]x'(\tau)d\tau ,$$

$$\left[x'''(\tau)x''(\tau) + \tfrac{a}{2}x''^2(\tau) + \tfrac{c}{2}x'^2(\tau)\right]\Big|_{T_x}^{t} - \int_{T_x}^{t} x'''^2(\tau)d\tau +$$

$$+ b\int_{T_x}^{t} x''^2(\tau)d\tau = \int_{T_x}^{t}\left[p(\tau) - h[x(\tau)]\right]x''(\tau)d\tau ,$$

$$\left[\tfrac{1}{2}x'''^2(\tau) + \tfrac{b}{2}x''^2(\tau) + cx''(\tau)x'(\tau)\right]\Big|_{T_x}^{t} + a\int_{T_x}^{t} x'''^2(\tau)d\tau -$$

$$- c\int_{T_x}^{t} x''^2(\tau)d\tau = \int_{T_x}^{t}\left[p(\tau) - h[x(\tau)]\right]x'''(\tau)d\tau ,$$

respectively. Denoting

$$D_o'(T_x,t) = \left[x'''(\tau)x(\tau) - x''(\tau)x'(\tau) + ax''(\tau)x(\tau) - \tfrac{a}{2}x'^2(\tau) + bx'(\tau)x(\tau) + \tfrac{c}{2}x^2(\tau)\right]\Big|_{T_x}^{t} ,$$

$$D_1'(T_x,t) = \left[x'''(\tau)x'(\tau) - \tfrac{1}{2}x''^2(\tau) + ax''(\tau)x'(\tau) + \tfrac{b}{2}x'^2(\tau)\right]\Big|_{T_x}^{t} ,$$

$$D_2'(T_x,t) = \left[x'''(\tau)x''(\tau) + \tfrac{a}{2}x''^2(\tau) + \tfrac{c}{2}x'^2(\tau)\right]\Big|_{T_x}^{t} ,$$

$$D_3'(T_x,t) = \left[\tfrac{1}{2}x'''^2(\tau) + \tfrac{b}{2}x''^2(\tau) + cx''(\tau)x'(\tau)\right]\Big|_{T_x}^{t} ,$$

we arrive at the following system

$$- b\int_{T_x}^{t} x'^2(\tau)d\tau + \int_{T_x}^{t} x''^2(\tau)d\tau = \int_{T_x}^{t} \left[p(\tau) - h[x(\tau)]\right] x(\tau)d\tau - D_o'(T_x, t)$$

$$c\int_{T_x}^{t} x'^2(\tau)d\tau - a\int_{T_x}^{t} x''^2(\tau)d\tau = \int_{T_x}^{t} \left[p(\tau) - h[x(\tau)]\right] x'(\tau)d\tau - D_1'(T_x, t)$$

$$b\int_{T_x}^{t} x''^2(\tau)d\tau - \int_{T_x}^{t} x'''^2(\tau)d\tau = \int_{T_x}^{t} \left[p(\tau) - h[x(\tau)]\right] x''(\tau)d\tau - D_2'(T_x, t)$$

$$c\int_{T_x}^{t} x''^2(\tau)d\tau + a\int_{T_x}^{t} x'''^2(\tau)d\tau = \int_{T_x}^{t} \left[p(\tau) - h[x(\tau)]\right] x'''(\tau)d\tau - D_3'(T_x, t)$$

for the integrals $\int_{T_x}^{t} x^{(j)2}(\tau)d\tau$, j=1,2,3, and consequently

$$\int_{T_x}^{t} x'^2(\tau)d\tau = - \frac{1}{ab-c}\left\{\int_{T_x}^{t} p(\tau)\left[ax(\tau) + x'(\tau)\right]d\tau - a\int_{T_x}^{t} h\left[x(\tau)\right]x(\tau)d\tau - \int_{T_x}^{t} h\left[x(\tau)\right]dx(\tau)\right\} + K_o'(T_x, t) ,$$

$$\int_{T_x}^{t} x''^2(\tau)d\tau = - \frac{1}{ab-c}\left\{\int_{T_x}^{t} p(\tau)\left[cx(\tau) + bx'(\tau)\right]d\tau -\right.$$

$$- c \int_{T_x}^{t} h[x(\tau)]\, x(\tau)\, d\tau - b \int_{T_x}^{t} h[x(\tau)]\, dx(\tau) \Big\} + K_1'(T_x, t)$$

or

$$\int_{T_x}^{t} x''^2(\tau)\, d\tau = \frac{1}{ab-c} \Big\{ \int_{T_x}^{t} p(\tau)\big[a x''(\tau) + x'''(\tau)\big]\, d\tau -$$

$$- a \int_{T_x}^{t} h[x(\tau)]\, x''(\tau)\, d\tau - \int_{T_x}^{t} h[x(\tau)]\, x'''(\tau)\, d\tau \Big\} +$$

$$+ K_2'(T_x, t),$$

$$\int_{T_x}^{t} x'''^2(\tau)\, d\tau = \frac{1}{ab-c} \Big\{ \int_{T_x}^{t} p(\tau)\big[c x''(\tau) + b x'''(\tau)\big]\, d\tau -$$

$$- c \int_{T_x}^{t} h[x(\tau)]\, x''(\tau)\, d\tau - b \int_{T_x}^{t} h[x(\tau)]\, x'''(\tau)\, d\tau \Big\} +$$

