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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 32 (1993), No. 1, 111--122

Persistent URL: http://dml.cz/dmlcz/120287

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS FACULTAS RERUM NATURALIUM
1993 Mathematica XXXII Vol. 110

FUNDAMENTAL CENTRAL DISPERSIONS OF THE PHASE FUNCTION WITH THE FINAL OSCILLATION

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(Received May 10, 1992)
Dedicated to Prof. O.Borůvka to his 93 birthday

Abstract

Phase functions and phases of ordered pairs of independent solutions of second-order linear differential equations $y'' = q(t)y$ have fundamental importance in the theory of second-order linear differencial transformations [1], [2] and in two-dimensional spaces of continuous functions [5], [6].

In this paper we shall study the phase function α with the final oscillation [4] and respective fundamental central dispersion φ. We shall deal with the relation between them which is expressed by the Abel functional equation [3].

Key words: Phase function, fundamental central dispersion.

MS Classification: 39A10

In this paper we denote N the set of all natural numbers, Z the set of all integers, $\mathbb{R}$ the set of all real numbers, $j = (a, b)$ an open interval, $C_0(j)$ the set of all continuous functions defined in the interval j. We suppose that the interval $j = (-\infty, \infty)$ that is $a = -\infty, b = \infty$ throughout the paper.

1 Phase functions

Now we give the definition of phase function from [1] and its fundamental characteristics.

Definition 1.1 The phase function in the interval j is a real function $\alpha = \alpha(t)$ with the following properties:
1. $\alpha \in C_0(j)$,
2. $\alpha(t)$ is increasing or decreasing in the interval j.

We denote by $M_1(M_2)$ the set of all increasing (decreasing) phase functions and by M the set of all phase functions. Thus $M = M_1 \cup M_2$.

Let $\alpha \in M$. The continuity of the phase function α yields an existence of proper or improper limits

$$c = \lim \alpha(t) \quad \text{for} \quad t \to -\infty,$$
$$d = \lim \alpha(t) \quad \text{for} \quad t \to \infty.$$

Definition 1.2 The proper or improper limit c (d) is called the left (right) boundary value of the phase function α.

To denote that c (d) is the left (right) boundary value of phase function α we write also c_α (d_α).

If the phase function α increases (decreases) in the interval j we have

$$c_\alpha < d_\alpha \qquad (c_\alpha > d_\alpha).$$

Let J denotes the set of all functional values of the phase function α. It is clear that J is an open interval with boundary values c_α, d_α. When the phase function α increases (decreases), it holds $J = (c_\alpha, d_\alpha)$ ($J = (d_\alpha, c_\alpha)$).

Definition 1.3 The number $|c - d|$, where c (d) is the left (right) boundary value of the phase function α, we call the oscillation of the phase function α in the interval j and denote by $O(\alpha/j)$ or briefly $O(\alpha)$. Thus $O(\alpha) = |c - d|$.

2 Conjugate numbres according to the phase function

Definition 2.1 Let $\alpha \in M$. Let $t \in j$ be any number. The number $x \in j$ we call conjugate with the number t according to the phase function α, if

$$\alpha(x) - \alpha(t) = \mu\pi, \tag{2.1}$$

where $\mu \in Z$.

The definition follows that every number is self-conjugate. This trivial case is excluded from other considerations.

Definition 2.2 Let (2.1) holds and $n = |\mu|$, $\mu \neq 0$. For $\mu > 0$ ($\mu < 0$) the number x we call n-th right (left) conjugate number with the number t.

The definition yields that for $O(\alpha) \leq \pi$ there are no conjugate numbers according to the phase function α. For $O(\alpha) > \pi$ there are different conjugate numbers according to the phase function α. We also say that the phase function α induces conjugate numbers in the interval j.

