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Algebras Satisfying Certain Quasiorder Identities

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Abstract

We characterize varieties of algebras having distributive lattices of quasiorders and those having quasiorders permutable with factor or decomposing congruences.

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By a *quasiorder* on an algebra $\mathcal{A} = (A, F)$ is meant a reflexive and transitive binary relation on A having the substitution property with respect to F. The set $\mathrm{Quord}\,\mathcal{A}$ of all quasiorders on $\mathcal{A}$ forms an algebraic lattice with respect to set inclusion, see [1], [5]. Hence, if n is an odd integer and $c_0, c_1, \dots, c_n$ are elements of $\mathcal{A}$, there exists the least quasiorder on $\mathcal{A}$ containing the pairs $\langle c_0, c_1\rangle$, $\langle c_2, c_3\rangle$, …, $\langle c_{n-1}, c_n\rangle$; denote it by $Q(\langle c_0, c_1\rangle, \dots, \langle c_{n-1}, c_n\rangle)$. If $n = 1$, we denote $Q(\langle c_0, c_1\rangle)$ briefly by $Q(c_0, c_1)$. Moreever, denote by $R(c_0, c_1)$ the least reflexive binary relation on A having the substitution property w.r.t. F and containing the pair $\langle c_0, c_1\rangle$ (the set of all reflexive relations on A with the substitution property forms a complete lattice also, see e.g. [1]).

Of course, meet in the lattice $\mathrm{Quord}\,\mathcal{A}$ concides with intersection, the identity relation ω is the least element of $\mathrm{Quord}\,\mathcal{A}$ and the greatest element of $\mathrm{Quord}\,\mathcal{A}$ is $A \times A$. Denote by $\vee$ join in $\mathrm{Quord}\,\mathcal{A}$.

Lemma 1 *Let n be an odd integer and $c_0, c_1, \ldots, c_n$ be elements of an algebra $\mathcal{A} = (A, F)$. Let $R, S \in \operatorname{Quord} \mathcal{A}$. Then*

(a) $Q(\langle c_0, c_1 \rangle, \ldots, \langle c_{n-1}, c_n \rangle) = Q(c_0, c_1) \vee \cdots \vee Q(c_{n-1}, c_n)$;

(b) $\langle a, b \rangle \in Q(c_0, c_1)$ *if and only if there exists an integer $k \geq 1$ such that*

$$\langle a, b \rangle \in R(c_0, c_1) \circ R(c_0, c_1) \circ \cdots \circ R(c_0, c_1) \quad (k \text{ factors});$$

(c) $\langle a, b \rangle \in Q(c_0, c_1)$ *if and only if there exists an integer $m \geq 1$ and unary polynomials $p_1, \ldots, p_m$ over $\mathcal{A}$ such that*

$$a = p_1(c_0), \ b = p_m(c_1) \text{ and } p_i(c_1) = p_{i+1}(c_0)$$

for $i = 1, \ldots, m-1$;

(d) $R \vee S = \cup_{i=1}^{\infty} R \circ S \circ R \circ \cdots \quad$ *(i factors in the i-th member).*

Proof The proof of (a) is almost evident. For (b), see e.g. [1], for (c) see e.g. [5]. The assertion (d) is also well-known. □

We say that a variety $\mathcal{V}$ is *quasiorder-distributive* if $\operatorname{Quord} \mathcal{A}$ is a distributive lattice for each $\mathcal{A} \in \mathcal{V}$. It was shown in [4] that $\mathcal{V}$ is quasiorder-distributive whenever $\mathcal{V}$ contains a majority term $m(x, y, z)$, i.e. m satisfies

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x \, .$$

G. Czédli and A. Lenkehegyi [5] found a weak Mal'cev condition characterizing quasiorder-distributive varieties of ordered algebras. We are going to give a similar characterization which works for any variety and it is a bit more simpler than that of [5].