$$+ K_3'(T_x, t) ,$$

where

$$K_0'(T_x, t) = \frac{1}{ab-c} \big[a D_0'(T_x, t) + D_1'(T_x, t)\big]$$

$$K_1'(T_x, t) = \frac{1}{ab-c} \big[c D_0'(T_x, t) + b D_1'(T_x, t)\big]$$

$$K_2'(T_x, t) = - \frac{1}{ab-c} \big[a D_2'(T_x, t) + D_3'(T_x, t)\big]$$

$$K_3'(T_x, t) = - \frac{1}{ab-c} \big[c D_2'(T_x, t) + b D_3'(T_x, t)\big] .$$

Using the Schwarz inequality, we come to the estimate

$$\Big| \int_{T_x}^{t} h[x(\tau)]\, x^{(j)}(\tau)\, d\tau \Big| = \Big| h[x(\tau)]\, x^{(j-1)}(\tau) \Big|_{T_x}^{t} -$$

$$- \int_{T_x}^{t} h'[x(\tau)]x'(\tau)x^{(j-1)}(\tau)d\tau \Big| \leqq \Big|\int_{T_x}^{t} h'[x(\tau)]x'(\tau)x^{(j-1)}(\tau)d\tau\Big| +$$

$$+ \Big|h[x(\tau)]x^{(j-1)}(\tau)\Big|_{T_x}^{t}\Big| \leqq H'\sqrt{\int_{T_x}^{t} x'^2(\tau)d\tau \int_{T_x}^{t} x^{(j-1)2}(\tau)d\tau} +$$

$$+ \Big|h[x(\tau)]x^{(j-1)}(\tau)\Big|_{T}^{t}\Big|, \quad j=2,3 \tag{7}$$

with a suitable positive constant H' and [cf.(2)]

$$\Big|\int_{T_x}^{t} h[x(\tau)]x'(\tau)d\tau\Big| = \Big|\int_{x(T_x)}^{x(t)} h(s)ds\Big| \leqq H|x(t) - x(T_x)| \tag{7.1}$$

for $j=1$,

$$- \int_{T_x}^{t} h[x(\tau)]x''(\tau)d\tau = \int_{T_x}^{t} h'[x(\tau)]x'^2(\tau)d\tau - h[x(\tau)]x'(\tau)\Big|_{T_x}^{t} =$$

$$= - \int_{T_x}^{t} |h'[(\tau)]|x'^2(\tau)d\tau - h[x(\tau)]x'(\tau)\Big|_{T_x}^{t} \leqq$$

$$\leqq \Big|h[x(\tau)]x'(\tau)\Big|_{T_x}^{t}\Big| \tag{7.2}$$

Furthermore, we have [cf.(5)]

$$\Big|\int_{T_x}^{t} p(\tau)x^{(j)}(\tau)d\tau\Big| \leqq \int_{T_x}^{t} |p(\tau)x^{(j)}(\tau)|d\tau \leqq D_jP_o \tag{8}$$

$(j=1,2,3)$

and

$$\Big|\int_{T_x}^{t} p(\tau)x(\tau)d\tau\Big| \leqq \int_{T_x}^{t} |p(\tau)x(\tau)|d\tau \leqq P_oM \tag{8.1}$$

for $t\in\langle T_x, t\rangle$, where $M = \max_{t\in\langle T_x,t\rangle}|x(t)|$.

It is evident that

$$D_j'(T_x, t) \longrightarrow D_j'(T_x) \in I \quad \text{for } t \to \infty \quad (j=0,1,2,3) \tag{9}$$

with respect to (4), and consequently $K_j'(T_x) \in I$ for $t \to \infty$ as well.

Now, using two first identities from above and (7.1), (8), (8.1), (9) $[h'(x) < 0 \Rightarrow h[x(t)]\,x(t) \leq 0$ for all $t \in I_1 = \langle 0, +\infty)]$, we get immediately the estimates

$$\int_{T_x}^{t} x'^2(\tau)d\tau \leq \frac{1}{ab-c}\left\{H\overline{M} + P_o(aM + D_1)\right\} + |K_o'| := R_1$$

and

$$\int_{T_x}^{t} x''^2 x(\tau)d\tau \leq \frac{1}{ab-c}\left\{cH\overline{M} + P_o(cM + bD_1)\right\} + |K_1'| := R_2 ,$$

where $\overline{M} = |x(t) - x(T_x)|$, so that $x'(t), x''(t) \in L_2 \langle 0, +\infty)$.

Hence, in view of this and (7), (7.2), we obtain from the third identity the estimate

$$\int_{T_x}^{t} x'''^2(\tau)d\tau \leq \frac{1}{ab-c}\left\{P_o(cD_2 + bD_3) + H'b\, R_1R_2 + \tilde{H}_1 + \tilde{H}_2 + \right.$$
$$\left. + |K_3'| := R_3 \right. ,$$

where $\tilde{H}_j = \left| h[x(\tau)x^{(j)}(\tau)\Big|_{T_x}^{t} \right|$, $j=1,2$, so that $x'''(t) \in L_2 \langle 0, +\infty)$ as well.

