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GENOMORPHISMS OF LATTICES AND SEMILATTICES

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Abstract

A concept of genomorphism was introduced by E. K. Blum. It is a congruence and subalgebra preserving mapping. We characterize all such mappings between two lattices or semilattices.

Key words: lattice, semilattice, isomorphism, homomorphism, genomorphism, isogenomorphism.

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The concept of homomorphism was generalized by numerous authors. In any case, it is reasonable to ask about preservation of subalgebras and induced congruences to make it applicable for algebraic constructions. One of the most general modifications was introduced by E. K. Blum and D. R. Estes [1], [2]:

Definition 1 Let $\mathcal{A} = (A, F)$, $\mathcal{B} = (B, G)$ be algebras (not necessary of the same type). For $M \subseteq A$ or $N \subseteq B$, denote by $A(M)$ or $B(N)$ the subalgebra of A or B generated by the set M or N, respectively. A mapping $\varphi : A \to B$ is called *generative* if for each n-ary $f \in F$ and every $a_1, \dots, a_n$ of A it holds

$$\varphi(f(a_1, \dots, a_n)) \in B(\varphi(a_1), \dots, \varphi(a_n)).$$

A mapping $\psi : A \to B$ is called *congruential* if $\psi(a_i) = \psi(b_i)$ for $i = 1, \dots, n$ imply $\psi(f(a_1, \dots, a_n)) = \psi(f(b_1, \dots, b_n))$ for each n-ary $f \in F$ and $a_1, \dots, a_n$, $b_1, \dots, b_n \in A$. A mapping $\varphi : A \to B$ is called a *genomorphismus* of an algebra $\mathcal{A}$ into $\mathcal{B}$ if it is both generative and congruential. A mapping $\psi : A \to B$ is called an *isogenomorphism* of $\mathcal{A}$ onto $\mathcal{B}$ if it is bijective genomorphism.

Remark 1 A generative mapping need not be congruential: if $\mathbb{Z} = (Z; +, -, O)$ is the group of all integers and $\varphi : \mathbb{Z} \to \mathbb{Z}$ is defined by $\varphi(a) = |a|$ (the absolute value) then φ is generative since $|x+y| \in \mathbb{Z}\ (|x|, |y|)$, $|-y| \in \mathbb{Z}\ (|x|, |y|)$ and $O \in \mathbb{Z}\ (|x|, |y|)$ but it is not congruential:

$$|1| = |1| \quad \text{and} \quad |1| = |-1| \quad \text{but} \quad |1+1| \neq O = |1+(-1)|.$$

On the other hand, for a bijection φ, if φ is generative then it is congruential and hence an isogenomorphism.

The following lemma is called a Genomorphism Theorem in [1]:

Lemma 1 *Let $\mathcal{A} = (A, F)$, $\mathcal{B} = (B, G)$ be algebras and $\varphi : A \to B$ be a surjective genomorphism of the algebra $\mathcal{A}$ onto $\mathcal{B}$. Then the kernel Θ of φ is a congruence and $\mathcal{A}/\Theta$ is isogenomorphic to $\mathcal{B}$.*

Hence, every surjective genomorphism φ of $\mathcal{A}$ onto $\mathcal{B}$ can be expressed in the form

$$\varphi = h \circ \psi,$$

where $h : A \to A/\Theta$ is the natural (cannonical) homomorphism and $\psi : A/\Theta \to B$ is an isogenomorphism. Hence, it is sufficient for our aims only to describe all isogenomorphisms of lattices and semilattices.

Remark 2 If φ is a genomorphism of an algebra $\mathcal{A}$ onto $\mathcal{B}$ and $\mathcal{C}$ is a subalgebra of $\mathcal{A}$ then $\varphi(\mathcal{C})$ need not be a subalgebra of $\mathcal{B}$, see e.g. Note 1 in [1]. However, for each subset M of A, we have

$$\varphi(A(M)) \subseteq B(\varphi(M))$$

(the subalgebra of $\mathcal{A}$ or $\mathcal{B}$ generated by M or $\varphi(M)$, respectively) thus φ is a "subalgebra preserving mapping".

