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DYNAMIC SYSTEMS DESCRIBED BY DIFFERENTIAL EQUATIONS WITH SINGULARITIES OF THE TYPE $\frac{0}{0}$ AND THEIR MACHINE SOLVING

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Abstract: The work treats of a machine solving of differential equations with singularities of the type $\frac{0}{0}$. The limit of the indefinite expression is determined by developing the solution in a power series. The technical application of the equations are given there.

Key words: a differential equation, singularity, a machine solving.

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In the technical practise we often meet with differential equations in which indefinite expressions of the type $\frac{0}{0}$ occur. These expressions occur in some differential equations with variable coefficients or in some nonlinear differential equations. The general form of differential equation with variable coefficients has in this case the form

$$\sum_{k=o}^{n} \frac{u_k(t)y^{(k)}}{g_k(t)} = F(t) \;, \tag{1}$$

the general form of the nonlinear differential equation of the given type is

$$\sum_{k=o}^{n} \frac{u_k(t)f_k(y^{(k)})}{v_k(t)g_j(y^{(j)})} = F(t) \;, \tag{2}$$

$j = 0,1,2,\dots n$, where in the equations (1) and (2) some of the expressions $u_k(t)$, $v_k(t)$, $g_k(t)$, and $g_j(y^{(j)})$ may be constants. From now we assume that only one broken expression is an indefinite expression of the type $\frac{0}{0}$ in equations (1) and (2). As it is given further, this condition is not necessary, such equations may be solved by the given methods, in vhich several indefinite expressions of the type $\frac{0}{0}$ occur.

For example, the differential equation (3) is the type of the equation (1)

$$ty''' + 2y'' + ty' = 0 \tag{3}$$

programmed in the form

$$y''' + \frac{2}{t}y'' + y' = 0 \tag{4}$$

with initial conditions $y(o) = 0$, $y'(o) = 1$, $y''(o) = 0$, the known function $y = \int i(t)dt = \int_0^t \frac{\sin t}{t}\,dt$ is its solution.

Bessel differential equation (5) is the next type of the equation (1)

$$t^2y'' + ty' + (t^2 - m^2)y = 0 \;. \tag{5}$$

with initial conditions $y^{(k)}(o) = 0$ for $k < n$, $y^{(n)}(o) = \frac{1}{2^n}$, $y^{(n+1)}(o) = 0$, which for $m = 0$ has the form

$$y'' + \frac{y}{t} + y = 0 \tag{5a}$$

with initial conditions $y(o) = 1$, $y'(o) = 0$. The expression $\frac{y'}{t}$ becomes an indefinite expression for $t \longrightarrow 0$. Similarly the equation (6) is the equation of the type (1)

$$\varphi'' + \frac{\varphi'}{x} - \frac{\varphi}{x^2} = -\frac{qx}{2D} \tag{6}$$

with initial conditions $(o) = \varphi'(o) = 0$, describing the angle of slight turning of section of uniformly loaded freely supported circular plate.

For example the equation (7) is the type of the equation (2)

$$t^2y'' - t^2y' - y^2 = -2t^2 \tag{7}$$

with initial conditions $y(o) = 0$, $y'(o) = 1$, programmed in the form

$$y'' - y' - \frac{y^2}{t^2} = -2 \quad . \tag{7a}$$

Let us assume a solution of (3) in the form of a power series and determine the derivatives of the solution up to the highest order, i.e. up to the third derivative. i.e.