It remains to show that $\int_{T_x}^{t} x^2(\tau)d\tau \in L_2 \langle 0, +\infty)$.

As mentioned before, it can be proved quite analogically to [6] that under our assumptions either $\lim\limits_{t\to\infty} x(t) = 0$ or $\liminf\limits_{t\to\infty}|x(t)| = 0 < \limsup\limits_{t\to\infty}|x(t)| < \infty$

can appear.

First of all we will prove that $\lim\limits_{t\to\infty} x(t) = 0$, only. (10)

Assume on the contrary that

$$\limsup_{t\to\infty}|x(t)| > 0 . \tag{11}$$

Multiplying (1) by $x(t)$ and integrating by parts again from $T_x \geqq 0$ to $t \geqq T_x$, we arrive at the identity leading to

$$\limsup_{t\to\infty} \int_{T_x}^{t} |h[x(\tau)]x(\tau)|\,d\tau \leqq \limsup_{t\to\infty} \Big\{ \int_{T_x}^{t} p(\tau)x(\tau)d\tau +$$

$$+ b\int_{T_x}^{t} x'^2(\tau)d\tau - \int_{T_x}^{t} x''^2(\tau)d\tau - D_0'(T_x,t)\Big\} \leqq P_0 \limsup_{t\to\infty}|x(t)| +$$

$$+ bD_1 + D_2 + D_0' < \infty . \tag{12}$$

Therefore it must be not only $\liminf\limits_{t\to\infty}|x(t)| = 0$, but with respect to (11) (for more details see [6]) $\limsup\limits_{t\to\infty}|x'(t)| = \infty$ as well, a contradiction to (3). Thus, (10) is true, only.

Since

$$\lim_{t\to\infty} \frac{h[x(t)]}{x(t)} = \lim_{t\to\infty} \frac{h'(x)x'(t)}{x'(t)} = h'(0) \leqq -\varepsilon < 0,$$

with a suitable $\varepsilon > 0$ [just according to (10)], and consequently

$$\left|\frac{h[x(t)]}{x(t)}\right| \geqq \frac{\varepsilon}{2} \quad \text{for } t \geqq T_x, \ (T_x \text{ - a great enough number}),$$

we get finally

$$\frac{\varepsilon}{2}\int_{T_x}^{\infty} x^2(t)dt \leqq \int_{T_x}^{\infty} \left|\frac{h[x(t)]}{x(t)}\right| x^2(t)dt = \int_{T_x}^{\infty} \left|h[x(t)]x(t)\right| dt < \infty$$

with respect to (12).

Now we can summarize all the above investigations into the following

<u>Theorem 1</u>. Under the assumptions

(i) $a > 0,\ ab > c > 0$,

(ii) $0 > h'(x)$ for all $x \in I$ with $h(0) = 0$,

(iii) $P < \liminf_{|x|\to\infty} |h(x)| \leqq \limsup_{|x|\to\infty} |h(x)| < \infty$,

(iv) $\limsup_{t\to\infty} \int_0^t |p(\tau)d\tau| < \infty$ $(\limsup_{t\to\infty} |p(t)| < \infty)$

there exist a solution x(t) of (1) satisfying (6).

II. Now, consider the fifth-order differential equation

$$x^{V}(t) + ax^{IV}(t) + bx'''(t) + cx''(t) + dx'(t) + h[x(t)] = \\ = p(t), \qquad (13)$$

where $a,b,c,d \in R^+$ are contants with $ab > c$, $(ab-c)c > a^2d$ and the same assumptions concerning the functions $h[x(t)]$, $p(t)$ are valid as for (1), i.e. (2), (3), where moreover $j = 1,\ldots$ $\ldots,4$, so that

$$D_1 = \frac{H + P}{d},\ D_2 = \frac{2(H + P)}{c},\ D_3 = \frac{3(H + P)}{b},\ D_4 = \frac{4(H + P)}{a},$$

and (4), (5), (6) for $j = 1,\ldots,4$.

Let us note that the existence result as the one for (1) is true also here by the same reasons and that is why we proceed directly to the verification of the analogy to (6) with $j = 0,1,\ldots,4$. Hence, multiplying (13) by x(t) and $x^{(j)}(t)$,

$j = 1,\dots,4$, successively and integrating the obtained identities by parts from a suitable $T_x \geq 0$ to $t \geq T_x$ (a sufficiently large number again), we receive

$$\Big[x^{IV}(\tau)x'(\tau) - x'''(\tau)x'(\tau) + \frac{1}{2}\,x''^2(\tau) + a(x'''(\tau)x(\tau) - x''(\tau)x'(\tau)) + b(x''(\tau)x(\tau) - \frac{1}{2}\,x'^2(\tau)) + cx'(\tau)x(\tau) - \frac{d}{2}\,x^2(\tau)\Big]\Big|_{T_x}^{t} + a\int_{T_x}^{t} x''^2(\tau)d\tau - c\int_{T_x}^{t} x'^2(\tau)d\tau = \int_{T_x}^{t}\big[p(\tau) - h[x(\tau)]\big]\,x(\tau)d\tau \quad ,$$

