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Natural Tensor Fields of Type (0,2) on the Tangent and Cotangent Bundles of a Semi-Riemannian Manifold *

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Abstract

To any (0,2)-tensor field on the tangent and cotangent bundles of a semi-Riemannian manifold, we associate a global matrix function 'mutatis mutandis' as in the Riemannian case. Based on this fact, natural (0,2)-tensor fields on these bundles are defined and characterized.

Key words: Connection map, tangent bundle, tensor field.

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1 Introduction

In [1] and [2] we lifted to suitable bundles (0,2)-tensor fields defined on the tangent and cotangent bundles over manifolds endowed with Riemannian metrics so that to look at them as global matrix functions. These matrix representations allowed us to define and classify natural (0,2)-tensor fields with respect to Riemannian metrics from a simple point of view. The main result that let

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us characterize these tensor fields is Lemma 3.1 of [1], which is related to the orthogonal group of $\mathbb{R}^n$. In this paper the main result is Theorem 2.1, which is a generalized version of this Lemma to non-degenerate symmetric bilinear forms on $\mathbb{R}^n$. We apply this result to charaterize natural (0,2)-tensor fields—defined in the sense of [1], [2]—on tangent (Proposition 3.1) and cotangent (Proposition 4.1) bundles over semi-Riemannian manifolds.

Throughout, all geometric objects are assumed to be differentiable, i.e. C^∞

2 The main result

For any integer $n \geqslant 2$ and $\nu = 0, 1, \ldots, n-1$, let us define the following diagonal $n \times n$ matrices:

$$I_0 = I \text{ (the unit matrix)} \qquad I_\nu = \begin{pmatrix} -1 & & & & & \\ & \overset{\nu}{\ddots} & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \overset{n-\nu}{\ddots} & \\ & & & & & 1 \end{pmatrix}$$

Let $\mathbf{q} : \mathbb{R}^n \to \mathbb{R}$ be the quadratic form defined by $\mathbf{q}(x) = x \,.\, I_\nu \,.\, x^t$, where x stands for a row vector. Let $\mathcal{O}_\nu(n)$ be the orthogonal group associated to $\mathbf{q}$, i.e., $a \in \mathcal{O}_\nu(n)$ if and only if $a \,.\, I_\nu \,.\, a^t = I_\nu$.

Theorem 2.1 *With the notations introduced above, if a differentiable map $A : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ satisfies $A(x \,.\, a) = a^t \,.\, A(x) \,.\, a$ for any $a \in \mathcal{O}_\nu(n)$ and $x \in \mathbb{R}^n$, then there exist differentiable functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ such that*

$$A(x) = \alpha(\mathbf{q}(x)) \,.\, I_\nu + \beta(\mathbf{q}(x)) \,.\, x^t \,.\, x \,.$$

Proof If $\nu = 0$, the result is Lemma 3.1 of [1], so we assume that $\nu = 1, \ldots, n-1$. We will divide the proof in three steps.

Step 1. Let $<,>$ be the bilinear form associated to $\mathbf{q}$. Since $\nu = 1 \ldots, n-1$, there exist two linearly independent vectors $u, v \in \mathbb{R}^n$ such that $\mathbf{q}(u) = \mathbf{q}(v) = 0$ and $< u, v >= \frac{1}{2}$. Hence, if we define $\theta(t) = u + t \,.\, v$, one gets that $\mathbf{q}(\theta(t)) = t$ for any $t \in \mathbb{R}$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuous function at zero such that $F(x \,.\, a) = F(x)$ for any $x \in \mathbb{R}^n$ and $a \in \mathcal{O}_\nu(n)$. If $F \circ \theta : \mathbb{R} \to \mathbb{R}$ is differentiable we conclude that there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = F(u) + \mathbf{q}(x) \,.\, f(\mathbf{q}(x)) \tag{2.1}$$

for any $x \in \mathbb{R}^n$. In fact, let $x \in \mathbb{R}^n$ be a vector satisfying $\mathbf{q}(x) = t \neq 0$. Since $\mathcal{O}_\nu(n)$ acts irreducibly on $\mathbb{R}^n$ and $x \neq 0$, $\theta(t) \neq 0$, the condition $\mathbf{q}(x) = \mathbf{q}(\theta(t))$ implies that there exists a matrix $a \in \mathcal{O}_\nu(n)$ such that $x \,.\, a = \theta(t)$. Setting $h = F - F(u)$, one gets:

$$\frac{h(x)}{\mathbf{q}(x)} = \frac{h(x \,.\, a)}{\mathbf{q}(x \,.\, a)} = \frac{h(\theta(t))}{\mathbf{q}(\theta(t))} = \frac{h(\theta(t))}{t} = \int_0^1 \frac{d}{ds}(h \circ \theta)(s \,.\, t) ds$$

Hence, if we define

$$f(t) = \int_0^1 \frac{d}{ds}(h \circ \theta)(s \,.\, t)\, ds$$

for any $t \in \mathbb{R}$, it follows that f is differentiable and satisfies (2.1).

Step 2. Here we prove that a differentiable map $B : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ satisfying

$$B(x \,.\, a) = a^{-1} \,.\, B(x) \,.\, a \tag{2.2}$$

for any $x \in \mathbb{R}^n$ and $a \in \mathcal{O}_\nu(n)$ may be written in the form

$$B(x) = \alpha(\mathbf{q}(x)) \,.\, I + \beta(\mathbf{q}(x)) \,.\, I_\nu \,.\, x^t \,.\, x \tag{2.3}$$

where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are differentiable functions.

We observe that the function $F : \mathbb{R}^n \to \mathbb{R}$ defined by

$$F(x) = \operatorname{tr}(B(x)) \tag{2.4}$$

satisfies $F(x \,.\, a) = F(x)$ for any $a \in \mathcal{O}_\nu(n)$. Hence, by the first step, there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = F(u) + \mathbf{q}(x) \,.\, f(\mathbf{q}(x)) \tag{2.5}$$

On the other hand, for any $x \in \mathbb{R}^n$ satisfying $\mathbf{q}(x) \neq 0$, let us denote, respectively, with $\mathbb{R} \,.\, x$, $(\mathbb{R} \,.\, x)^\perp$, the subspace generated by x, and the orthogonal complement with respect to $<,>$; hence, $\mathbb{R}^n = \mathbb{R} \,.\, x \oplus (\mathbb{R} \,.\, x)^\perp$. If $\mathcal{O}_x \subset \mathcal{O}_\nu(n)$ denotes the isotropy group at x, for any $a \in \mathcal{O}_x$ one gets that $B(x) = B(x \,.\, a) = a^{-1} \,.\, B(x) \,.\, a$. Let T_x be the orthogonal symmetry with respect to $\mathbf{q}$ associated to the 1-dimensional subspace generated by x. It is clear that $T_x(y) = y \,.\, a$ with $a \in \mathcal{O}_x$; consequently $\mathbb{R} \,.\, x$ and $(\mathbb{R} \,.\, x)^\perp$ are invariant under $B(x)$. We may identify $\mathcal{O}_x$ with the orthogonal group associated to the quadratic form $\mathbf{q}$ restricted to $(\mathbb{R} \,.\, x)^\perp$. Consequently, there exist real numbers $G(x)$ and $\tilde{\alpha}(x)$ such that

$$x \,.\, B(x) = G(x) \,.\, x \tag{2.6}$$

and

$$y \,.\, B(x) = \tilde{\alpha}(x) \,.\, y \tag{2.7}$$

for any $y \in (\mathbb{R} \,.\, x)^\perp$. Since $G(x)$ is given by

$$G(x) = \frac{x \,.\, B(x) \,.\, x^t}{x \,.\, x^t} \tag{2.8}$$

let $G : \mathbb{R}^n - \{0\} \to \mathbb{R}$ be the differentiable map defined by (2.8). Condition $B(0) = a^{-1} \,.\, B(0) \,.\, a$ for any $a \in \mathcal{O}_\nu$ implies that $B(0) = \mu \,.\, I$ for some $\mu \in \mathbb{R}$, and since

$$\lim_{x \to 0} \frac{x \,.\, B(x) \,.\, x^t}{x \,.\, x^t} = \mu$$

the map G can be continuously extended to $\mathbb{R}^n$ by defining $G(0) = \mu$. This function satisfies

$$G(x \, . \, a) = G(x) \tag{2.9}$$

for any $x \in \mathbb{R}^n$ and $a \in \mathcal{O}_\nu(n)$. In fact, since the set of vectors $x \in \mathbb{R}^n$ such that $\mathbf{q}(x) \neq 0$ is dense in $\mathbb{R}^n$, we only need to check (2.9) on vectors belonging to this subset. But this is an immediate consequence of (2.2), (2.6) and (2.8).