For a variety $\mathcal{V}$, denote by $F_v(x_0, \ldots, x_n)$ the free algebra of $\mathcal{V}$ generated by the free generators $x_0, \ldots, x_n$. For the sake of brevity, we denote by $\overline{x}$ the sequence $x_0, \ldots, x_n$.

Theorem 1 *A variety $\mathcal{V}$ is quasiorder-distributive if and only if for any even integer $k > 0$ there exists an integer n and $(k+2)$-ary terms $t_0, t_1, \ldots, t_n$ and $(k+2)$-ary terms $v_0^i, v_2^i, \ldots, v_k^i$ for i even and terms $w_1^i, w_3^i, \ldots, w_{k+1}^i$ for i odd such that*

$$t_0(x_0, \overline{x}) = x_0, \quad t_n(x_k, \overline{x}) = x_k, \quad t_i(x_0, \overline{x}) = t_{i-1}(x_k, \overline{x}) \quad \text{for } i = 1, \ldots, n$$
$$t_i(x_0, \overline{x}) = v_0^i(x_0, \overline{x}), \quad t_i(x_k, \overline{x}) = v_{k-2}^i(x_{k-1}, \overline{x}),$$
$$v_j^i(x_j, \overline{x}) = v_{j-2}^i(x_{j-1}, \overline{x}) \quad \text{for } i \text{ even and } j = 2, 4, \ldots, k$$
$$t_i(x_0, \overline{x}) = w_1^i(x_1, \overline{x}), \quad t_i(x_k, \overline{x}) = w_{k-1}^i(x_k, \overline{x}),$$
$$w_j^i(x_j, \overline{x}) = w_{j-2}^i(x_{i-1}, \overline{x}) \quad \text{for } i \text{ odd and } j = 3, 5, \ldots, k+1.$$

Proof Let $\mathcal{V}$ be a quasiorder-distributive variety. Let $\mathcal{A} = F_v(x_0, \ldots, x_k)$, $Q = Q(x_0, x_k)$, $R = Q(\langle x_0, x_1 \rangle, \langle x_2, x_3 \rangle, \ldots, \langle x_{k-2}, x_{k-1} \rangle)$ and $S = Q(\langle x_1, x_2 \rangle, \langle x_3, x_4 \rangle, \ldots, \langle x_{k-1}, x_k \rangle)$ for even integer k. Applying our Lemma 1, we have

$\langle x_0, x_k \rangle \in Q \cap (R \cup S)$ thus also $\langle x_0, x_k \rangle \in (Q \cap R) \vee (Q \cap S)$. Applying (d) of Lemma 1, there exist an integer $n \geq 1$ and elements $c_0, c_1, \dots, c_n$ of $\mathcal{A}$ such that $x_0 = c_0$, $x_k = c_n$ and $\langle c_i, c_{i+1} \rangle \in Q \cap R$ for i even and $\langle c_i, c_{i+1} \rangle \in Q \cap S$ for i odd, i.e. $\langle c_i, c_{i+1} \rangle \in Q$ for $i = 1, 2, \dots, n-1$, $\langle c_i, c_{i+1} \rangle \in R$ for i even and $\langle c_i, c_{i+1} \rangle \in S$ for i odd. Hence, by (c) of Lemma 1, there exists $(k+2)$-ary terms $t_0, t_1, \dots, t_n$ such that $t_0(x_0, \overline{x}) = x_0$, $t_n(x_k, \overline{x}) = x_k$ and $t_{i-1}(x_k, \overline{x}) = t_i(x_0, \overline{x})$ for $i = 1, \dots, n$. Further, $\langle c_i, c_{i+1} \rangle \in R$ for i even yields the existence of elements $d_0^i, d_2^i \dots, d_k^i$ which $c_i = d_0^i$, $c_{i+1} = d_k^i$ and $\langle d_0^i, d_2^i \rangle \in Q(x_0, x_1)$, $\langle d_2^i, d_4^i \rangle \in Q(x_2, x_3), \dots, \langle d_{k-2}^i, d_k^i \rangle \in Q(x_{k-2}, x_{k-1})$.