$$\Big[x^{IV}(\tau)x'(\tau) - x'''(\tau)x''(\tau) + a(x'''(\tau)x'(\tau) - \frac{1}{2}\,x''^2(\tau)) + bx''(\tau)x'(\tau) + \frac{c}{2}\,x'^2(\tau)\Big]\Big|_{T_x}^{t} + \int_{T_x}^{t} x'''^2(\tau)d\tau - b\int_{T_x}^{t} x''^2(\tau)d\tau + d\int_{T_x}^{t} x'^2(\tau)d\tau = \int_{T_x}^{t}\big[p(\tau) - h[x(\tau)]\big]\,x'(\tau)d\tau \quad ,$$

$$\Big[x^{IV}(\tau)x''(\tau) - \frac{1}{2}\,x'''^2(\tau) + ax'''(\tau)x''(\tau) + \frac{b}{2}\,x''^2(\tau) + \frac{d}{2}\,x'^2(\tau)\Big]\Big|_{T_x}^{t} - a\int_{T_x}^{t} x'''^2(\tau)d\tau + c\int_{T_x}^{t} x''^2(\tau)d\tau = \int_{T_x}^{t}\big[p(\tau) - h[x(\tau)]\big]\,x''(\tau)d\tau \quad ,$$

$$\left[x^{IV}(\tau)x'''(\tau) + \frac{a}{2}x'''^2(\tau) + \frac{c}{2}x''^2(\tau) + dx''(\tau)x'(\tau)\right]\Bigg|_{T_x}^{t} -$$

$$-\int_{T_x}^{t} x^{IV\,2}(\tau)d\tau + b\int_{T_x}^{t} x'''^2(\tau)d\tau -$$

$$- d\int_{T_x}^{t} x''^2(\tau)d\tau = \int_{T_x}^{t}\left[p(\tau) - h[x(\tau)]\right]x'''^2(\tau)d\tau ,$$

$$\Big[\frac{1}{2}x^{IV\,2}(\tau) + \frac{b}{2}x'''^2(\tau) + cx'''(\tau)x''(\tau) + d(x'''(\tau)x'(\tau) -$$

$$- \frac{1}{2}x''^2(\tau))\Big]\Bigg|_{T_x}^{t} + a\int_{T_x}^{t} x^{IV\,2}(\tau)d\tau - c\int_{T_x}^{t} x'''^2(\tau)d\tau =$$

$$= \int_{T_x}^{t}\left[p(\tau) - h[x(\tau)]\right]x^{IV\,2}(\tau)d\tau ,$$

respectively. Denoting

$$D_o'(T_x,t) = \Big[x^{IV}(\tau)x'(\tau) - x'''(\tau)x'(\tau) + \frac{1}{2}x''^2(\tau) +$$

$$+ a(x'''(\tau)x(\tau) - x''(\tau)x'(\tau)) + b(x''(\tau)x(\tau) -$$

$$- \frac{1}{2}x'^2(\tau)) + cx'(\tau)x(\tau) - \frac{d}{2}x^2(\tau)\Big]\Bigg|_{T_x}^{t}$$

$$D_1'(T_x,t) = \Big[x^{IV}(\tau)x'(\tau) - x'''(\tau)x''(\tau) + a(x'''(\tau)x'(\tau) -$$

$$- \frac{1}{2}x''^2(\tau)) + bx''(\tau)x'(\tau) + \frac{c}{2}x'^2(\tau)\Big]\Bigg|_{T_x}^{t}$$

$$D_2'(T_x,t) = \Big[x^{IV}(\tau)x''(\tau) - \frac{1}{2}x'''^2(\tau) + ax'''(\tau)x''(\tau) +$$

$$+ \frac{b}{2}x''^2(\tau) + \frac{d}{2}x'^2(\tau)\Big]\Bigg|_{T_x}^{t}$$

$$D_3'(T_x,t) = \Big[x^{IV}(\tau)x'''(\tau) + \frac{a}{2}\, x'''^2(\tau) + \frac{c}{2}\, x''^2(\tau) + $$
$$+ \, dx''(\tau)x'(\tau)\Big]\Bigg|_{T_x}^{t}$$

$$D_4'(T_x,t) = \Big[\frac{1}{2}\, x^{IV\,2}(\tau) + \frac{b}{2}\, x'''^2(\tau) + cx'''(\tau)x''(\tau) + $$
$$+ \, d\big(x'''(\tau)x'(\tau) - \frac{1}{2}\, x''^2(\tau)\big)\Big]\Bigg|_{T_x}^{t}$$

we arrive at the following system

$$- c\int_{T_x}^{t} x'^2(\tau)d\tau + a\int_{T_x}^{t} x''^2(\tau)d\tau = \int_{T_x}^{t} \big[p(\tau) - h[x(\tau)]\big]\, d\tau \; -$$
$$- \, D_0'(T_x,t)$$

$$d\int_{T_x}^{t} x'^2(\tau)d\tau - b\int_{T_x}^{t} x''^2(\tau)d\tau + \int_{T_x}^{t} x'''^2(\tau)d\tau =$$
$$= \int_{T_x}^{t} \big[p(\tau) - h[x(\tau)]\big]\, x'(\tau)d\tau - D_1'(T_x,t)$$

$$c\int_{T_x}^{t} x''^2(\tau)d\tau - a\int_{T_x}^{t} x'''^2(\tau)d\tau = \int_{T_x}^{t} \big[p(\tau) - h[x(\tau)]\big]\, x''(\tau)d\tau \; -$$
$$- \, D_2'(T_x,t)$$

