Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Władysław Kierat

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 32 (1993), No. 1, 63--68

Persistent URL: `http://dml.cz/dmlcz/120299`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS FACULTAS RERUM NATURALIUM
1993 Mathematica XXXII Vol. 110

A NOTE ON APPLICATIONS OF THE ALGEBRAIC DERIVATIVE TO SOLVING OF SOME DIFFERENTIAL EQUATIONS

WŁADYSŁAW KIERAT

(Received May 20, 1992)

Abstract

In this note by means of Mikusinski operational calculus the differential equations with linear coefficients are investigated.

Key words: operational calculus, power series, solutions of linear differential equations

MS Classification: 34A25

1 Introduction

K. Yosida in [4] solved the Laplace differential equation

$$a_2tx''(t) + (a_1t + b_1)x'(t) + (a_0t + b_0)x(t) = 0 \tag{1}$$

where $a_i, b_i \in \mathbb{C}$, $i = 0,1,2$ and $t \in [0,\infty)$ by means of Mikusinski operational calculus assuming that the algebraic equations

$$a_2z^2 + a_1z + a_0 = 0$$

has two distinct roots z_1 and z_2.

The purpose of this note is to show that Yosida method presented in [4] may be used to determining particular solutions of the differential equations with linear coefficients.

$$a_ntx^{(n)}t) + (a_{n-1}t + b_{n-1})x^{(n-1)}(t) + \ldots + (a_0t + b_0)x(t) = 0 \tag{2}$$

$a_i, b_i \in \mathbb{C}$, $i = 0, 1, 2, \ldots, n$, $t \in [0, \infty)$ and without the restriction that the polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

has distinct roots.

We shall use in this note notations introduced by J. Mikusinski ([2],[3]).

Let $\mathcal{C}$ be the ring of continuous complex-valued functions in $[0, \infty)$ with ordinary addition and convolution as multiplication. The formal quotiens p/q, p and q are in $\mathcal{C}$ will be called Mikusinski operators. By $\mathcal{M}$, we shall denote the field of Mikusinski operators. The basic idea of our considerations is connected with the concept of an algebraic derivative D ([1],[2] p.294). The algebraic derivative D is defined as follows

$$Df = D\{f(t)\} = \{-tf(t)\} \quad \text{for} \quad f \in \mathcal{C},$$
$$D(p/q) = (Dpq - pDq)/q^2 \quad \text{for} \quad p, q \in \mathcal{C}, \quad q \neq 0.$$

For the positive integers n we have

$$Ds^n = ns^{n-1}, \tag{3}$$

s denotes the differential operator.

This equation suggests that the operation D may be considered as derivation with respect to the differential operator s.

2 Operational differential equations

K. Yosida in his paper has shown the more general formula

$$D(s - \alpha)^\gamma = \gamma(s - \alpha)^{\gamma-1}, \quad \alpha, \gamma \in \mathbb{C}. \tag{4}$$

Moreover, we shall need a formula of differentiation of the exponential operational functions

$$\exp[\gamma(s - \alpha)^{-k}], \quad \alpha, \gamma \in \mathbb{C}, \quad k \in \mathbb{N}. \qquad ([2], \text{p.}202).$$

Obviously, the operator $(s - \alpha)^{-k}$, $k \in \mathbb{N}$ is in $\mathcal{C}$. The above exponential operational function can be defined by means of the power series

$$\exp[\gamma(s - \alpha)^{-k}] = 1 + \frac{\gamma(s - \alpha)^{-k}}{1!} + \frac{\gamma(s - \alpha)^{-2k}}{2!} + \ldots. \qquad ([2], \text{p.}206)$$

Now, differentiating this power series term by term with respect to s we obtain

$$D \exp[\gamma(s - \alpha)^{-k}] = \exp[\gamma(s - \alpha)^{-k}]\gamma(-k)(s - \alpha)^{-k-1}. \tag{5}$$

The formulas (4) and (5) allow to solve the operational differential equation

$$\frac{Dx}{x} = \frac{Q(s)}{P(s)}, \tag{6}$$

where P and Q are polynomials with complex coefficients and deg. $Q <$ deg. P.

The equation (6) can be written as follows

$$\frac{Dx}{x} = \sum_{i=1}^{r}\sum_{j=1}^{k_i} \frac{\gamma_{ij}}{(s-\alpha_i)^j}, \quad \alpha_i, \gamma_{ij} \in \mathbb{C} \tag{7}$$

The complex numbers α_i are the roots of the polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0, \quad k_i \in \mathbb{N}, \quad k_1 + \ldots + k_r = \text{deg. } P.$$

Let $v_1, \ldots, v_n$, $v_i \neq 0$ for $i = 1, \ldots, n$ be arbitrary elements of $\mathcal{M}$. It can easily be observed that

$$\frac{D(v_1 v_2 \ldots v_n)}{v_1 v_2 \ldots v_n} = \frac{Dv_1}{v_1} + \frac{Dv_2}{v_2} + \ldots + \frac{Dv_n}{v_n}$$

This indicates that each equation

$$\frac{Dx}{x} = \frac{\gamma_{ij}}{(s-\alpha_i)^j} \tag{8}$$

may be solved separately.

Let v_{ij} be solution of the differential equation (8). Then the element

$$v = \prod_{i=1}^{r}\prod_{j=1}^{k_i} v_{ij}$$

is a solution of the differential equation (7).