Our further considerations are devoted to phase functions with a final oscillation. It means a case of the phase functions $\alpha \in M$ such that $O(\alpha) \in \mathbb{R}$. The definition of the oscillation follows that this case sets in if and only if the left and right boundary values of the phase function α are proper numbers. Thus $c_\alpha, d_\alpha \in \mathbb{R}$.

3 Fundamental numbers and fundamental sequences

Theorem 3.1 *Let $\alpha \in M$ and $m \in N$, $m \geq 2$. Let $(m-1)\pi < O(\alpha/j) \leq m\pi$. Let $c_\alpha(d_\alpha)$ be left (right) boundary value of the phase function α. Then there are numbers $a_\mu, b_{-\mu}, \mu = 1, 2, \ldots, m-1$, in the interval j such that*

$$\alpha(a_\mu) = c_\alpha + \varepsilon_\alpha \mu\pi, \tag{3.1}$$

$$\alpha(b_{-\mu}) = d_\alpha - \varepsilon_\alpha \mu\pi, \tag{3.2}$$

where $\varepsilon_\alpha = 1$ for α increasing and $\varepsilon_\alpha = -1$ for α decreasing in the interval j. The numbers $a_\mu, b_{-\mu}$, $\mu = 1, 2, \ldots, m-1$ are just the numbers from the interval j such that $\alpha(a_\mu), \alpha(b_{-\mu}) \in J$, $\mu \in Z$.

Proof The assumption

$$O(\alpha/j) \in ((m-1)\pi, m\pi\rangle$$

yields that the boundary points c_α, d_α of the open interval J, which means the set of all values of the phase function α, hold the following relations

$$c_\alpha + (m-1)\pi < d_\alpha \leq c_\alpha + m\pi, \qquad \text{if } \alpha \text{ increases in } j,$$

$$c_\alpha - m\pi \leq d_\alpha < c_\alpha - (m-1)\pi, \qquad \text{if } \alpha \text{ decreases in } j.$$

So the interval J possesses exactly $(m-1)$ points a_μ and exactly $(m-1)$ points $b_{-\mu}$ holding the formulae (3.1) and (3.2).

Definition 3.1 The final sequence of points

$$\{a_\mu\}_{\mu=1}^{m-1}$$

we call the left fundamental sequence of points induced in the interval j by the phase function α.

The final sequence of points

$$\{b_{-\mu}\}_{\mu=1}^{m-1}$$

we call the right fundamental sequence of points induced in the interval j by the phase function α.

It is obvious that:

1. The points of left (right) fundamental sequence are conjugate numbers according to the phase function α.

2. For fundamental sequences of points in j induced by the phase function α there hold the relations

$$a_1 < a_2 < \ldots < a_{m-1}, \tag{3.3}$$

$$b_{-1} > b_{-2} > \ldots > b_{-(m-1)}. \tag{3.4}$$

3. If we set $a_0 = a,\ b_0 = b$ then

$$a_0 < b_{-(m-1)} \leq a_1 < b_{-(m-2)} \leq \ldots < b_{-1} \leq a_{m-1} < b_0. \tag{3.5}$$

   If $O(\alpha) \in ((m-1)\pi, m\pi)$ then in (3.5) signs of inequality are valid there.

   If $O(\alpha) = m\pi$ then in (3.5) signs of equality are valid there and thus the left and right fundamental sequences coinside.

4. Let $O(\alpha) \in ((m-1)\pi, m\pi)$. Then we have

$$0 < |\alpha(b_{-(m-\mu)}) - \alpha(a_\mu)| < \pi.$$

   Let $O(\alpha) = m\pi$. Then we have

$$|\alpha(b_{-(m-\mu)}) - \alpha(a_\mu)| = 0.$$

Definition 3.2 Let $m \in M,\ m \geq 2$. Let $O(\alpha) \in ((m-1)\pi, m\pi),\ \alpha \in M$. The phase function α we call of finite type (m) general.

Let $m \in M,\ m \geq 2$. Let $O(\alpha) = m\pi,\ \alpha \in M$. The phase function α we call of finite type (m) special.