$$- \, d\int_{T_x}^{t} x''^2(\tau)d\tau + b\int_{T_x}^{t} x'''^2(\tau)d\tau - \int_{T_x}^{t} x^{IV^2}(\tau)d\tau = \int_{T_x}^{t} \big[\, p(\tau) \; -$$
$$- \, h[x(\tau)]\big]x'''(\tau)d\tau - D_3'(T_x,t)$$

$$- c\int_{T_x}^{t} x'''^2(\tau)d\tau + a\int_{T_x}^{t} x^{IV^2}(\tau)d\tau = \int_{T_x}^{t} [p(\tau) -$$

$$- h[x(\tau)]] x^{IV}(\tau)d\tau - D_4'(T_x,t)$$

for the integrals $\int_{T_x}^{t} x^{(j)^2}(\tau)d\tau$, j=1,...,4, and consequently

$$\int_{T_x}^{t} x'^2(\tau)d\tau = \frac{1}{(ab-c)c-a^2d}\Big\{ -\int_{T_x}^{t} p(\tau)[(ab-c)x(\tau) + a^2x'(\tau) +$$

$$+ ax''(\tau)]d\tau + (ab-c)\int_{T_x}^{t} h[x(\tau)]x(\tau)d\tau +$$

$$+ a^2\int_{T_x}^{t} h[x(\tau)]x'(\tau)d\tau + a\int_{T_x}^{t} h[x(\tau)]x''(\tau)d\tau\Big\} +$$

$$+ K_o'(T_x,t) ,$$

$$\int_{T_x}^{t} x''^2(\tau)d\tau = -\frac{1}{(ab-c)c-a^2d}\Big\{\int_{T_x}^{t} p(\tau)[adx(\tau) + acx'(\tau) +$$

$$+ cx''(\tau)]d\tau - ad\int_{T_x}^{t} h[x(\tau)]x(\tau)d\tau -$$

$$- ac\int_{T_x}^{t} h[x(\tau)]x'(\tau)d\tau - c\int_{T_x}^{t} h[x(\tau)]x''(\tau)d\tau\Big\} +$$

$$+ K_1'(T_x,t) ,$$

$$\int_{T_x}^{t} x'''^2(\tau)d\tau = -\frac{1}{(ab-c)c-a^2d}\Big\{\int_{T_x}^{t} p(\tau)\{ acdx(\tau) + ac^2x'(\tau) +$$

$$+ \left[c^2+(ab-c)c-a^2d\right]x''(\tau)\Big\} d\tau -$$

$$- acd\int_{T_x}^{t} h[x(\tau)]x(\tau)d\tau - ac^2\int_{T_x}^{t} h[x(\tau)]x'(\tau)d\tau -$$

$$- \left[c^2+(ab-c)c-a^2d\right]\int_{T_x}^{t} h[x(\tau)]x''(\tau)d\tau\Big\} + K_2'(T_x,t)$$

or

$$\int_{T_x}^{t} x'''^2(\tau)d\tau = -\frac{1}{\left[(ab-c)c-a^2d\right](ab-c)}\Big\{\int_{T_x}^{t} p(\tau)\Big\{a^2d^2x(\tau) +$$

$$+ a^2cdx'(\tau) + acdx''(\tau) - a\left[(ab-c)c-a^2d\right]x'''(\tau) -$$

$$- \left[(ab-c)c-a^2d\right]x^{IV}(\tau)\Big\} d\tau - a^2d^2\int_{T_x}^{t} h[x(\tau)]x(\tau)d\tau -$$

$$- a^2cd\int_{T_x}^{t} h[x(\tau)]x'(\tau)d\tau - acd\int_{T_x}^{t} h[x(\tau)]x''(\tau)d\tau +$$

$$+ a\left[(ab-c)c-a^2d\right]\int_{T_x}^{t} h[x(\tau)]x'''(\tau)d\tau +$$

$$+ \left[(ab-c)c-a^2d\right]\int_{T_x}^{t} h[x(\tau)]x^{IV}(\tau)d\tau\Big\} + K_3'(T_x,t) ,$$

$$x^{IV^2}(\tau)d\tau = \frac{1}{a\left[(ab-c)c-a^2d\right]}\Big\{- \left[c^2+(ab-c)c-a^2d\right]\int_{T_x}^{t} p(\tau)\left[dx(\tau) +\right.$$

$$\left.+ cx'(\tau)\right]d\tau - \Big\{\frac{c}{a}\left[c^2+(ab-c)c-a^2d\right] + b\left[(ab-c)c -\right.$$

$$\left.- a^2d\right]\Big\}\int_{T_x}^{t} p(\tau)x''(\tau)d\tau - a\left[(ab-c)c -\right.$$

$$- a^2d\Big] \int_{T_x}^{t} p(\tau)x'''(\tau)d\tau + d\Big[c^2+(ab-c)c-a^2d\Big] \cdot$$

$$\cdot \int_{T_x}^{t} h\Big[x(\tau)\Big] x(\tau)d\tau + c\Big[c^2+(ab-c)c -$$

$$- a^2d\Big] \int_{T_x}^{t} h\Big[x(\tau)\Big] x'(\tau)d\tau + \Big\{ \frac{c}{a}\Big[c^2+(ab-c)c-a^2d\Big] +$$

