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Annihilators and Ideals in Distributive and Modular Ordered Sets

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Abstract

The aim of this paper is to show a connection between indexed annihilators and ideals in distributive and modular ordered sets.

Key words: Indexed annihilator, ideal, s-ideal, distributive set, modular set.

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M. Mandelker [6] introduced the concept of annihilator in a lattice. He proved that a lattice L is distributive if and only if every its annihilator is an ideal of L. Annihilators in lattices were intensively studied by B. Davey and J. Nieminen, see [2], [3]. Recently this concept has been generalized also for ordered sets, see [4]. Let us recall some basic concepts.

Let $(S, \leq)$ be an ordered set and X be a subset of S. Denote by

$$L(X) = \{y \in S;\ y \leq x \text{ for each } x \in X\}$$

and

$$U(X) = \{y \in S;\ x \leq y \text{ for each } x \in X\}.$$

If $X = \{a, b\}$ or $X = A \cup B$ or $X = A \cup \{b\}$, we will write briefly $L(a, b)$ or $L(A, B)$ or $L(A, b)$, respectively and, analogously, $U(a, b)$ or $U(A, B)$ or $U(A, b)$. We will also use the notation $UL(X)$ instead of $U(L(X))$ and $LU(X)$ instead of $L(U(X))$.

An ordered set $(S, \leq)$ is called *distributive (modular)* (see [5], [7]) if

$$L(U(a,b),c) = LU(L(a,c),L(b,c))$$
$$\Big(a \leq c \Rightarrow L(U(a,b),c) = LU(a,L(b,c))\Big)$$

holds for each $a, b, c \in S$.

For an ordered set S let $N_0(S)$ denotes a sublattice of the Dedekind–MacNeill hull $N(S)$ of S generated by the set $\{L(x); x \in S\}$.

The following lemma describes distributivity of $N_0(S)$ (see [4]):

Lemma 1 *Let S be an ordered set. Then $N_0(S)$ is distributive iff the following condition holds:*

for every $j, m, x \in S$, if $L(j,x) \subseteq L(m)$, $U(m,x) \subseteq U(j)$, then $j \leq m$.

Using Lemma 1 we obtain the following useful characterizations of distributive ordered sets:

Theorem 1 *For an ordered set S the following conditions are equivalent:*

(1) S is distributive

(2) $N_0(S)$ is distributive

(3) $\forall n \in \mathbb{N}\ \forall x, x_1, \ldots, x_n \in S$:

$$L(x, U(x_1, \ldots, x_n)) = LU(L(x,x_1), \ldots, L(x,x_n))$$

(4) $\forall x, y, z \in S: \ U(L(x,y),z) = UL(U(x,z),U(y,z))$

(5) $\forall a, b, c \in S: \ L(U(a,b),c) \subseteq LU(a, L(b,c))$.

Proof (1) $\Rightarrow$ (2) Let $j, m, x \in S$ be such that $L(j,x) \subseteq L(m), U(m,x) \subseteq U(j)$. Then we have

$$L(j) = L(j) \cap LU(m,x) = L(j, U(m,x)) =$$
$$= LU(L(j,m), L(j,x)) \subseteq LU(L(j,m),m) = L(m),$$

so $j \leq m$. By Lemma 1, $N_0(S)$ is distributive.

(2) $\Rightarrow$ (3) Evidently, $L(x), L(x_1), \ldots, L(x_n) \in N_0(S)$. The join of elements $L(x_1), \ldots, L(x_n)$ in $N_0(S)$ is equal to

$$L(x_1) \vee \ldots \vee L(x_n) = LU(L(x_1), \ldots, L(x_n)) = LU(x_1, \ldots, x_n).$$

Using distributivity of $N_0(S)$ we get

$$L(x, U(x_1, \ldots, x_n)) = L(x) \cap LU(x_1, \ldots, x_n) = L(x) \cap (L(x_1) \vee \ldots \vee L(x_n)) =$$
$$= L(x,x_1) \vee \ldots \vee L(x,x_n) = LU\left(L(x,x_1), \ldots, L(x,x_n)\right).$$