On the other hand, since $\theta(t) \neq 0$ for any $t \in \mathbb{R}$, the function $G \circ \theta : \mathbb{R} \to \mathbb{R}$ is differentiable; hence, there exists a differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that

$$G(x) = G(u) + \mathbf{q}(x) \, . \, g(\mathbf{q}(x)) \tag{2.10}$$

for any $x \in \mathbb{R}^n$. From (eq:26) and (eq:27) it follows that for any $x \in \mathbb{R}^n$ such that $\mathbf{q}(x) \neq 0$, the real number $\tilde{\alpha}(x)$ is given by

$$\tilde{\alpha}(x) = \frac{F(x) - G(x)}{n - 1} \tag{2.11}$$

Let us define—via (2.11)—the differentiable function $\tilde{\alpha} : \mathbb{R}^n - \{0\} \to \mathbb{R}$ and

$$\tilde{\beta}(x) = \frac{n \, . \, G(x) - F(x)}{(n-1) \, . \, \mathbf{q}(x)} \tag{2.12}$$

if $\mathbf{q}(x) \neq 0$. Then, one gets

$$B(x) = \tilde{\alpha}(x) \, . \, I + \tilde{\beta}(x) \, . \, I_\nu \, . \, x^t \, . \, x \tag{2.13}$$

for any $x \in \mathbb{R}^n$ such that $\mathbf{q}(x) \neq 0$. From (2.4), (2.5) and (2.10) it follows that $F(u) = n \, . \, \mu$ and $G(u) = \mu$. Hence, if we define the differentiable mappings $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ by

$$\alpha(t) = \mu + t \, . \, \frac{f(t) - g(t)}{n - 1} \tag{2.14}$$

and

$$\beta(t) = \frac{n \, . \, g(t) - f(t)}{n - 1} \tag{2.15}$$

then $\tilde{\alpha}(x) = \alpha(\mathbf{q}(x))$ and $\tilde{\beta}(x) = \beta(\mathbf{q}(x))$ if $\mathbf{q}(x) \neq 0$. Consequently, from (2.13) one gets

$$B(x) = \alpha(\mathbf{q}(x)) \, . \, I + \beta(\mathbf{q}(x)) \, . \, I_\nu \, . \, x^t \, . \, x \tag{2.16}$$

Now, equality (2.16) holds in $\mathbb{R}^n$ because both members are differentiable maps on $\mathbb{R}^n$ and coincide in a dense subset of $\mathbb{R}^n$.

Step 3. Let A be a matrix satisfying the hypothesis of the theorem. The result now follows from the second step applied to $B = I_\nu \, . \, A$. □

3 Natural (0,2)-tensor fields on tangent bundles

Let (M, g) be a semi-Riemannian manifold of dimension $n \geqslant 2$ and index $\nu = 0, \ldots, n-1$. Let $\pi : TM \to M$ be the tangent bundle over M, $\mathcal{O}_\nu(M)$ the bundle of orthonormal frames over (M, g) and $\psi : \mathbf{N} = \mathcal{O}_\nu(M) \times \mathbb{R}^n \to TM$ the map defined by

$$\psi(p, u, \xi) = \sum_{i=1}^{n} \xi^i \, . \, u_i$$

where the orthonormal basis $u = \{u_1, \ldots, u_n\}$ of M_p (the tangent space to M at p), is assumed to be ordered so that $g(u_1, u_1) = \ldots = g(u_\nu, u_\nu) = -1$ if $\nu \geqslant 1$ and $\xi = (\xi^1, \ldots, \xi^n)$.