Now, we turn our attention to the case of lattices. Let L be a lattice and $\leq$ its induced order. Elements $a, b \in L$ are *comparable* if $a \leq b$ or $b \leq a$, elements a, b are *incomparable* in the oposite case; this fact is expressed by the symbol $a \parallel b$. If φ is a mapping of L into another lattice and C is a subset of L, denote by $\varphi|C$ the restriction of φ onto C. By $L(a, b)$ is denoted the sublattice generated by a, b. Evidently, for $a \neq b$ we have:
if a, b are comparable then $L(a, b) = \{a, b\}$, i.e. card $L(a, b) = 2$,
if a, b are incomparable then $L(a, b) = \{a, b, a \wedge b, a \vee b\}$, i.e. card $L(a, b) = 4$.

Lemma 2 *Let L_1, L_2 be lattices and $\varphi : L_1 \to L_2$ a bijection. Then for $a, b \in L_1$ we have*

(a) if a, b are comparable then

$$\varphi(a \wedge b) \in L_2(\varphi(a), \varphi(b)) \quad \textit{and} \quad \varphi(a \vee b) \in L_2(\varphi(a), \varphi(b));$$

(b) if $a \parallel b$ and φ is an isogenomorphism then also $\varphi(a) \parallel \varphi(b)$ and the restriction $\varphi|L_1(a,b)$ is either an isomorhism or a dual isomorphism of $L_1(a,b)$ onto $L_2(\varphi(a),\varphi(b))$.

Proof (a) For comparable a, b we have $\{a \vee b, a \wedge b\} = \{a, b\}$ whence the assertion is trivial.

(b) Suppose $a \parallel b$. Then $a \neq a \wedge b \neq b$. If $\varphi(a) \leq \varphi(b)$ then

$$L_2(\varphi(a),\varphi(b)) = \{\varphi(a),\varphi(b)\}.$$

Since φ is an isogenomorphism, we have $\varphi(a \wedge b) \in L_2(\varphi(a),\varphi(b))$ thus either

$$\varphi(a \wedge b) = \varphi(a) \quad \text{or} \quad \varphi(a \wedge b) = \varphi(b)$$

which contradicts to the fact that φ is a bijection. Hence, we have also

$$\varphi(a) \parallel \varphi(b), \quad \text{i.e.} \quad \operatorname{card} L_2(\varphi(a),\varphi(b)) = 4$$

thus

$$\varphi(a) \neq \varphi(b) \neq \varphi(a) \vee \varphi(b) \neq \varphi(a) \neq \varphi(a) \wedge \varphi(b) \neq \varphi(b).$$

Hence, it remains only

$$\varphi(a \vee b) = \varphi(a) \vee \varphi(b) \quad \text{and} \quad \varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$$

or

$$\varphi(a \vee b) = \varphi(a) \wedge \varphi(b) \quad \text{and} \quad \varphi(a \wedge b) = \varphi(a) \vee \varphi(b)$$

proving (b). □

Corollary 1 *Let L_1, L_2 be lattice and $\varphi : L_1 \longrightarrow L_2$ a bijection. If L_1 is a chain then φ is an isogenomorphism of L_1 onto L_2.*

It is a trivial consequence of Lemma 2 (a).

Remark 3 If $\varphi : L_1 \longrightarrow L_2$ is an isogenomorphism then $\varphi^{-1} : L_2 \longrightarrow L_1$ need not be an isogenomorphism, see the following:

Example 1 Let L_1, L_2 be four element lattices such that L_1 is a chain but L_2 not. Then, by the Corollary, $\varphi : L_1 \longrightarrow L_2$ visualized in Fig. 1 is an isogenomorphism but, by (b) of Lemma 2, φ^{-1} is not an isogenomorphism.

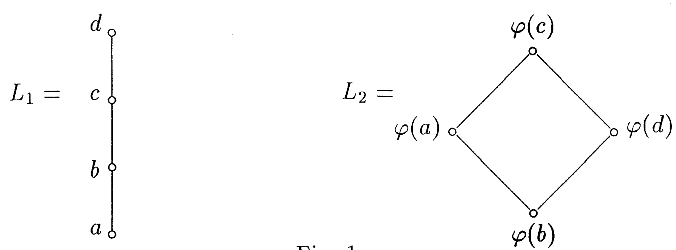

Fig. 1

Example 2 If $\varphi : L_1 \longrightarrow L_2$ is an isogenomorphism then φ^{-1} need not be an isogenomorphism even if L_1 is not a chain. One can it check in the following Fig. 2, where $\varphi : N_5 \longrightarrow M_3$:

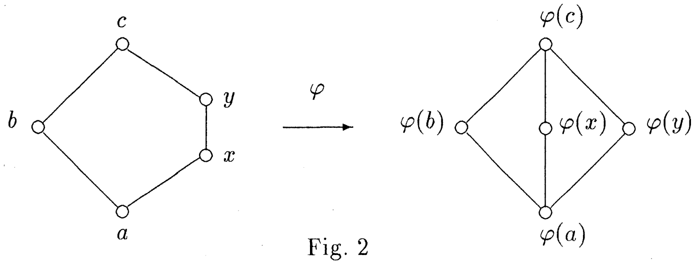

Fig. 2

Theorem 1 *Let L_1, L_2 be lattices and $\varphi : L_1 \longrightarrow L_2$ a bijection. The following conditions are equivalent:*

(1) φ is an isogenomorphism;

(2) if $\varphi(x) \leq \varphi(y)$ then x, y are comparable and if $\varphi(x) \parallel \varphi(y)$ and $x \parallel y$ then $\varphi|L_1(x,y)$ is either an isomorphism or a dual isomorphism of $L_1(x,y)$ onto $L_2(\varphi(x), \varphi(y))$.

Proof (1) $\Rightarrow$ (2): If $\varphi(x) \leq \varphi(y)$ then $\operatorname{card} L_2(\varphi(x), \varphi(y)) = 2$. Suppose $x \parallel y$. Then $\operatorname{card} L_1(x,y) = 4$ but, by (b) of Lemma 2, $\varphi|L_1(x,y)$ is a bijection of $L_1(x,y)$ onto $L_2(\varphi(x), \varphi(y))$, a contradiction. Thus x, y are comparable.

If $\varphi(x) \parallel \varphi(y)$ and $x \parallel y$ then it follows immediately by (b) of Lemma 2.

(2) $\Rightarrow$ (1): Suppose $\varphi : L_1 \longrightarrow L_2$ be a bijective mapping satisfying (2) and $x, y \in L_1$:

(i) if x, y are comparable then

$$\varphi(x \wedge y) \in L_2(\varphi(x), \varphi(y)), \qquad \varphi(x \vee y) \in L_2(\varphi(x), \varphi(y)) \qquad (*)$$

by (a) of Lemma 1.

(ii) if $x \parallel y$ and $\varphi(x) \parallel \varphi(y)$ then $(*)$ follows directly by (2).

(iii) the case $x \parallel y$ and $\varphi(x), \varphi(y)$ comparable is excluded by the first condition of (2).

In all possible cases, φ satisfies $(*)$ thus it is an isogenomorphism. □

For semilattices, the results are similar. Denote by $S(a,b)$ the semilattice generated by a, b, i.e. $S(a,b) = \{a, b, a \wedge b\}$ for $a \parallel b$ and $S(a,b) = \{a, b\}$ for a, b comparable.

Lemma 3 *Let S_1, S_2 be $\wedge$-semilattices and $\varphi : S_1 \to S_2$ be a bijection. For $a, b \in S_1$, we have:*

(a) if a, b are comparable then $\varphi(a \wedge b) \in S_2(\varphi(a), \varphi(b))$;

(b) if $a \parallel b$ and φ is an isogenomorphism then $\varphi(a) \parallel \varphi(b)$ and $\varphi|S_1(a,b)$ is an isomorphism of $S_1(a,b)$ onto $S_2(\varphi(a), \varphi(b))$.

The proof of (a) is the same as those of Lemma 2 for lattices. For (b), we also use that of Lemma 2 but the case of dual isomorphism is excluded because we have only one binary operation.

By using Lemma 3 instead of Lemma 2, we can modify the proof of Theorem 1 to obtain:

Theorem 2 *Let S_1, S_2 be semilattices and $\varphi : S_1 \to S_2$ a bijection. The following conditions are equivalent:*

(1) φ is an isogenomorphism;

(2) if $\varphi(x) \leq \varphi(y)$ then $x \leq y$ and if $\varphi(x) \parallel \varphi(y)$ and $x \parallel y$ then $\varphi|S_1(x,y)$ is an isomorphism of $S_1(x,y)$ onto $S_2(\varphi(x), \varphi(y))$.

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