We notice now some properties of fundamental sequences of points in the interval j.

In the case that $O(\alpha) \in ((m-1)\pi, m\pi)$ points of fundamental sequences $\{a_\mu\}_{\mu=1}^{m-1}$, $\{b_{-\mu}\}_{\mu=1}^{m-1}$ divide the interval j into intervals j_μ, i_ν where

$$j_\mu = (a_\mu, b_{-(m-1-\mu)}), \qquad \mu = 0, 1, 2, \ldots, m-1, \tag{3.6}$$

$$i_\nu = (b_{-(m-\nu)}, a_\nu), \qquad \nu = 1, 2, \ldots, m-1. \tag{3.7}$$

In the case that $O(\alpha) = m\pi$ the intervals i_ν, $\nu = 1, 2, \ldots, m-1$, are empty sets and

$$j_\mu = (a_\mu, a_{\mu+1}), \qquad \mu = 0, 1, \ldots, m-1. \tag{3.8}$$

Definition 3.3 The number $a_1(b_{-1})$ defined by (3.1) ((3.2)) we call left (right) fundamental number in the interval j and denote by r (s). Thus

$$r = a_1, \qquad s = b_{-1}.$$

In the case that the phase function α is of finite type (m) general the numbers r, s are not conjugate. In the case that the phase function α is of finite type (m) special the numbers r, s are conjugate.

Let $O(\alpha) \in ((m-1)\pi, m\pi)$. Let $t_0 \in j_\mu$ for one of the numbers $\mu = 0, 1, \ldots, m-1$. Then in any interval j_μ for the other μ there is exactly one conjugate number with the number t_0. We have m conjugate numbers here.

Let $t_0 \in i_\nu$ for one of the numbers $\nu = 1, 2, \ldots, m-1$. Then in any interval i_ν for the other ν there is exactly one conjugate number with the number t_0. We have $m-1$ conjugate numbers here.

Let $O(\alpha) = m\pi$. Let $t_0 \in j_\mu$ for one of the numbers $\mu = 0, 1, \ldots, m-1$. Then in any interval j_μ for the other μ there is exactly one conjugate number with the number t_0. We have m conjugate numbers in this case.

We note that left and right fundamental sequence of points of the phase function α, $O(\alpha) \in ((m-1)\pi, m\pi\rangle$ possesses always $m-1$ conjugate numbers so even in a case that both fundamental sequences of points coinside.

4 Equivalent phase functions

In the set M of all phase functions we define an equivalence relation which we denote by $\sim$.

Definition 4.1 We say that phase functions $\alpha, \beta \in M$ are equivalent in M and write $\alpha \sim \beta$ if in the interval j with the exception of singularities of functions $\mathrm{tg}\alpha$, $\mathrm{tg}\beta$ there holds the relation

$$\mathrm{tg}\beta(t) = \frac{a_{11}\mathrm{tg}\alpha(t) + a_{12}}{a_{21}\mathrm{tg}\alpha(t) + a_{22}} \tag{4.1}$$

where

$$a_{11}a_{22} - a_{12}a_{21} \neq 0, \quad t \in (-\infty, \infty).$$

It is evident that

1. The equivalence relation $\sim$ in M is reflexive, symmetric and transitive.

2. Let $a_{ik} \in \mathbb{R}$, $i, k = 1, 2$. Let $a_{11}a_{22} - a_{12}a_{21} \neq 0$.
Let $\delta = \mathrm{sgn}(a_{11}a_{22} - a_{12}a_{21})$. The linear rational function

$$\frac{a_{11}t + a_{12}}{a_{21}t + a_{22}}$$

increases (decreases) by parts in j if and only if $\delta = 1$ ($\delta = -1$).