The family of maps $\mathbf{R}_a : \mathbf{N} \to \mathbf{N}$, $a \in \mathcal{O}_\nu(n)$, given by

$$\mathbf{R}_a(p, u, \xi) = (p, ua, \xi \, . \, a^{-1})$$

where $ua = \left\{\sum_{i=1}^n a_1^i \, . \, u_i, \ldots, \sum_{i=1}^n a_n^i \, . \, u_i\right\}$ if

$$a = \begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \vdots & & \vdots \\ a_n^1 & \cdots & a_n^n \end{pmatrix},$$

define the action of $\mathcal{O}_\nu(n)$ on $\mathbf{N}$. Clearly $\psi \circ \mathbf{R}_a = \psi$.

Let ∇ be the Levi-Civita connection of g and $\mathbf{K} : TTM \to TM$ the connection map induced by ∇. For any $p \in M$ and any $v \in M_p$, let $\pi_{*v} : (TM)_v \to M_p$ the differential map of π at v, and $\mathbf{K}_v : (TM)_v \to M_p$ the restriction of $\mathbf{K}$ to $(TM)_v$. Since the linear map $\pi_{*v} \times \mathbf{K}_v : (TM)_v \to M_p \times M_p$ defined by $\pi_{*v} \times \mathbf{K}_v(b) = (\pi_{*v}(b), \mathbf{K}(b))$ is an isomorphism that maps isomorphically the horizontal subspace H_v (=kernel of $\mathbf{K}_v$) onto $M_p \times \{0_p\}$ and the vertical subspace V_v (= kernel of π_{*v}) onto $\{0_p\} \times M_p$, where 0_p denotes the zero vector, we define as in [1], the differentiable mappings $\mathbf{e}_i, \mathbf{e}_{n+i} : \mathbf{N} \to T(TM)$ for $1 \leqslant i \leqslant n$ by

$$\mathbf{e}_i(p, u, \xi) = (\pi_{*v} \times \mathbf{K}_v)^{-1}(u_i, 0_p) \quad \text{and} \quad \mathbf{e}_{n+i}(p, u, \xi) = (\pi_{*v} \times \mathbf{K}_v)^{-1}(0_p, u_i)$$

where $v = \psi(p, u, \xi)$.

Since $(TM)_v = H_v \oplus V_v$, any vector field X on TM may be written in the form $X = X^{\mathbf{h}} + X^{\mathbf{v}}$, where

$$X^{\mathbf{h}}(v) = (\pi_{*v} \times \mathbf{K}_v)^{-1} \left(\pi_{*v}(X(v)), 0_p\right)$$

and

$$X^{\mathbf{v}}(v) = (\pi_{*v} \times \mathbf{K}_v)^{-1} \left(0_p, \mathbf{K}(X(v))\right)$$

if $v \in M_p$. Hence, the mappings $\mathbf{e}_i, \mathbf{e}_{n+i}$ let us view X as the function ${}^{\nabla}\!X = (x^1, \ldots, x^{2n}) : \mathbf{N} \to \mathbb{R}^{2n}$ where $x^\ell : \mathbf{N} \to \mathbb{R}$ are determined—for $v = \psi(p, u, \xi)$—by

$$\begin{aligned} x^i(p, u, \xi) &= -g(\pi_{*v}(X(v)), u_i) & \qquad 1 \leqslant i \leqslant \nu \\ x^i(p, u, \xi) &= g(\pi_{*v}(X(v)), u_i) & \qquad \nu + 1 \leqslant i \leqslant n \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} x^{n+i}(p,u,\xi) &= -g(\mathbf{K}(X(v)),u_i) \qquad & 1\leqslant i\leqslant \nu \\ x^{n+i}(p,u,\xi) &= g(\mathbf{K}(X(v)),u_i) \qquad & \nu+1\leqslant i\leqslant n \end{aligned} \tag{3.2}$$

if $\nu \geqslant 1$.