From (4) it follows that the operator

$$c(s-\alpha_i)^{\gamma_{i1}}, \quad c \in \mathbb{C}$$

is a solution of the operational equation

$$\frac{Dx}{x} = \frac{\gamma_{i1}}{(s-\alpha_i)^1}$$

According to (5) the operator

$$c \exp[\frac{\gamma_{ij}}{-j+1}(s-\alpha_i)^{-j+1}], \quad c \in \mathbb{C}$$

is a solution of the operational equation

$$\frac{Dx}{x} = \frac{\gamma_{ij}}{(s-\alpha_i)^j} \quad j = 2, \ldots, k_i.$$

Finally, we conclude that the element

$$v = c\prod_{i=1}^{r}(s-\alpha_i)^{\gamma_{i1}}\prod_{j=2}^{k_i}\exp[\frac{\gamma_{ij}}{-j+1}(s-\alpha_i)^{-j+1}] \tag{9}$$

satisfies the equation (7).

Naturally, if

1° there exists an opèrator v such that $Dv = w$ and

2° there exists the exponential function $e^{v(\cdot)}$ ([2] p.203),

then according to ([1]) e^v is a solution of the operator equation

$$\frac{Dx}{x} = w.$$

It is known, if $w = P(s)/Q(s)$, where P and Q are polynomials, then there exists an operator v such that $Dv = w$. If deg. $P >$ deg. Q, then the exponential function $e^{v(\cdot)}$ does not exist ([3] p.217). If deg. $P <$ deg. Q and $\gamma_{11} = \gamma_{21} = = \ldots = \gamma_{r1} = 0$ in the equation (7), then the exponential function $e^{v(\cdot)}$ exists. Generally, the existence of an exponential function is an open problem. For example, there exists the exponential function $e^{-s(\cdot)}$ for real arguments but the exponential function $e^{i\,s(\cdot)}$, $i = \sqrt{-1}$ does not exist. In the both cases the series

$$1 + \frac{-s}{1!} + \frac{(-s)^2}{2!} + \cdots$$
$$1 + \frac{i\,s}{1!} + \frac{(i\,s)^2}{2!} + \cdots$$

are divergent ([3] p.160,168).

The operator

$$v = \{-te^{\alpha t}\ln t\}/\{te^{\alpha t}\}$$

is a solution of the operator equation $Dx = (s-\alpha)^{-1}$. Now, the existence of the exponential function $e^{v(\cdot)}$ is an open problem. Finally, the formula (9) can be expended in a power series.

3 Applications

We know that if a function $x \in \mathcal{C}$ has a n-th derivative $\{x^n(t)\}$, which is continuous in the interval $[0,\infty)$, then

$$x^n = s^n x - s^{n-1}x(0) - \cdots - sx^{n-2}(0) - x^{n-1}(0). \qquad \text{([2] p.51)}$$

Assuming that $x \neq 0$ and it is n-time continously differentiable we can rewrite the equation (2) as follows

$$-a_n D[s^n x - s^{n-1}x(0) - \cdots - sx^{n-2}(0) - x^{n-1}(0)] + (a_{n-1}D + b_{n-1})$$

$$[s^{n-1}x - s^{n-2}x(0) - \cdots - sx^{n-3}(0) - x^{n-2}(0)] + \cdots + (-a_0 D + b_0)x = 0$$

This equation can be written in the compact form

$$-P(s)Dx + Q(s)x = R(s), \tag{10}$$

where $P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$ and deg. Q and deg. R are less n. For some initials dates the right hand side of the equation (10) vanishes. We shall consider in this note only such situation. Finally, we conclude that if the element

$$x = c\prod_{i=1}^{r}(s-\alpha_i)^{\gamma_{i1}}\prod_{j=2}^{k_i}\exp[\frac{\gamma_{ij}}{-j+1}(s-\alpha_i)^{-j+1}] \tag{11}$$

is in $\mathcal{C}$, then it is a solution of the differential equation (2).

4 Example

Let us consider the differential equation

$$tx^{(2)}(t) + (-2\alpha t + 1)x^{(1)}(t) + (\alpha^2 t + \beta)x(t) = 0$$

Further we have

$$\frac{Dx}{x} = \frac{-1}{s-\alpha} + \frac{\alpha+\beta}{(s-\alpha)^2}.$$

From (9) it follows that

$$x = (s-\alpha)^{-1}\exp[-(\alpha+\beta)(s-\alpha)^{-1}].$$

The solution x can be represented by the power series

$$x = \frac{1}{s-\alpha} + \frac{-(\alpha+\beta)}{1!(s-\alpha)^2} + \frac{(\alpha+\beta)^2}{2!(s-\alpha)^3}t + \cdots.$$

Finally we obtain

$$x(t) = e^{\alpha t}(1 + \frac{-(\alpha+\beta)}{[1!]^2}t + \frac{(\alpha+\beta)^2}{[2!]^2}t^2 + \cdots).$$

References

[1] Mikusinski, J.: *Remarks on the algebraic derivative in the operational calculus,* Stud. Math. T. XIX. 1960.

[2] Mikusinski, J.: *Operational Calculus,* Vol. I, PWN-Pergamon Press, Warszawa, 1983.

[3] Mikusinski, J.: *Operational Calculus,* Vol. I,II, PWN-Pergamon Press, Warszawa, 1987.

[4] Yosida, K.: *The Algebraic Derivative and Laplace's Differential Equation,* Proc. Japan. Acad. 59 ser A, 1983.

Author's address: Institute of Mathematics
Silesian University
Bankowa 14
40 007 Katowice
Poland