Theorem 4.1 *Let $\alpha, \beta \in M$. Let $\beta \sim \alpha$. Then fundamental sequences of points of function α are fundamental sequences of points of the phase function β.*

Proof Let $O(\alpha) \in ((m-1)\pi, m\pi)$, $m \geq 2$. Let us consider left and right fundamental sequence of conjugate points according to the phase function α, that means points $a_\mu, b_{-\mu}$ defined by (3.1) and (3.2). Then (4.1) yields

$$\mathrm{tg}\beta(a_\mu) = \frac{a_{11}\mathrm{tg}\alpha(a_\mu) + a_{12}}{a_{21}\mathrm{tg}\alpha(a_\mu) + a_{22}} = \frac{a_{11}\mathrm{tg}c_\alpha + a_{12}}{a_{21}\mathrm{tg}c_\alpha + a_{22}}$$

for $\mu = 1, \dots, m-1$.

From here we have

$$\beta(a_\mu) = c_\beta + \delta\varepsilon_\alpha \mu\pi, \quad \mu = 1, \dots, m-1, \tag{4.2}$$

since from (4.1) for $t \to -\infty$ we get

$$\mathrm{tg}c_\beta = \frac{a_{11}\mathrm{tg}c_\alpha + a_{12}}{a_{21}\mathrm{tg}c_\alpha + a_{22}}$$

Further from (4.1) we have

$$\mathrm{tg}\beta(b_{-\mu}) = \frac{a_{11}\mathrm{tg}\alpha(b_{-\mu}) + a_{12}}{a_{21}\mathrm{tg}\alpha(b_{-\mu}) + a_{22}} = \frac{a_{11}\mathrm{tg}d_\alpha + a_{12}}{a_{21}\mathrm{tg}d_\alpha + a_{22}}$$

for $\mu = 1, \dots, m-1$.

From here we have

$$\beta(b_{-\mu}) = d_\beta - \delta\varepsilon_\alpha \mu\pi, \quad \mu = 1, \dots, m-1, \tag{4.3}$$

since from (4.1) for $t \to \infty$ we get

$$\mathrm{tg}d_\beta = \frac{a_{11}\mathrm{tg}d_\alpha + a_{12}}{a_{21}\mathrm{tg}d_\alpha + a_{22}}$$

The formulas (4.2) and (4.3) certify the validity of the theorem.

Corollary 4.1 *Let $O(\alpha) \in ((m-1)\pi, m\pi)$. Let $\beta \sim \alpha$. Then*

$$0 < |\beta(b_{-(m-\mu)}) - \beta(a_\mu)| < \pi.$$

Proof The assertion follows from the above theorem and the assertion 4. of paragraph 3.

Theorem 4.2 *Let $\alpha, \beta \in M$. Let $\beta \sim \alpha$. Let $O(\alpha) \in ((m-1)\pi, m\pi)$. Then also*

$$O(\beta) \in ((m-1)\pi, m\pi).$$

Proof From (4.2) and (4.3) we get

$$\begin{aligned} \beta(a_\mu) &= c_\beta + \delta\varepsilon_\alpha \mu\pi, \qquad (4.4) \\ \beta(b_{-(m-\mu)}) &= d_\beta - \delta\varepsilon_\alpha (m-\mu)\pi. \end{aligned}$$

From here

$$\beta(a_\mu) - \beta(b_{-(m-\mu)}) = c_\beta - d_\beta + \delta\varepsilon_\alpha m\pi.$$

As a consequence of (4.1) and (3.5) it holds

$$\begin{aligned} -\pi < \beta(b_{-(m-\mu)}) - \beta(a_\mu) < 0, &\quad \text{if} \quad \delta\varepsilon_\alpha = 1, \\ 0 < \beta(b_{-(m-\mu)}) - \beta(a_\mu) < \pi, &\quad \text{if} \quad \delta\varepsilon_\alpha = -1, \end{aligned}$$

or

$$0 < \beta(a_\mu) - \beta(b_{-(m-\mu)}) < \pi, \quad \text{if} \quad \delta\varepsilon_\alpha = 1, \qquad (4.5)$$

$$-\pi < \beta(a_\mu) - \beta(b_{-(m-\mu)}) < 0, \quad \text{if} \quad \delta\varepsilon_\alpha = -1. \qquad (4.6)$$