Hence, there exist $(k+2)$-ary terms v_j^i satisfying $v_j^i(x_j, \overline{x}) = v_{j-2}^i(x_{j-1}, \overline{x})$ for i even and $j = 2, 4, \dots, k$ and

$$t_i(x_0, \overline{x}) = c_i = d_0^i = v_0^i(x_0, \overline{x}),$$
$$t_i(x_k, \overline{x}) = c_{i+1} = d_k^i = v_k^i(x_{k-1}, \overline{x}).$$

Analogously, $\langle c_i, c_{i+1} \rangle \in S$ for i odd yields the existence of elements $e_1^i, e_3^i, \dots, e_{k+1}^i$ of $\mathcal{A}$ such that $c_i = e_1^i$, $c_{i+1} = e_{k+1}^i$ and $\langle e_1^i, e_3^i \rangle \in Q(x_1, x_2)$, $\langle e_3^i, e_5^i \rangle \in Q(x_3, x_4), \dots, \langle e_{k-1}^i, e_{k+1}^i \rangle \in Q(x_{k-1}, x_k)$, whence we infer the existence of $(k+2)$-ary terms w_j^i satisfying $w_j^i(x_j, \overline{x}) = w_{j-2}^i(x_{j-1}, \overline{x})$ for i odd and $j = 3, 5 \dots, k+1$ and $t_i(x_0, \overline{x}) = w_1^i(x_1, \overline{x})$, $t_i(x_k, \overline{x}) = w_{k+1}^i(x_k, \overline{x})$.

Conversely, let $\mathcal{V}$ be a variety satisfying the term identities of the assumption and let $\mathcal{A} \in \mathcal{V}$ and $Q, R, S \in \operatorname{Quord} \mathcal{A}$. Suppose $\langle a, b \rangle \in Q \wedge (R \vee S)$. Then $\langle a, b \rangle \in Q$ and, by Lemma 1, $\langle a, b \rangle \in R \circ S \circ \dots \circ R \circ S$ with k factors (without loss of generality we can suppose that k is even since R, S are reflexive). Hence, there exist elements $a_0, a_1, \dots, a_k$ of $\mathcal{A}$ with $a = a_0$, $b = a_k$ and $\langle a_j, a_{j+1} \rangle \in R$ for j even and $\langle a_j, a_{j+1} \rangle \in S$ for j odd. Denote by $\overline{a}$ the sequence $a_0, \dots, a_k$ and set $c_i = t_i(a_0, \overline{a})$ and

$$d_j^i = v_j^i(a_j, \overline{a}) \quad \text{for } i \text{ even}, \ j = 0, 2, \dots, k-2, \quad d_k^i = v_k^i(a_{k-1}, \overline{a})$$
$$e_j^i = w_j^i(a_j, \overline{a}) \quad \text{for } i \text{ odd}, \ j = 1, 3, \dots, k-1, \quad e_{k+1}^i = w_{k+1}^i(a_k, \overline{a}).$$

Applying the identities, we have $c_0 = a$, $c_n = b$ and

$$\langle c_i, c_{i+1} \rangle = \langle t_i(a_0, \overline{a}), t_i(a_k, \overline{a}) \rangle = \langle t_i(a, \overline{a}), t_i(b, \overline{a}) \rangle \in Q$$

for $i = 0, \dots, n-1$.

For i even we obtain $c_i = d_0^i$, $c_{i+1} = d_k^i$ and

$$\langle d_j^i, d_{j+1}^i \rangle \in Q(a_j, a_{j+1}) \subseteq R$$

for $j = 0, 2, \dots, k-2$. In account of transitivity, it yields $\langle c_i, c_{i+1} \rangle \in R$.

