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AN APROXIMATIVE SOLUTION OF THE GENERALIZED EIGENVALUE PROBLEM

TOMÁŠ KOJECKÝ

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In this note an algorithm and its convergence to the solution of the equation

$$A\, x = \lambda\, B\, x \tag{1}$$

is shown, where $x \in X$, X is a Hilbert space and A and B are linear operators from X onto X. There are used some terms of the theory of the spectral representation of normal operator [1].

Let $(.,.)$ denote the scalar product in X, the norm in X be defined $\| . \|_X^2 = (.,.)$. Let $[X]$ be the space of linear bounded operators on X, the norm of $T \in [X]$ be defined as usual : $\| T \| = \sup_{\|x\|=1} \| T\, x \|$. Let C be the open complex plane, we denote the spectrum of T by $\sigma(T)$ and its spectral radius by $r(T)$. Let spectral radius circle be a set of $\lambda \in C$, for which $|\lambda| = r(T)$. We denote it by the letter ω. We define to $T \in [X]$ its adjoint T^* for which $(Tx, y) = (x, T^*y)$ holds for every $x, y \in X$. We say that T is self-adjoint, if $T = T^*$ and that T is normal, if $T T^* = T^* T$. We assume that $r(T) > 0$ not to repeat it in most of statements. We will use the convergence theorem from [3], which is shown here only for the sake of completeness (without the proof). Iterations are constructed in the following way ([5], [2])

$$x^{(n+1)} = T\, x^{(n)} , \qquad x_n = \frac{x^{(n)}}{k_n} \tag{2}$$

$$\mu_n = \frac{(x^{(n+1)}, y_n)}{(x^{(n)}, z_n)} \tag{3}$$

where $x^{(o)} \in X$, the sequences $\{y_n\}$, $\{z_n\}$ of elements of X and the number sequence $\{k_n\}$ are such that the denominators in (2) and (3) are not equal to zero and

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = y \quad , \tag{4}$$

where $y \in X$. We remark to the next theorem, that no assumptions about the isolations of points from $\sigma(T) \cap \omega(T)$ are made.

Theorem 1. Let $T \in [X]$ be a normal operator and $x^{(o)} \in X$ be a fixed vector such that

$$(E(\omega)\, x^{(o)}, y) = (E(\{\mu_0\})\, x^{(o)}, y) \neq 0 \quad ,$$

where $\mu_0 \in \omega(T) \cap \sigma(T)$ and E is the spectral measure generated by T. Let (4) hold for y_n , z_n and $y \in X$. Further let k_n be such that

$$\lim_{n\to\infty} \prod_{i=1}^{n} \mu_0\, k_n^{-1} = \beta \neq 0 \ , \ |\beta| < \infty \ . \tag{5}$$

We denote $x_0 = \beta E(\{\mu_0\})\, x^{(o)}$. Then

$$\lim_{n\to\infty} \mu_n = \mu_0 \qquad \text{and} \qquad \lim_{n\to\infty} x_n = x_0$$

hold, where x_0 is the eigenvector of T corresponding to the eigenvalue μ_0.

This iterative process is applied for seeking an eigenvalue and its corresponding eigenvector of the equation (2). The procedure is similar as in [2] or [5]. The iterations are constructed in this way

$$v^{(n)} = B\,u^{(n)}, \quad A\,u^{(n+1)} = v^{(n)}, \quad u^{(o)} = x^{(o)} \tag{6}$$

and further

$$u_n = \frac{u^{(n)}}{k_n} \tag{7}$$

$$\lambda_n = \frac{(u^{(n)}, y_n)}{(u^{(n+1)}, z_n)} \tag{8}$$

where k_n , y_n and z_n are those as in theorem 1.

Now we can state the main assertion.

Theorem 2. Let the linear operators A and B are such **that** A^{-1} exists and $T = A^{-1}B$ satisfies together with y_n , z_n , k_n , y and $x^{(o)}$ assumptions of the theorem 1. Then u_n converges to u_o in the norm in X and λ_n converges to λ_o where u_o is the eigenvector of the equation (2) and λ_o is its corresponding eigenvalue.