From (4.4) and (4.5) we get

$$0 < c_\beta - d_\beta + m\pi = \beta(a_\mu) - \beta(b_{-(m-\mu)}) < \pi, \quad \text{if} \quad \delta\varepsilon_\alpha = 1$$

and then the phase function β increases so

$$-m\pi < c_\beta - d_\beta < -(m-1)\pi$$

or

$$(m-1)\pi < d_\beta - c_\beta = O(\beta) < m\pi.$$

If $\delta\varepsilon_\alpha = -1$then the phase function β decreases and (4.4) and (4.6) follows that

$$-\pi < c_\beta - d_\beta - m\pi = \beta(a_\mu) - \beta(b_{-(m-\mu)}) < 0$$

or

$$(m-1)\pi < c_\beta - d_\beta = O(\beta) < m\pi.$$

Thus in both cases we have

$$O(\beta) \in ((m-1)\pi, m\pi).$$

Theorem 4.3 *Let $\alpha, \beta \in M$. Let $\beta \sim \alpha$. Let $O(\alpha) = m\pi$. Then*

$$O(\beta) = m\pi.$$

Proof In this case both fundamental sequences of points coinside and we have

$$b_{-(m-\mu)} = a_\mu.$$

Thus

$$\begin{aligned} \beta(a_\mu) &= c_\beta + \delta\varepsilon_\alpha m\pi, \\ \beta(a_\mu) &= \beta(b_{-(m-\mu)}) = d_\beta - \delta\varepsilon_\alpha(m-\mu)\pi. \end{aligned}$$

After subtracting

$$c_\beta - d_\beta = \delta\varepsilon_\alpha m\pi$$

and then

$$O(\beta) = |c_\beta - d_\beta| = m\pi.$$

5 Fundamental central dispersions

Let $\alpha \in M$. Let $O(\alpha) \in ((m-1)\pi, m\pi)$, $m \in N$, $m \geq 2$. Let $\varepsilon_\alpha = 1 (\varepsilon_\alpha = -1)$, when α increases (decreases) in the interval j.

The fundamental points

$$a_\mu, b_{-\mu}, \mu = 1, \ldots, m-1$$

defined by (3.1) and (3.2) and the intervals j_μ, i_ν defined by (3.6) and (3.7):

$$\begin{aligned} j_\mu &= (a_\mu, b_{-(m-1-\mu)}), & \mu &= 0, 1, \ldots, m-1, \\ i_\nu &= (b_{-(m-\nu)}, a_\nu), & \nu &= 1, 2, \ldots, m-1, \end{aligned}$$

form a partition of the interval j.

Let $r = a_1$, $s = b_{-1}$. The fundamental points will be completed by points $a_0 = a = -\infty$, $b_0 = b = \infty$.

Now we define a function $\varphi = \varphi(t)$, $t \in j$, which maps each point $t \in j$ on a point $\varphi(t) \in j$, where $\varphi(t)$ is the first rigth conjugate point with the point t according to the phase function α. If the point t does not possess any right conjugate point then φ maps the point t on the smallest conjugate point with t in the interval j according to the phase function α. That means: The function φ maps points

$$\begin{aligned} & a_0 \text{ on } a_1,\ a_1 \text{ on } a_2, \ldots, a_{m-2} \text{ on } a_{m-1},\ a_{m-1} \text{ on } a_0, \text{ etc.}, \\ & b_0 \text{ on } b_{-(m-1)},\ b_{-(m-1)} \text{ on } b_{-(m-2)}, \ldots, b_{-2} \text{ on } b_{-1},\ b_{-1} \text{ on } b_0, \text{ etc.}, \end{aligned}$$

and intervals

$$\begin{aligned} & j_0 \text{ on } j_1, & & j_1 \text{ on } j_2, & & \ldots, & & j_{m-1} \text{ on } j_0, & & \text{etc.}, \\ & i_1 \text{ on } i_2, & & i_2 \text{ on } i_3, & & \ldots, & & i_{m-1} \text{ on } i_1, & & \text{etc.} \end{aligned}$$