$$y = a_o + a_1t + a_2t^2 + \ldots + a_nt^n \tag{8}$$

$$y' = a_1 + 2a_2t + \ldots + na_nt^{(n-1)} \tag{8a}$$

$$y'' = 2a_2 + 6a_3t + 12a_4t^2 + \ldots + n(n-1).\, a_nt^{(n-2)} \tag{8b}$$

$$y''' = 6a_3 + 24a_4t + \ldots + n(n-1)(n-2)\, a_nt^{(n-3)} \tag{8c}$$

after substitution into (3) we obtain

$$6a_3t + 24a_4t^2 + 60a_5t^3 + \ldots + n(n-1)(n-2)a_nt^{n-2} +$$
$$+ 4a_2 + 12a_3t + 24a_4t^2 + 40a_5t^3 + \ldots + 2n(n-1)a_nt^{n-2} +$$
$$+ a_1t + 2a_2t^2 + 3a_3t^3 + 4a_4t^4 + \ldots + na_nt^n = 0 \quad .$$

The coefficients a_1 through a_n will be obtained by comparing the coefficients in particular powers of the independent variable t. In view of the fact that for determining $\lim\limits_{t \to o} \frac{y''}{t}$ determining $y'''(o)$ is sufficient, it is sufficient for us - as it follows from (8c) - to determine the coefficient a_3 at the most. On the ground of the given initial conditions and the relations (8) and (8a) and (8b) it holds:

$$a_o = 0$$

$$a_1 = 1$$

$$a_2 = 0 \ ,$$

by comparing the coefficients in the first power of t we get

$$6a_3 + 12a_3 + a_1 = 0$$

i.e.

$$18a_3 = - a_1 \ ,$$

$$a_3 = - \frac{a_1}{18} = - \frac{1}{18} \ .$$

By (8c) it holds $y'''(o) = 6a_3 = -\frac{1}{3}$, from (4) we come to

$$\lim_{t \to o} \frac{y''}{t} = \frac{- y'''(o) - y'(o)}{2} = \frac{\frac{1}{3} - 1}{} = - \frac{1}{3} \ .$$

The value of the expression $\lim\limits_{t \to o} \frac{y''}{t}$ can be determined by substitution for y'' according to (8b), i.e.

$$\lim_{t \to o} \frac{y''}{t} = \lim_{t \to o} \frac{6a_3t + 12a_4t^2 + \ldots + n(n-1)a_nt^{n-2}}{t} = 6a_3 \ .$$

After substitution into (4) we get

$$6a_3 + 12a_3 + a_1 = 0 \ ,$$

hence in the case of $a_1 = 1$ we determine the value of

$$a_3 = - \frac{1}{18}$$

and then

$$\lim \frac{y''}{t} = 6a_3 = -\frac{1}{3} \quad .$$

In the same way we proceed in Bessel equation ((5), (5a)).

$$y = a_o + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

$$y' = a_1 + 2a_2 t + 3a_3 t^2 + \ldots + na_n t^{n-1}$$

$$y'' = 2a_2 + 6a_3 t + \ldots + n(n-1)a_n t^{n-2}$$

From the initial conditions it holds $a_o = 1$, $a_1 = 0$. After substitution into (5) (where $m = 0$) we get

$$2a_2 t^2 + 6a_3 t^3 + \ldots + n(n-1)a_n t^n + 2a_2 t^2 + 3a_3 t^3 + \ldots$$

$$\ldots + na_n t^n + t^2 + a_2 t^3 + \ldots + a_n t^{n+2} = 0$$

By comparing of the coefficients in the power t^2 it holds

$$2a_2 + 2a_2 + 1 = 0 \quad ,$$

i.e.

$$a_2 = -\frac{1}{4}$$

and $\lim\limits_{t \to o} \frac{y'}{t} = 2a_2 = -\frac{1}{2}$.

The above method can be also used in programming nonlinear differential equations (see (7)), but some difficulties can appear in determining of the coefficients a_k, because equation, by which solution these coefficients are determined, is an nonlinear one, it has a relevant number of roots and a solution of the given differential equation need not be one-to-one.