Analogously, for i odd we have $c_i = e_1^i$, $c_{i+1} = e_{k+1}^i$ and

$$\langle e_j^i, e_{j+1}^i \rangle \in Q(a_j, a_{j+1})$$

for $j = 1, 3, \dots, k-1$ whence $\langle c_i, c_{i+1} \rangle \in S$.

Together, we obtain

$$\langle a, b\rangle \in (Q \cap R) \vee (Q \cap S)$$

finishing the proof. □

Now, we turn our attention to another problem which has been recently studied for congruences in [3]. Let $\mathcal{A}_1, \mathcal{A}_2$ be two algebras of the same type. The congruences Π_1, Π_2 of $\mathcal{A}_1 \times \mathcal{A}_2$ induced by the projections pr_1, pr_2 of $\mathcal{A}_1 \times \mathcal{A}_2$ onto $\mathcal{A}_1$ or $\mathcal{A}_2$, respectively, are called *factor congruences.* A variety $\mathcal{V}$ has *quasiorders permutable with factor congruences* if for any $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{V}$ and each $Q \in \operatorname{Quord} \mathcal{A}$

$$Q \cdot \Pi_1 = \Pi_1 \cdot Q \quad \text{and} \quad Q \cdot \Pi_2 = \Pi_2 \cdot Q$$

Theorem 2 *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) $\mathcal{V}$ has quasiorders permutable with factor congruences;

(2) there exist $(n+1)$-ary terms $t_1, \ldots, t_k$ and binary terms $r_1, \ldots, r_n$ and ternary terms $s_1, \ldots, s_n$ such that

$$\begin{gathered} x = t_1(x, r_1(x,y), \ldots, r_n(x,y)) \\ y = t_k(y, r_1(x,y), \ldots, r_n(x,y)) \\ z = t_k(y, s_1(x,y,z), \ldots, s_n(x,y,z)) \\ t_j(y, r_1(x,y), \ldots, r_n(x,y)) = t_{j+1}(x, r_1(x,y), \ldots, r_n(x,y)) \\ t_j(y, s_1(x,y,z), \ldots, s_n(x,y,z)) = t_{j+1}(x, s_1(x,y,z), \ldots, s_n(x,y,z)) \end{gathered}$$

for $j = 1, \ldots, k-1$.

Proof

(1) ⇒ (2): Let $\mathcal{A}_1 = F_v(x,y)$ and $\mathcal{A}_2 = F_v(x,y,z)$. Let $Q = Q((x,x),(y,y)) \in \operatorname{Quord} \mathcal{A}_1 \times \mathcal{A}_2$. Then

$$(x,x) Q (y,y) \Pi_1 (y,z)$$

and, by (1), there is d of $\mathcal{A}_2$ such that

$$(x,x) \Pi_1 (x,d) Q (y,z)$$

thus $\langle (x,d),(y,z)\rangle \in Q((x,x),(y,y))$. Applying Lemma 1, there are unary polynomials $\varphi_1, \ldots, \varphi_k$ over $\mathcal{A}_1 \times \mathcal{A}_2$ such that

$$(x,d) = \varphi_1((x,x)), \quad (y,z) = \varphi_k((y,y))$$

and

$$\varphi_j((y,y)) = \varphi_{j+1}((x,x)) \quad \text{for } j = 1, \ldots, k. \tag{$*$}$$

Since $\mathcal{A}_1, \mathcal{A}_2$ are free algebras, their elements are binary or ternary terms, respectively. Hence, there are $(n+1)$-ary terms $t_1, \ldots, t_k$ and binary terms $r_1, \ldots, r_n$ and ternary terms $s_1, \ldots, s_n$ such that

$$\varphi_j(v) = t_j(v, (r_1(x,y), s_1(x,y,z)), \ldots, (r_n(x,y), s_n(x,y)))$$

for each $v \in \mathcal{A}_1 \times \mathcal{A}_2$. Applying it in $(*)$ and reading it coordinatewise, we obtain (2).