So we get following relations:

$$\begin{aligned}
\alpha[\varphi(t)] &= \alpha(t)+\varepsilon_\alpha\pi \\
&\quad \text{for } t\in j_0\cup b_{-(m-1)}\cup i_1\cup a_1\cup j_1\cup\ldots\cup b_{-2}\cup i_{m-2}\cup a_{m-2}\cup j_{m-2}, \\
\alpha[\varphi(t)] &= \alpha(t)-\varepsilon_\alpha(m-2)\pi \qquad \text{for } t\in i_{m-1}, \\
\alpha[\varphi(t)] &= \alpha(t)-\varepsilon_\alpha(m-1)\pi \qquad \text{for } t\in j_{m-1},
\end{aligned}$$

where

$$\varphi(s-)=b_0=b=\infty, \qquad \varphi(s+)=b_{-(m-1)}, \tag{5.1}$$

and the symbol $\varphi(s-)$ resp. $\varphi(s+)$ means the limit of the function $\varphi(t)$ at the point s on the left resp. on the right

$$\varphi(a_{m-1}-)=a_1=r, \qquad \varphi(a_{m-1}+)=a_0=a=-\infty, \tag{5.2}$$

where the symbol $\varphi(a_{m-1}-)$ resp. $\varphi(a_{m-1}+)$ has the above given meaning,

$$\varphi(a_0)=a_1=r, \qquad \varphi(b_0)=b_{-(m-1)} \tag{5.3}$$

and the symbol $\varphi(a_0)$ resp. $\varphi(b_0)$ means the limit $\varphi(t)$ for $t\to-\infty$ resp. for $t\to\infty$.

Definition 5.1 Let $\alpha\in M$, $O(\alpha)\in((m-1)\pi, m\pi)$, $m\in N$, $m\geq 2$. Let α^{-1} denotes the inverse function of the phase function α. We define the function φ in the interval j as follows

$$\varphi(t)=\begin{cases}\alpha^{-1}(\alpha(t)+\varepsilon_\alpha\pi) & \text{for } t\in(-\infty,s), \\ \alpha^{-1}(\alpha(t)-\varepsilon_\alpha(m-2)\pi) & \text{for } t\in i_{m-1}=(b_{-1},a_{m-1}),\ b_{-1}=s, \\ \alpha^{-1}(\alpha(t)-\varepsilon_\alpha(m-1)\pi) & \text{for } t\in j_{m-1}=(a_{m-1},b_0),\end{cases} \tag{5.4}$$

and in the points of discontinuity $s=b_{-1}$, a_{m-1} and in the points a_0, b_0 there are the limits of the function φ given by (5.1), (5.2) and (5.3).

We call the function φ the fundamental central dispersion of the phase function α.

This function φ is continuous in the interval j with the exception of points s, a_{m-1}.

So the fundamental properties of the function φ are:

1. φ is defined and continuous in the interval j with the exeption of the points s, a_{m-1}.
2. $\lim\limits_{t\to s-}\varphi(t)=\infty, \qquad \lim\limits_{t\to s+}\varphi(t)=b_{-(m-1)},$
3. $\lim\limits_{t\to a_{m-1}-}\varphi(t)=r, \qquad \lim\limits_{t\to a_{m-1}+}\varphi(t)=-\infty,$

4. $\lim_{t\to-\infty} \varphi(t) = r, \qquad \lim_{t\to+\infty} \varphi(t) = b_{-(m-1)}\ (< r),$

5. φ increases by parts in the interval j, namely
from r to ∞ in the interval $(-\infty, s)$,
from $b_{-(m-1)}$ to r in the interval (s, a_{m-1}),
from $-\infty$ to $b_{-(m-1)}$ in the interval (a_{m-1}, ∞).