$$+ b\Big[(ab-c)c-a^2d\Big]\Big\} \int_{T_x}^{t} h\Big[x(\tau)\Big] x''(\tau)d\tau +$$

$$+ a\Big[(ab-c)c-a^2d\Big] \int_{T_x}^{t} h\Big[x(\tau)\Big] x'''(\tau)d\tau \Bigg\} + K_4'(T_x, t)$$

or

$$\int_{T_x}^{t} x^{IV^2}(\tau)d\tau = - \frac{c}{a^2\Big[(ab-c)c-a^2d\Big]} \Bigg\{ ac \int_{T_x}^{t} p(\tau)\Big[dx(\tau) + cx'(\tau)\Big] d\tau +$$

$$+ \Big[c^2+(ab-c)c-a^2d\Big] \int_{T_x}^{t} p(\tau)x''(\tau)d\tau - a\Big[(ab-c)c -$$

$$- a^2d\Big] \int_{T_x}^{t} p(\tau)x^{IV}(\tau)d\tau - ac\Big[d \int_{T_x}^{t} h\Big[x(\tau)\Big] x(\tau)d\tau +$$

$$+ c \int_{T_x}^{t} h\Big[x(\tau)\Big] x'(\tau)d\tau\Big] - \Big[c^2+(ab-c)c -$$

$$- a^2d\Big] \int_{T_x}^{t} h\Big[x(\tau)\Big] x''(\tau)d\tau + a\Big[(ab-c)c -$$

$$- a^2d\Big] \int_{T_x}^{t} h\Big[x(\tau)\Big] x^{IV}(\tau)d\tau \Bigg\} + K_5'(T_x, t)$$

or

$$\int_{T_x}^{t} x^{IV^2}(\tau)d\tau = \frac{1}{ab-c}\left\{-\frac{cd}{(ab-c)c-a^2d}\left\{\int_{T_x}^{t} p(\tau)\left[adx(\tau) + \right.\right.\right.$$

$$\left. + acx'(\tau) + cx''(\tau)\right]d\tau - ad\int_{T_x}^{t} h\left[x(\tau)\right]x(\tau)d\tau -$$

$$- ac\int_{T_x}^{t} h\left[x(\tau)\right]x'(\tau)d\tau - c\int_{T_x}^{t} h\left[x(\tau)\right]x''(\tau)d\tau\right\} +$$

$$+ \int_{T_x}^{t} p(\tau)\left[cx'''(\tau) + bx^{IV}(\tau)\right]d\tau -$$

$$\left. - c\int_{T_x}^{t} h\left[x(\tau)\right]x'''(\tau)d\tau - b\int_{T_x}^{t} h\left[x(\tau)\right]x^{IV}(\tau)d\tau\right\} +$$

$$+ K_6'(T_x, t) ,$$

where

$$K_o'(T_x,t) = -\frac{1}{(ab-c)c-a^2d}\left[(ab-c)D_o'(T_x,t) + a^2D_1'(T_x,t) + \right.$$

$$\left. + aD_2'(T_x,t)\right]$$

$$K_1'(T_x,t) = \frac{1}{(ab-c)c-a^2d}\left[adD_o'(T_x,t) + acD_1'(T_x,t) + cD_2'(T_x,t)\right]$$

$$K_2'(T_x,t) = \frac{1}{a\left[(ab-c)c-a^2d\right]}\left\{acdD_o'(T_x,t) + ac^2D_1'(T_x,t) + \right.$$

$$\left. + \left[c^2+(ab-c)c-a^2d\right]D_2'(T_x,t)\right\}$$

$$K_3'(T_x,t) = \frac{1}{\left[(ab-c)c-a^2d\right](ab-c)}\left\{a^2d^2D_o'(T_x,t) + a^2cdD_1'(T_x,t) + \right.$$

$$+ acdD_2'(T_x,t) - a\left[(ab-c)c-a^2d\right]D_3'(T_x,t) -$$

$$- \left[(ab-c)c-a^2d\right] D_4'(T_x,t)\Big\}$$

$$K_4'(T_x,t) = \frac{1}{a\left[(ab-c)c-a^2d\right]} \Big\{ \left[c^2+(ab-c)c-a^2d\right] \left[dD_0'(T_x,t) + \right.$$

$$\left. + cD_1'(T_x,t)\right] + \Big\{ \frac{c}{a} \left[c^2+(ab-c)c-a^2d\right] +$$

$$+ b\left[(ab-c)c-a^2d\right]\Big\} D_2'(T_x,t) + a\left[(ab-c)c-a^2d\right] D_3'(T_x,t)\Big\}$$

$$K_5'(T_x,t) = \frac{c}{a^2\left[(ab-c)c-a^2d\right]} \Big\{ ac\left[dD_0'(T_x,t) + cD_1'(T_x,t) + \right.$$

$$\left. + \left[c^2+(ab-c)c-a^2d\right] \cdot D_2'(T_x,t) - a\left[(ab-c)c - \right.\right.$$