(2) $\Rightarrow$ (1): Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{V}$ and $Q \in \operatorname{Quord} \mathcal{A}_1 \times \mathcal{A}_2$. Suppose

$$(a_1, a_2) Q (c_1, b_2) \Pi_1 (c_1, c_2).$$

Set $d = t_1(a_2, b_2, s_1(a_2, b_2, c_2), \ldots, s_n(a_2, b_2, c_2))$. Then

$$(a_1, d_1) = t_1((a_1, a_2), (r_1(a_1, c_1), s_1(a_2, b_2, c_2)), \ldots, (r_n(a_1, c_1), s_n(a_2, b_2, c_2)))$$
$$(c_1, c_2) = t_k((c_1, b_1), (r_1(a_1, c_1), s_1(a_2, b_2, c_2)), \ldots, (r_n(a_1, c_1), s_n(a_2, b_2, c_2)))$$

and

$$t_j((c_1, b_2), (r_1(a_1, c_1), s_1(a_2, b_2, c_2)), \ldots) = \\ = t_{j+1}((a_1, a_2), (r_1(a_1, c_1), s_1(a_2, b_2, c_2) \ldots)$$

for $j = 1, \ldots, k-1$, whence $(a_1, d) Q (c_1, c_2)$. Thus $(a_1, a_2) \Pi_1 (a_1, d) Q (c_1, c_2)$ proving $Q \cdot \Pi_1 \subseteq \Pi_1 \cdot Q$. Conversely,

$$\Pi_1 \cdot Q \subseteq \Pi_1 \cdot (Q \cdot \Pi_1) \subseteq \Pi_1 \cdot (\Pi_1 \cdot Q) = \Pi_1 \cdot Q.$$

Thus also $Q \cdot \Pi_1 = \Pi_1 \cdot Q$.

By interchanging of the first and second coordinate, we infer also $Q \cdot \Pi_2 = \Pi_2 \cdot Q$ thus $\mathcal{V}$ has quasiorders permutable with factor congruences. □

Example 1 For a variety $\mathcal{V}$ of lattices, take $n = 2$, $k = 1$ and $t_1(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$ and $r_1(x, y) = x \vee y$, $r_2(x, y) = x \wedge y$, $s_1(x, y, z) = y \wedge z$, $s_2(x, y, z) = z$.

Then

$$t_1(x, r_1(x, y), r_2(x, y)) = (x \wedge (x \vee y)) \vee (x \wedge y) = x$$
$$t_1(y, r_1(x, y), r_2(x, y)) = (y \wedge (x \vee y)) \vee (x \wedge y) = y$$
$$t_1(y, s_1(x, y, z), s_2(x, y, z)) = (y \wedge (y \wedge z)) \vee z = z$$

thus $\mathcal{V}$ has quasiorders permutable with factor congruences. Let us note that a nontrivial lattice variety have not permutable quasiorders since due to [4], $\mathcal{V}$ has permutable quasiorders if and only if $\mathcal{V}$ has permutable congruences which is not the case of $\mathcal{V}$ in general.

Let $\mathcal{A}_1, \mathcal{A}_2$ be algebras of the same type and $\mathcal{B}$ be a subalgebra of $\mathcal{A}_1 \times \mathcal{A}_2$. By the *decomposing congruences of* $\mathcal{B}$ are called the congruences θ_1, θ_2 defined by the setting

$$\theta_1 = \{\langle (b_1, b_2), (c_1, c_2) \rangle \in B^2;\ b_1 = c_1\}$$
$$\theta_2 = \{\langle (b_1, b_2), (c_1, c_2) \rangle \in B^2;\ b_2 = c_2\}$$

A variety $\mathcal{V}$ has *quasiorders permutable with decomposing congruences* if for any $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{V}$, every subalgebra $\mathcal{B}$ of $\mathcal{A}_1 \times \mathcal{A}_2$ and each $Q \in \operatorname{Quord} \mathcal{A}_1 \times \mathcal{A}_2$ it holds $Q \cdot \theta_1 = \theta_1 \cdot Q$ and $Q \cdot \theta_2 = \theta_2 \cdot Q$.