As in [1], for any (0,2)-tensor field G on TM we define the differentiable function

$$^{\nabla}G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix} : \mathbf{N} \to \mathbb{R}^{2n\times 2n}$$

as follows: if $(p,u,\xi)\in \mathbf{N}$ and $v=\psi(p,u,\xi)$, let $^{\nabla}G(p,u,\xi)$ be the matrix of the bilinear form $G_v : (TM)_v \times (TM)_v \to \mathbb{R}$ induced by G on $(TM)_v$ with respect to the basis $\{\mathbf{e}_1(p,u,\xi),\ldots,\mathbf{e}_{2n}(p,u,\xi)\}$. Hence, each $A_i : \mathbf{N}\to\mathbb{R}^{n\times n}$ satisfies the following $\mathcal{O}_\nu(n)$-invariance property

$$A_i \circ \mathbf{R}_a = a \,.\, A_i \,.\, a^t \qquad (i=1,2,3,4) \tag{3.3}$$

and for any pair of vector fields X, Y on TM one gets

$$G(X,Y)\circ\psi = {}^{\nabla}X \,.\, {}^{\nabla}G \,.\, ({}^{\nabla}Y)^t \tag{3.4}$$

Just as we did in [1], we define G to be natural with respect to g if $^{\nabla}G$ only depends on the variable ξ. Here also, we shall refer to $^{\nabla}G$ as the matrix of G with respect to g.

Proposition 3.1 *Let G be a (0,2) tensor field on TM and $^{\nabla}G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}$ the matrix of G with respect to g. Then G is natural with respect to g if there exist differentiable functions $\alpha_i,\beta_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2,3,4)$ such that*

$$A_i(p,u,\xi) = \alpha_i(\mathbf{q}(\xi)) \,.\, I_\nu + \beta_i(\mathbf{q}(\xi)) \,.\, (\xi . I_\nu)^t \,.\, (\xi . I_\nu)$$

or, equivalently, if for any vector fields X,Y on TM the following equalities are satisfied

$$\begin{aligned} G(X^{\boldsymbol{h}},Y^{\boldsymbol{h}})(v) &= \alpha_1(|v|^2) \,.\, g(\pi_*(X(v)),\pi_*(Y(v))) \\ &\quad + \beta_1(|v|^2) \,.\, g(\pi_*(X(v)),v) \,.\, g(\pi_*(Y(v)),v) \\ G(X^{\boldsymbol{h}},Y^{\boldsymbol{v}})(v) &= \alpha_2(|v|^2) \,.\, g(\pi_*(X(v)),\boldsymbol{K}(Y(v))) \\ &\quad + \beta_2(|v|^2) \,.\, g(\pi_*(X(v)),v) \,.\, g(\boldsymbol{K}(Y(v)),v) \\ G(X^{\boldsymbol{v}},Y^{\boldsymbol{h}})(v) &= \alpha_4(|v|^2) \,.\, g(\boldsymbol{K}(X(v)),\pi_*(Y(v))) \\ &\quad + \beta_4(|v|^2) \,.\, g(\boldsymbol{K}(X(v)),v) \,.\, g(\pi_*(Y(v)),v) \\ G(X^{\boldsymbol{v}},Y^{\boldsymbol{v}})(v) &= \alpha_3(|v|^2) \,.\, g(\boldsymbol{K}(X(v)),\boldsymbol{K}(Y(v))) \\ &\quad + \beta_3(|v|^2) \,.\, g(\boldsymbol{K}(X(v)),v) \,.\, g(\boldsymbol{K}(Y(v)),v) \end{aligned}$$

where $|v|^2 = g(v,v)$.

Proof According to (3.3), if G is natural, each matrix function A_i can be viewed as a function $B : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ which satifies $B(x \,.\, a^{-1}) = a \,.\, B(x) \,.\, a^t$ for any $a \in \mathcal{O}_\nu(n)$ or, equivalently, $B(x \,.\, a) = a^{-1} \,.\, B(x) \,.\, (a^{-1})^t$. Setting

$$A(x) = I_\nu \,.\, B(x) \,.\, I_\nu \tag{3.5}$$

this matrix function verifies $A(x.a) = a^t \,.\, A(x) \,.\, a$ for any $a \in \mathcal{O}_\nu(n)$; hence, by the theorem above, there exist differentiable functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ such that

$$A(x) = \alpha(\mathbf{q}(x)) \,.\, I_\nu + \beta(\mathbf{q}(x)) \,.\, x^t \,.\, x$$