6. $\varphi(t) > t$ for $t \in (-\infty, s)$, $\varphi(t) < t$ for $t \in (s, a_{m-1})$, $t \in (a_{m-1}, \infty)$.
Indeed, in the interval $(-\infty, s)$ we have

$$\varphi(t) = \alpha^{-1}(\alpha(t) + \varepsilon_\alpha \pi) > \alpha^{-1}(\alpha(t)) = t,$$

since for $\varepsilon_\alpha = 1$ the function α^{-1} increases and for $\varepsilon_\alpha = -1$ it decreases.
Similarly we show that in the interval $(s, a_{m-1}) = i_{m-1}$ we have
for $m \in N, m \geq 2$

$$\varphi(t) = \alpha^{-1}(\alpha(t) - \varepsilon_\alpha(m-2)\pi) < \alpha^{-1}(\alpha(t)) = t;$$

if $m = 2$ then

$$\varphi(t) \equiv t$$

and in the interval $(a_{m-1}, \infty) = j_{m-1}$ we have for $m \in N,\ m \geq 2$

$$\varphi(t) = \alpha^{-1}(\alpha(t) - \varepsilon_\alpha(m-1)\pi) < \alpha^{-1}(\alpha(t)) = t.$$

7. For $t \in (-\infty, s)$ it holds $\alpha(\varphi(t)) = \alpha(t) + \varepsilon_\alpha \pi$,
for $t \in (s, a_{m-1})$ it holds $\alpha(\varphi(t)) = \alpha(t) - \varepsilon_\alpha(m-2)\pi$,
for $t \in (a_{m-1}, \infty)$ it holds $\alpha(\varphi(t)) = \alpha(t) - \varepsilon_\alpha(m-1)\pi$.

Definition 5.2 Let $\alpha \in M$, $O(\alpha) = m\pi$, $m \in N$, $m \geq 2$. Let α^{-1} denotes the inverse of the function α. We define the function φ by the following relation

$$\varphi(t) = \begin{cases} \alpha^{-1}(\alpha(t) + \varepsilon_\alpha \pi) & \text{for } t \in (-\infty, s),\ s = b_{-1}, \\ \alpha^{-1}(\alpha(t) - \varepsilon_\alpha(m-1)\pi) & \text{for } t \in (s, \infty), \end{cases} \tag{5.5}$$

and we call it the fundamental central dispersion of the phase function $\alpha(t)$, $t \in j$.

The following relations hold

$$\varphi(s-) = \infty, \qquad \varphi(s+) = -\infty, \qquad \varphi(a_0) = r, \qquad \varphi(b_o) = r.$$

The function φ is not continuous in the point s.

Thus we can summarize properties of the function φ:

1. φ is defined and continuous in j with the exception of the point s.

2. $\lim_{t \to s-} \varphi(t) = \infty, \qquad \lim_{t \to s+} \varphi(t) = -\infty,$

3. $\lim_{t \to -\infty} \varphi(t) = r, \qquad \lim_{t \to +\infty} \varphi(t) = r,$

4. φ increases by parts in the interval j, namely
   from r to ∞ in the interval $(-\infty, s)$ and
   from $-\infty$ to r in the interval (s, ∞).

5. $\varphi(t) > t$ for $t \in (-\infty, s)$, $\qquad \varphi(t) < t$ for $t \in (s, \infty)$.

6. $\alpha(\varphi(t)) = \alpha(t) + \varepsilon_\alpha \pi$ for $t \in (-\infty, s)$
   $\alpha(\varphi(t)) = \alpha(t) - \varepsilon_\alpha (m-1)\pi$ for $t \in (s, \infty)$.

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