$$\left.\left. - a^2d\right] D_4'(T_x,t)\Big\}$$

$$K_6'(T_x,t) = \frac{1}{ab-c} \Big\{ \frac{cd}{(ab-c)c-a^2d} \Big\{ a\left[dD_0'(T_x,t) + cD_1'(T_x,t)\right] +$$

$$+ cD_2'(T_x,t)\Big\} - cD_3'(T_x,t) - bD_4'(T_x,t) \Big\} .$$

Now, using the first identity from above and (7.1), (8), (8.1), (9) $\left[h'(x) < 0 \Rightarrow h[x(t)]x(t) \leqq 0 \text{ for all } t \in I_1 = \langle 0,+\infty)\right]$, we get again

$$\int_{T_x}^{t} x'^2(\tau)d\tau \leqq \frac{1}{(ab-c)c-a^2d} \Big\{ P_o[(ab-c)M + a^2D_1' + aD_2'] + a^2H\overline{M} +$$

$$+ aH' \int_{T_x}^{t} x'^2(\tau)d\tau + a\tilde{H}_1\Big\} + |K_o'| , \quad \text{i.e.}$$

$$\left(1 - \frac{aH'}{(ab-c)c-a^2d}\right) \int_{T_x}^{t} x'^2(\tau)d\tau \leqq \frac{1}{(ab-c)c-a^2d} \Big\{ P_o[(ab-c)M +$$

$$+ a(aD_1'+D_2')] + a(aH\overline{M}+\tilde{H}_1)\Big\} + |K_o'| ,$$

from which we have for $0 < H' < \frac{(ab-c)c-a^2d}{a}$ the following estimate

$$\int_{T_x}^{t} x'^2(\tau)d\tau \leqq \frac{1}{(ab-c)c-a^2d-aH'} \Big\{ P_o[(ab-c)M + a(aD_1' + D_2')] + $$
$$+ a(aH\bar{M} + \tilde{H}_1) + [(ab-c)c-a^2d]\,|K_o'|\Big\} := R_1 \quad .$$

Using the second and the third identities, we get furthermore

$$\int_{T_x}^{t} x''^2(\tau)d\tau \leqq \frac{1}{(ab-c)c-a^2d} \Big\{ P_o[a(dM + cD_1') + cD_2'] + acH\bar{M} + $$
$$+ c(H'R_1 + \tilde{H}_1)\Big\} + |K_1'| := R_2$$

and

$$\int_{T_x}^{t} x'''^2(\tau)d\tau \leqq \frac{1}{a[(ab-c)c-a^2d]} \Big\{ P_o[ac(dM + cD_1') + [c^2 + $$
$$+ (ab-c)c-a^2d]\, D_2'] + ac^2H\bar{M} + [c^2 + (ab-c)c - $$
$$- a^2d][H'R_1 + \tilde{H}_1]\Big\} + |K_2'| := R_3 \; .$$

Taking the fourth inequality from above, we have by virtue of (7)

$$\int_{T_x}^{t} x^{IV^2}(\tau)d\tau \leqq \frac{1}{a[(ab-c)c-a^2d]} \Big\{[[c^2 + (ab-c)c-a^2d](dM + cD_1') + $$
$$+ \Big\{\frac{c}{a}[c^2 + (ab-c)c-a^2d] + b[(ab-c)c-a^2d]\Big\}\, D_2' + $$
$$+ a[(ab-c)c-a^2d]D_3']P_o + c[c^2 + (ab-c)c-a^2d]H\bar{M} + $$
$$+ \Big\{\frac{c}{a}\,[c^2 + (ab-c)c-a^2d] + b[(ab-c)c-a^2d]\Big\}[H'R_1 + $$
$$+ \tilde{H}_1] + a[(ab-c)c-a^2d][H'\sqrt{R_1R_2} + \tilde{H}_2]\Big\} + |K_4'| := R_4;$$

so that $x^{(j)}(t) \in L_2 \langle 0,+\infty)$ for $j=1,\ldots,4$.

Coming back to the first equation of the system above, we can readily check that

$$\limsup_{t\to\infty} \int_{T_x}^{t} |h[x(\tau)x(\tau)| d\tau < \infty ,$$

this time by means of $x''(t) \in L_2 \langle 0,+\infty)$, again.
Hence, one can prove just by the same manner as in Part I that

$$x(t) \in L_2 \langle 0,+\infty)$$

as well.

Summarizing the results of our investigation from Part II, we can given

Theorem 2. Under the assumptions

(i) $a > 0,\ ab > c > 0,\ (ab-c)c > a^2 d > 0$,

(ii) $0 > h'(x) > -\frac{1}{a}\,[(ab-c)c-a^2 d]$ for all $x \in I$ with $h(0)=0$,

(iii) $P < \liminf_{|x|\to\infty} |h(x)| \leqq \limsup_{|x|\to\infty} |h(x)| < \infty$,

(iv) $\limsup_{t\to\infty} \int_0^t |p(\tau)| d\tau < \infty$ $(\limsup_{t\to\infty} |p(t)| < \infty)$

there exists a solution $x(t)$ of (13) satisfying (6) with $j=0,1,\ldots,4$.

Final remark.