Let us also assume the solution of (7) in the form of a power series and determine the derivatives of the solution up to the highest order, i.e. up to the second derivation.

$$y = a_o + a_1 t + a_2 t^2 + \ldots + a_n t^n \quad ,$$

$$y' = a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \ldots + na_nt^{n-1} \quad ,$$

$$y'' = 2a_2 + 6a_3t + \ldots + n(n-1)a_nt^{n-2} \quad .$$

From the given initial conditions $y(o) = 0$, $y'(o) = 1$ it holds $a_o = 0$, $a_1 = 1$. For determining the limit of the expression $\lim\limits_{t\to o} \frac{y^2}{t^2}$ it is sufficient in according to (7a) to determine the value of $y''(o)$, i.e. to determine the coefficient a_2. After substitution of the expressions for y, y' and y'' into (7) we get

$$2a_2t^2 + 6a_3t^3 + \ldots + n(n-1)a_nt^n - a_1t^2 - 2a_2t^3 - 3a_3t^4 -$$
$$- \ldots - a_nnt^{n+1} - a_o^2 - a_1^2t^2 - a_2^2t^4 - \ldots - a_n^2t^{2n} -$$
$$- 2a_oa_1t - 2a_oa_2t^2 - 2a_oa_3t^3 - 2a_1a_2t^3 - \ldots -$$
$$- 2a_{n-1}a_nt^{2n-1} = - 2t^2 \quad .$$

where $a_o = 0$, $a_1 = 1$. By comparing of the coefficients in the second power of t we get

$$2a_2 - a_1 - a_1^2 = - 2$$

i.e. $a_2 = 0$ and $y''(o) = 0$. After substitution into (7a) we get

$$\lim_{t\to o} \frac{y^2}{t^2} = 2 + y''(o) - y'(o) = 1 \ .$$

The value of the expression $\lim\limits_{t\to o} \frac{y^2}{t^2}$ can be determine easier by substitution for y^2, i.e.

$$\lim_{t\to o} \frac{y^2}{t^2} = \lim_{t\to o} \frac{a_o + a_1^2t^2 + a_2^2t^4 + \ldots + 2a_oa_1t + 2a_oa_2t^2 + \ldots + a_n^2t^{2n}}{t^2} =$$
$$= a_1^2 = 1 \quad ,$$

i.e. $a_1 = \pm 1$. According to the given condition $y'(o)$ it holds $a_1 = 1$.

Similarly the above method can be used in programming of differential equations, where several indefinite expressions of the type $\frac{0}{0}$ occur, how it is in (6) solved in interval $x \in (0; x_1\rangle$. In this case it is necessary to determine relations for limits of indefinite expressions and then to proceed by comparing of coefficients. We again assume the solution of (6) in the form

$$\varphi = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \qquad (9)$$

$$\varphi' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1}$$

$$\varphi'' = 2a_2 + 6a_3 x + 12a_4 x^2 + \dots + n(n-1) a_n x^{n-2}$$

From the initial conditions it holds $a_0 = a_1 = 0$, so

$$\lim_{x \to 0} \frac{\varphi}{x^2} = a_2 \ , \qquad \lim_{x \to 0} \frac{\varphi'}{x} = 2a_2 \qquad (10)$$

Coefficient a_2 can be determined by substitution of relations (9) into equation (6), which is arranged for the form

$$x^2 \varphi'' + x \varphi' - \varphi = - \frac{q x^3}{D} \quad , \qquad (11)$$

i.e.

$$2a_2 x^2 + 6a_3 x^3 + \dots + n(n-1) a_n x^n + 2a_2 x^2 + 3a_3 x^3 + \dots +$$

$$+ n a_n x^n - a_2 x^2 - a_3 x^3 - \dots - a_n x^n = - \frac{q\, x^3}{D}$$

$$2a_2 + 2a_2 - a_2 = 0$$

$$a_2 = 0 \ .$$

After substitution into (10) we get

$$\lim_{x \to 0} \frac{\varphi}{x^2} = 0 \ , \qquad \lim_{x \to 0} \frac{\varphi'}{x} = 0 \ .$$

By the same way we also proceed in nonlinear differential equations, there also difficulties in determining of coefficients a_k may appear. If we programme the equation, for example

$$ty'^2y'' + y'^3 - 4ty = 3ty'^2 \qquad (12)$$

in the form

$$y'' + \frac{y'}{t} - \frac{4y}{y'^2} = 3 \qquad (13)$$

with initial conditions $y(o) = y'(o) = 0$, then two indefinite expressions of the type $\frac{0}{0}$ occur in (13). ($t \in (o;\ t_1 \rangle$).