Theorem 3 *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) $\mathcal{V}$ has quasiorders permutable with decomposing congruences;

(2) there exist 4-ary terms $t_1, \dots, t_k$ such that

$$x = t_1(x,x,y,y), \quad y = t_k(y,x,y,y), \quad z = t_k(y,x,y,z),$$
$$t_j(y,x,y,z) = t_{j+1}(x,x,y,z) \quad \text{for } j = 1,\dots,k-1.$$

Proof

(1) $\Rightarrow$ (2): Set $\mathcal{A}_1 = F_v(x,y)$, $\mathcal{A}_2 = F_v(x,y,z)$ and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}_1 \times \mathcal{A}_2$ generated by the elements $(x,x),(y,y),(y,z)$ of $\mathcal{A}_1 \times \mathcal{A}_2$. Let $Q = Q((x,x),(y,y)) \in \operatorname{Quord}\mathcal{B}$. Then

$$(x,x)Q(y,y)\theta_1(y,z)$$

and by (1), there exists an element (x,d) of $\mathcal{B}$ with

$$(x,x)\theta_1(x,d)Q(y,z)\,.$$

By Lemma 1, there exists unary polynomials $\tau_1,\dots,\tau_k$ over $\mathcal{B}$ such that

$$(x,d) = \tau_1((x,x)), \quad (y,z) = \tau_k((y,y))$$

and

$$\tau_j((y,y)) = \tau_{j+1}((x,x)) \quad \text{for } j = 1,\dots,k-1.$$

Since $\mathcal{B}$ has just three generators $(x,x),(y,y),(y,z)$, there exist 4-ary terms $t_1,\dots,t_k$ with

$$\tau_j((v_1,v_2)) = t_j((v_1,v_2),(x,x),(y,y),(y,z))\,.$$

Writing the foregoing identities componentwise, we obtain

$$x = t_1(x,x,y,y), \quad d = t_1(y,x,y,z),$$
$$t_j(y,x,y,y) = t_{j+1}(x,x,y,y),$$
$$t_j(y,x,y,z) = t_{j+1}(x,x,y,z) \quad \text{for } j = 1,\dots,k-1,$$
$$y = t_k(y,x,y,y), \quad z = t_k(y,x,y,z).$$

However, the 4-th identity implies the third one thus it can be omitted as well as the second one. The remaining identities are those of (2).

(2) $\Rightarrow$ (1): Let $\mathcal{V}$ be a variety satisfying (2) and let $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{V}$ and $\mathcal{B}$ be a subalgebra of $\mathcal{A}_1 \times \mathcal{A}_2$. Suppose $Q \in \operatorname{Quord}\mathcal{B}$ and

$$(a_1,a_2)Q(b_1,b_2)\theta_1(c_1,c_2)\,.$$

for some $(a_1,a_2),(b_1,b_2),(c_1,c_2)$ of $\mathcal{B}$. Take $d = t_1(a_2,a_2,c_2,b_2)$. We infer

$$(a_1,d) = t_1((a_1,a_2),(a_1,a_2),(c_1,c_2),(c_1,b_2))$$
$$t_j((c_1,b_2),(a_1,a_2),(c_1,c_2),(c_1,b_2)) =$$
$$= t_{j+1}((a_1,a_2),(a_1,a_2),(c_1,c_2),(c_1,b_2)) \quad \text{for } j = 1,\dots,k-1$$
$$(c_1,c_2) = t_k((c_1,b_2),(a_1,a_2),(c_1,c_2),(c_1,b_2))$$

whence $(a_1,d) \in \mathcal{B}$ and $(a_1,d)Q(c_1,c_2)$ proving $Q \cdot \theta_1 \subseteq \theta_1 \cdot Q$. The rest of the proof is a routine way. $\square$

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