It could be seen (cf. also [7]) that the square integrability of solutions and their derivatives up to the $(n-1)$-th order including to the equations under our consideration can be easily verified just until $n = 5$. Moreover, for $n < 5$ no special growth restrictions concerning $h'(x)$ have been needed, while for $n = 5$ an upper bound for $|h'(x)|$ has been already required. The situation becomes still much more complicated for $n > 5$ with respect to an explicit expression of the growth bound of $|h'(x)|$. Nevertheless, the "Hurwitz-structure" of the appropriate systems considered in the analogy to the above investigations survives for the general n.

Summary

It is considered the nonlinear fourth-order differential equation

$$x^{IV}(t) + ax'''(t) + bx''(t) + cx'(t) + h[x(t)] = p(t) \qquad (1)$$

with constants $a,b,c \in R^+$ satisfying the Routh-Hurwitz condition $ab > c$; $h(x) \in C^1(-\infty,+\infty)$, $h'(x) < 0$, $h(0) = 0$, $|h(x)| \leq H$, $|p(t)| \leq P$ and $\liminf_{|x| \to \infty} |h(x)| > P$ for all $t \in I = (-\infty,+\infty)$. Using the assumptions of the boundedness of $h'(x)$ on the interval $I_1 = \langle 0,+\infty)$ and

$$\limsup_{t \to \infty} \int_0^t |p(\tau)| d\tau < \infty$$

it is proved that $x^{(j)}(t) \in L_2 \langle 0,+\infty)$ holds for $j = 0,1,2,3$. For the 5-th order differential equation analogous to (1) with the corresponding assumptions on the constants and on the functions $h(x)$, $p(t)$ it is shown that the validity of $x^{(j)}(t) \in L_2 \langle 0,+\infty)$, $j = 0,1,\dots,4$, is attainable with a certain growth restriction of $|h'(x)|$ on the interval $I_1 = \langle 0,+\infty)$, only.

Souhrn

O EXISTENCI KVADRATICKY INTEGROVATELNÝCH ŘEŠENÍ A JEJICH DERIVACÍ DIFERENCIÁLNÍCH ROVNIC ČTVRTÉHO A PÁTÉHO ŘÁDU

Je uvažována nelineární diferenciální rovnice 4.řádu

$$x^{IV}(t) + ax'''(t) + bx''(t) + cx'(t) + h[x(t)] = p(t), \qquad (1)$$

kde konstanty $a,b,c \in R^+$ splňují Routh-Hurwitzovu podmínku $ab > c$; $h(x) \in C^1(-\infty,+\infty)$, $h'(x) < 0$, $h(0)=0$, $|h(x)| \leq H$,

$|p(t)| \leq P$ a $\liminf_{|x| \to \infty} |h(x)| > P$ pro všechna $t \in I = (-\infty, +\infty)$.

Užitím předpokladů o ohraničenosti $h'(x)$ na intervalu $I_1 = \langle 0, +\infty)$ a

$$\limsup_{t \to \infty} \int_0^t |p(\tau)| d\tau < \infty$$

se dokazuje, že pro j=0,1,2,3 platí $x^{(j)}(t) \in L_2 \langle 0, +\infty)$. U diferenciální rovnice 5.řádu analogické rovnici (1) je za obdobných předpokladů o konstantách a o funkcích $h(x)$, $p(t)$ dokázáno, že platnost $x^{(j)}(t) \in L_2 \langle 0, +\infty)$, $j=0,1,\dots,4$, lze dosáhnout pouze při jistém omezeném růstu $|h'(x)|$ na intervalu $I_1 = \langle 0, +\infty)$.

Резюме

О СУЩЕСТВОВАНИИ РЕШЕНИЙ И ИХ ПРОИЗВОДНЫХ ИНТЕГРИРУЕМЫХ С КВАДРАТОМ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ЧЕТВЕРТОГО И ПЯТОГО ПОРЯДКОВ

Рассматривается нелинейное дифференциальное уравнение 4-го порядка

$$x^{IV}(t) + ax'''(t) + bx''(t) + cx'(t) + h[x(t)] = p(t), \qquad (1)$$

с постоянными $a, b, c \in R^+$ исполняющими условие Рауса-Гурвица $ab > c$; $h(x) \in C^1(-\infty, +\infty)$, $h'(x) < 0$, $h(0) = 0$, $|h(x)| \leq H$, $|p(t)| \leq P$, $\liminf_{|x| \to \infty} |h(x)| > |p(o)|$ для всех $t \in I = (-\infty, +\infty)$. С учетом предположений об ограниченности $h'(x)$ на интервале $I_1 = \langle 0, \infty)$, $\limsup_{t \to \infty} |\int^t p(\tau) d\tau| < \infty$ доказывается, что $x^{(j)}(t) \in I_2 \langle 0, \infty)$ для $j = 0,1,2,3$.

Для дифференциального уравнения 5-го порядка аналогичного ти-

па при соответствующих предположениях о постоянных и функциях $h(x)$, $p(t)$ показывается, что $x^{(j)}t \in I_2(0,\infty), j = 0,1\ldots,4$, можно достигнуть только для ограниченного роста $|h'(x)|$ на интервале $I_1 = (0, \infty)$.

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RNDr. Jan Andres, CSc.,
RNDr. Vladimír Vlček, CSc.,

katedra matematické analýzy a numerické matematiky přírodovědecké fakulty Univerzity Palackého

Gottwaldova 15

771 46 Olomouc, ČSSR

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