(2) $\Rightarrow$ (4) Condition (4) is equivalent to

$$L(U(x,z),U(y,z)) = LU(L(x,y),z).$$

Using distributivity of $N_0(S)$ we obtain

$$L(U(x,z),U(y,z)) = LU(x,z) \cap LU(y,z) = (L(x) \vee L(z)) \cap (L(y) \vee L(z)) = \\ = L(z) \vee (L(x) \cap L(y)) = L(z) \vee L(x,y) = LU(L(x,y),z).$$

(2) $\Rightarrow$ (5) The following equalities hold in $N_0(S)$:

$$L(U(a,b),c) = LU(a,b) \cap L(c) = (L(a) \vee L(b)) \cap L(c) = L(a,c) \vee L(b,c) = \\ = LU(L(a,c),L(b,c)) \subseteq LU(L(a),L(b,c)) = LU(a,L(b,c)).$$

(3) $\Rightarrow$ (1) We put $n = 2$ in (3).

(5) $\Rightarrow$ (2) Let $L(j,x) \subseteq L(m)$, $U(m,x) \subseteq U(j)$ then

$$L(j) \cap LU(m,x) = L(j) = \\ = L(j,U(m,x)) \subseteq LU(m,L(j,x)) \subseteq LU(m,L(m)) = L(m),$$

so $j \leq m$ and, therefore, $N_0(S)$ is distributive.

(2) $\Rightarrow$ (1) The following equalities are valid in $N_0(S)$:

$$L(U(a,b),c) = LU(a,b) \cap L(c) = (L(a) \vee L(b)) \cap L(c) = \\ = L(a,c) \vee L(b,c) = LU(L(a,c),L(b,c))$$

(4) $\Rightarrow$ (2) Let $L(j,x) \subseteq L(m)$, $U(m,x) \subseteq U(j)$. Then

$$U(L(j,x),m) = U(m) = UL(U(j,m),U(x,m)) \subseteq UL(U(j,m),U(j)) = U(j),$$

so $U(m) \subseteq U(j)$ and $j \leq m$. □

Definition 1 (see [5]) Let $(S,\leq)$ be an ordered set. A subset $I \subseteq S$ is called an ideal of S if $x,y \in I$ imply $LU(x,y) \subseteq I$. An ideal I of $(S,\leq)$ is called *strong* (or *s-ideal*) if for every non-void finite subset $F \subseteq I$ also $LU(F) \subseteq I$. Let $a,b \in S$. By *principal annihilator* $\langle a,b\rangle$ is meant the set

$$\langle a,b\rangle = \{x \in S;\ UL(a,x) \supseteq U(b)\}.$$

Let us note that the set $\mathrm{Id}(S)$ of all ideals of S forms an algebraic lattice with respect to set inclusion.

Let us recall definitions used in [5]:

Definition 2 Let S be an ordered set, $A \subseteq S$, $B \subseteq S$. A *double generalized annihilator (d-annihilator)* in S is the set defined by:

$$\langle A,B\rangle = \{x \in S;\ UL(A,x) \supseteq U(B)\},$$

and, dually, a *double generalized dual annihilator (dual d-annihilator)* in S is:

$$\langle A,B\rangle_d = \{x \in S;\ LU(A,x) \supseteq L(B)\}.$$

If A is a one element set, then the (dual) d-annihilator is called the *(dual) annihilator.*

In [5] it has been shown that the set S is distributive iff every annihilator in S is an ideal. Moreover, in [1] the following lemma was proven:

Lemma 2 *Let S be an ordered set, $a \in S$, $B \subseteq S$. If $U(B) = \emptyset$ then $\langle a, B\rangle = S$, if $U(B) \neq \emptyset$, then*

$$\langle a, B\rangle = \cap\{\langle a, b_\gamma\rangle; b_\gamma \in U(B)\}.$$

Using Lemma 2 and results mentioned above we can prove

Theorem 2 *The set S is distributive iff every principal annihilator in S is an ideal.*

Proof It suffices only to show that if every principal annihilator is an ideal then every annihilator is an ideal. For every $a \in S$, $B \subseteq S$ we have by Lemma 2 $\langle a, B\rangle = S$ or $\langle a, B\rangle = \cap\{\langle a, b_\gamma\rangle; b_\gamma \in U(B)\}$. In the first case $\langle a, B\rangle$ is an ideal, in the second one, $\langle a, B\rangle$ is the intersection of ideals, so it is an ideal again. □