Proof: This will be denote alike the proof of theorem 3 in [2]. For $u^{(n+1)}$ from (6) holds

$$u^{(n+1)} = A^{-1}\,v^{(n)} = A^{-1}\,B\,u^{(n)} \tag{9}$$

so that

$$u^{(n)} = T^n\,x^{(o)} = x^{(n)} \quad .$$

Applying theorem 1 to the sequence $\{x^{(n)}\}$ finishes the proof of the convergence u_n to the eigenvector of the equation (2). Similarly we prove the convergence of the sequence $\{\lambda_n\}$. The equation $T\,x_o = \mu_o\,x_o$ for the limit elements $x^{(n)}$ and μ_n from theorem 1 hold and as $T = A^{-1}B$, we have

$$A^{-1}\,B\,u_o = \mu_o\,u_o \tag{10}$$

Owing to the fact, that

$$\lim_{n\to\infty} \mu_n = \lambda_o^{-1}$$

we obtain from (10) that

$$A\,u_o = \lambda_o\,B\,u_o$$

what finishes the proof.

Now we will investigate the generalized eigenvalue problem in case that A and B are linear operators from X into Y, where X and Y are complex Hilbert spaces. Under the term eigenvector of a couple of A and B we mean a vector for which $A\,x \neq \sigma$ and which solves the equation (1). We will denote by $[X,Y]$ the space of all linear bounded operators from X into Y. The pseudoinverse (Moore-Penrose) operator A^+ is defined by the next relations [6]

$$A\,A^+A = A$$

$$A^+A\,A^+ = A^+$$

$$(A^+A)^* = A^+A$$

$$(A\,A^+)^* = A\,A^+$$

The following iterative process

$$v^{(n)} = A\,u^{(n)}, \quad u^{(n+1)} = B^+\,v^{(n)}, \quad u^{(o)} = x^{(o)} \qquad (11)$$

is used together with (7) and (8).

Theorem 3. Let A be a linear operator from X into Y and $B \in [X,Y]$ such that $R(A) \subset R(B)$, where R denotes their corresponding ranges. Further let $T = B^+A$ satisfy together with y_n, z_n, k_n, y and $x^{(o)}$ assumption of theorem 1. Then u_n from (7) converges to eigenvector of a couple of A and B and μ_n from (3) to its corresponding eigenvalue.

Proof. We show firstly, that if x_o is an eigenvector and μ_o its corresponding eigenvalue of the operator $T = B^+A$, then x_o

and μ_0 also satisfy the equation (1). Let

$$B^+A\ x_0 = \mu_0\ x_0$$

hold, therefore we have

$$B\ B^+A\ x_0 = \mu_0\ B\ x_0 \quad .$$

$B\ B^+$ is the orthogonal projection from Y onto R(B) and the identity operator on R(B), so for every $y \in R(A) \subset R(B) \subset Y$

$$B\ B^+\ y = y$$

holds, from which we obtain

$$A\ x_0 = \mu_0\ B\ x_0 \quad .$$

Further $|\mu_0| = r(T)$ under the assumptions of theorem 2 and owing to this $A\ x_0 \neq \sigma$, i.e. x_0 is an eigenvector under our definition. We have from (11) similarly, that

$$u^{(n+1)} = B^+A\ u^{(n)}$$

holds, i.e.

$$x^{(n+1)} = T\ x^{(n)} \quad .$$

With respect to this fact the sequences (7) and (3) converge to the eigenvector x_0 and its corresponding eigenvalue μ_0 and under the first part of this proof x_0 and μ_0 are the solution of (1) and the assertion is proved.

Finally we remark, that no assumptions on the neighbourhood of spectral radius circle have been made, i.e. our investigation covers the case, when the dominant eigenvalue of T is not isolated. The situation of not isolated eigenvalue was studied by Kolomý for a self-adjoint nonnegative operator [4].

SOUHRN

PŘIBLIŽNÉ ŘEŠENÍ ZOBECNĚNÉHO PROBLÉMU VLASTNÍCH ČÍSEL

TOMÁŠ KOJECKÝ

Je zkoumáno řešení rovnice $A x = \lambda B x$ v Hilbertově prostoru; řešení se hledá pomocí iterací (6), (7), (8) a (11) v případě, že buď $B^{+}A$ nebo $A^{-1}B$ jsou normální ohraničené operátory. Je ukázána konvergence výše zmíněných iterací.

РЕЗЮМЕ

ПРИБЛИЖЕННОЕ РЕШЕНИЕ ОБОБЩЕННОЙ ПРОБЛЕМЫ СОБСТВЕННЫХ ЗНАЧЕНИЙ

Т. КОЕЦКИ

Показывается путь для нахождения решения уравнения (1) в пространстве Гильберта при помощи итерации (6), (7) и (8) в случае, когда B^+A или $A^{-1}B$ нормальные операторы. Доказывается стремление итерации.

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