If we develop the solution in a series, we get

$$y = a_o + a_1t + a_2t^2 + \ldots + a_nt^n ,$$

$$y' = a_1 + 2a_2t + 3a_3t^2 + \ldots + na_nt^{n-1} ,$$

$$y'' = 2a_2 + 6a_3 + \ldots + n(n-1)a\ n^{t^{n-2}} ,$$

where in according to the initial conditions $a_o = a_1 = 0$.

$$y'^2 = 4a_2^2t^2 + 9a_3^2t^4 + \ldots 12a_2a_3t^3 + \ldots + n^2a_n^2t^{2n-2} ,$$

$$y'^3 = 8a_2^3t^3 + 27a_3^3t^6 + \ldots + 3.4a_2^2t^2 . 3a_3t^2 + n^3a_n^3t^{3^{n-3}}$$

$$\lim_{t\to o} \frac{y'}{t} = 2a_2 , \qquad \lim_{t\to o} \frac{4y}{y'^2} = \frac{1}{a_2} .$$

After substitution the above relations into (12) and by comparing coefficients in t^3 we get

$$8a_2^3t^3 + \ldots + 8a_2^3t^3 + \ldots - 4a_2t^3 = 12a_2^2t^3 ,$$

i.e.

$$16a_2^3 - 12a_2^2 - 4a_2 = 0 ,$$

i.e.

$$a_{2\ 1,2,3} = 0;\ 1;\ -\frac{1}{4} .$$

The value $a_2 = 0$ is incorservient for machine solving from the given initial value, because $\lim\limits_{t \to o} \frac{4y}{y'^2} \longrightarrow \infty$, in the case of $a_2 = 1$ is $\lim\limits_{t \to o} \frac{y'}{t} = 2$, $\lim\limits_{t \to o} \frac{4y}{y'^2} = 1$ and the function $y = t^2$ is the solution, in the case $a_2 = -\frac{1}{4}$ is $\lim\limits_{t \to o} \frac{y'}{t} = -\frac{1}{2}$, $\lim\limits_{t \to o} \frac{4y}{y'^2} = -4$ and the function $y = -\frac{1}{4} t^2$.

In analog solution the indefinite expressions $z = \frac{f(t)}{g(t)}$, $f(o) = g(o) = 0$ are simulated in the form

$$z \doteq \frac{f(t) + z(o)ae^{-\alpha t}}{g(t) + ae^{-\alpha t}} , \qquad (14)$$

where $z(o) = \lim\limits_{t \to o} \frac{f(t)}{g(t)}$.

The coefficients a and α are as a rule chosen in such form to satisfy a relation $a\alpha = 1$.

The supplementary members quickly disappear with growing of the independent variable t, the characteristic course of error of approximation of the expression $z = \frac{f(t)}{g(t)}$ by the expression (14) is in the Fig.1.

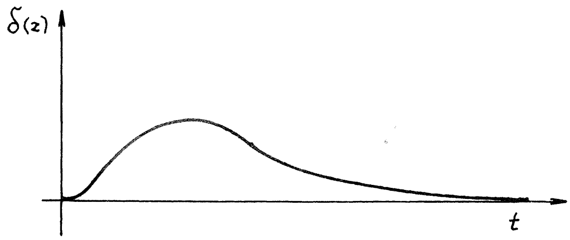

Fig. 1

In digital solution we start from limit in singular point, in both cases we have to determine the limits of singular expressions beforehand.

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