Similarly as distributivity also modularity of a given set can be characterized by the following annihilator condition:

Theorem 3 *The set S is modular iff the following condition (M) holds:*

if $x, a \in S$, $B \subseteq S$ with $B \subseteq L(a)$, $U(x) \supseteq U(B)$ and $y \in \langle a, B\rangle$, then
$$a \in \langle U(x, y), B\rangle.$$

Proof (i) Let S be modular and let a, x, B satisfy assumptions of condition (M). Then $y \in \langle a, B\rangle$ implies $UL(a, y) \supseteq U(B)$. Further, $a \in U(B)$ gives $a \in U(x)$. Due to modularity we have

$$L(a, U(x, y)) = LU(x, L(a, y)) = L(U(x) \cap UL(a, y)) \subseteq LU(B),$$

i.e. $UL(a, U(x, y)) \supseteq ULU(B) = U(B)$, henceforth $a \in \langle U(x, y), B\rangle$.

(ii) Conversely, let S satisfies condition (M) and let $x, z \in S$, $x \leq z$. Then $UL(y, z) \supseteq U(x, L(y, z))$, so $y \in \langle z, \{x\} \cup L(y, z)\rangle$. Further, for every $b \in \{x\} \cup L(y, z)$ it holds $b \leq z$ and, moreover, $U(x) \supseteq U(x, L(y, z))$. Due to condition (M) we obtain $z \in \langle U(x, y), \{x\} \cup L(y, z)\rangle$, hence $UL(z, U(x, y)) \supseteq U(x, L(y, z))$, i.e. $L(z, U(x, y)) \subseteq LU(x, L(y, z))$. Since the converse inclusion is true, S is modular. □

Example 1 The set S depicted in Fig. 1 is not modular. If we put $B = \{b\}$, $x = b$, $y = a$ in the condition (M), then $B \subseteq L(c)$, $U(x) \supseteq U(b) = \{b, c\}$, $a \in \langle c, b\rangle = \{a, b\}$, but $c \notin \langle U(x, a), b\rangle = \{b\}$.

Fig. 1

Let us note that if $(S, \leq)$ is a lattice, the concepts of ideal and strong ideal coincide with the lattice ideal and the concept of annihilator coincides with that of [6] or [2], [3].

As it was shown in [1], annihilators are important tools for some investigations of ordered sets. Unfortunately, there is an essential difference with the set of all ideals of an ordered set, namely the set of all annihilators of S does not form a lattice in a general case:

Example 2 Let $S = \{a, b, c, d, 1\}$ and the ordered set $(S, \leq)$ has the diagram as shown in Fig. 2.

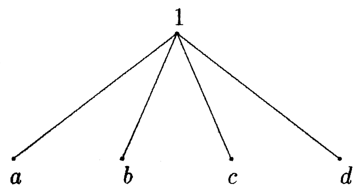

Fig. 2

Then we have

$$\langle a, c\rangle = \{b, c, d\} \quad \text{and} \quad \langle b, c\rangle = \{a, c, d\}$$

but for none $x, y \in S$ we have

$$\langle x, y\rangle = \{c, d\} = \langle a, c\rangle \cap \langle b, c\rangle.$$

To avoid this disadvantage, we can introduce the following new concept (see [1]):

Definition 3 Let $(S, \leq)$ be an ordered set and $a_\gamma, b_\gamma \in S$ for $\gamma \in \Gamma \neq \emptyset$. By an indexed annihilator determined by a_γ, b_γ $(\gamma \in \Gamma)$ is meant the set $\{z \in S;\ UL(z, a_\gamma) \supseteq U(b_\gamma),\ \gamma \in \Gamma\}$.

Remark 1 The set $IA(S)$ of all indexed annihilators of $(S, \leq)$ forms a complete lattice with respect to set inclusion. The greatest element of $IA(S)$ is equal to S and the operation meet coincides with set intersection.

We are able to give an explicite construction of the indexed annihilator $\mathscr{A}(X)$ generated by the set X (see [1]):

Construction 1 Let $X \subseteq S$. For each $a \in S$ denote by $B_a = \{b_{\gamma a}; \gamma_a \in \Gamma_a\}$ the so called polar of a, i.e. the set of all elements $b_{\gamma a} \in S$ satisfying the condition $UL(a, x) \supseteq U(b_{\gamma a})$ for each $x \in X$ $(B_a \neq \emptyset$ since $a \in B_a)$. Put

$$A_a = \cap\{\langle a, b_{\gamma a}\rangle;\ \gamma_a \in \Gamma_a\}.$$

Then

$$\mathscr{A}(X) = \cap\{A_a;\ a \in S\}.$$

Example 3 Let the diagram of $(S, \leq)$ be given in Fig. 3.

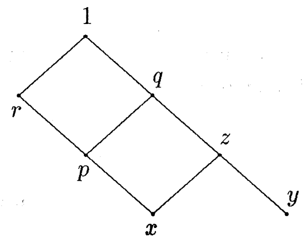

Fig. 3

For $X = \{x, y\}$ we have the polars

$$\begin{aligned} B_1 &= \{1, q, z\} = B_q = B_z \\ B_r &= \{1, r, q, p, z, x\} = B_p = B_r = B_x \\ B_y &= \{1, q, z, y\}. \end{aligned}$$

Hence

$$A_1 = \langle 1, 1\rangle \cap \langle 1, q\rangle \cap \langle 1, z\rangle = S \cap \{q, p, x, y, z\} \cap \{x, y, z\} = \{x, y, z\}$$
$$A_r = \{x, y, z\} = A_q = A_z = A_y = A_p = A_x$$

thus $\mathscr{A}(X) = \{x, y, z\}$.

Hence, it is a natural problem if every ideal of an ordered set S is an indexed annihilator at least in the case of distributive $(S, \leq)$.

Especially, if $(S, \leq)$ is a finite distributive lattice then $\mathrm{Id}(S) = IA(S)$ since every ideal J of S is a principal ideal, i.e. $J = L(x)$ for some $x \in S$ and $L(x) = \langle 1, x\rangle$, where 1 is the greatest element of S. We are proceed to show that the answer to our problem is negative (in infinite case).

Example 4 Let M be an infinite set. Consider the set $A = \operatorname{Exp} M$ of all subsets of M ordered by set inclusion (trivially, $(\operatorname{Exp} M, \subseteq)$ is a distributive lattice). The set J of all finite subsets of A forms an ideal of $(A, \subseteq)$. By using of the Construction of $\mathscr{A}(X)$, we obtain

$$\mathscr{A}(J) = A \neq J.$$

Hence J is not an indexed annihilator of A.

This motivates our investigation for which ideal J of $(S, \leq)$ we have $\mathscr{A}(J) = J$.

Proposition 1 *Let $(S, \leq)$ be a finite ordered set. Then $\mathscr{A}(J) = J$ for every strong ideal J of $(S, \leq)$ (see [1]).*

Let us show that converse is also true, i.e. it holds

Theorem 4 *Let S be a finite distributive set. If J is an ideal for which $\mathscr{A}(J) = J$, then J is an s-ideal.*

Proof We put

$$J = \{j_i; i \in I\}, \qquad B_a = \{b_{\gamma a} \in S;\ L(a, j_i) \subseteq L(b_{\gamma a}) \text{ for every } j_i \in J\}$$

by the Construction. From this we have

$$\cup\{L(a, j_i);\ i \in I\} \subseteq L(b_{\gamma a}).$$

Using LU operator to this inclusion we obtain

$$LU(\cup\{L(a, j_i);\ i \in I\}) \subseteq LUL(b_{\gamma a}) = L(b_{\gamma a}).$$

Since S is distributive, it holds

$$LU(\cup\{L(a, j_i);\ i \in I\}) = L(a, U(J)).$$

Now, let $z \in LU(J)$ be an arbitrary element. If we prove that $z \in \mathscr{A}(J) = J$, then $LU(J) = J$ and so J is an s-ideal.

If $z \in LU(J)$, then $L(z) \subseteq LU(J)$ and therefore

$$L(a, z) \subseteq L(a, U(J)) \subseteq L(b_{\gamma a})$$

for every $a \in S$, $b_{\gamma a} \in B_a$. By the Construction it means that $z \in \mathscr{A}(J)$. □

As consequence of Proposition and Theorem 4 we get

Corollary 1 *Let S be finite distributive set. Then for ideal $J \in \mathrm{Id}(S)$ $\mathscr{A}(J)=J$ holds iff J is an s-ideal.*

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