From (3.5) it follows that

$$B(x) = \alpha(\mathbf{q}(x)) \,.\, I_\nu + \beta(\mathbf{q}(x)) \,.\, (x \,.\, I_\nu)^t \,.\, (x.I_\nu)$$

The expression of G applied to vector fields, is now a consequence of (3.1), (3.2) and (3.4). □

4 Natural (0,2)-tensor fields on cotangent bundles

For any $p \in M$, let M_p^* be the dual space of M_p and let $\pi : T^*M \to M$ be the cotangent bundle of M. For any $(p, u) \in \mathcal{O}_\nu(M)$ we denote with (p, u^*) the dual basis and $\mathcal{O}_\nu^*(M)$ the bundle consisting of all those ordered dual basis. Set $\mathbf{N} = \mathcal{O}_\nu^*(M) \times \mathbb{R}^n$ and let $\psi : \mathbf{N} \to T^*M$ be the map defined by

$$\psi(p, u^*, \xi) = \sum_{i=1}^n \xi_i \,.\, u^i$$

if $u^* = \{u^1, \ldots, u^n\}$ and $\xi = (\xi_1, \ldots, \xi_n)$.

The family of maps $\mathbf{R}_a : \mathbf{N} \to \mathbf{N}$, $a \in \mathcal{O}_\nu(n)$, given by

$$\mathbf{R}_a(p, u^*, \xi) = (p, u^*a, \xi.a^t)$$

where $u^*a = (ua)^* = \{\sum_{i=1}^n b_i^1 \,.\, u^i, \ldots, \sum_{i=1}^n b_i^n \,.\, u^i\}$ if

$$a^{-1} = \begin{pmatrix} b_1^1 & \cdots & b_1^n \\ \vdots & & \vdots \\ b_n^1 & \cdots & b_n^n \end{pmatrix}$$

defines the action of $\mathcal{O}_\nu(n)$ on $\mathbf{N}$. Clearly, $\psi \circ \mathbf{R}_a = \psi$.

Let $\mathbf{K}^* : T(T^*M) \to T^*M$ be the dual connection map. We'll recall that for any $p \in M$ and any co-vector $w \in M_p^*$ the restriction $K_w^* = K^*|_{(T^*M)_w} : (T^*M)_w \to M_p^*$ is a surjective linear map, characterized by the fact that for any 1- form ω on M such that $\omega(p) = w$ and any vector $v \in M_p$, it satisfies $K_w^*(\omega_{*p}(v)) = \nabla_v \omega$, where $\omega_{*p} : M_p \to (T^*M)_w$ denotes the differential map of ω at p.

Since the linear map $\pi_{*w} \times \mathbf{K}_w^* : (T^*M)_w \to M_p \times M_p^*$ defined by $\pi_{*w} \times \mathbf{K}_w^*(b) = (\pi_{*w}(b), \mathbf{K}^*(b))$ is an isomorphism that maps the horizontal subspace H_w (= kernel of $\mathbf{K}_w^*$) onto $M_p \times \{0_p\}$ and the vertical subspace V_w (= kernel of π_{*w}) onto $\{0_p\} \times M_p^*$, where 0_p denotes indistinctly the zero vector and the zero co-vector, we define as in [2] the differentialbe mappings $e_i, e_{n+i} : \mathbf{N} \to T(T^*M)$ for $1 \leqslant i \leqslant n$ by $e_i(p, u^*, \xi) = (\pi_{*w} \times \mathbf{K}_w^*)^{-1}(u_i, 0_p)$ and $e_{n+i}(p, u^*, \xi) = (\pi_{*w} \times \mathbf{K}_w^*)^{-1}(0_p, u^i)$ where $w = \psi(p, u^*, \xi)$.

Since $(T^*M)_w = H_w \oplus V_w$, any vector field X on T^*M may be written in the form $X = X^{\mathbf{h}} + X^{\mathbf{v}}$, where

$$X^{\mathbf{h}}(w) = (\pi_{*w} \times \mathbf{K}_w^*)^{-1}(\pi_{*w}(X(w)), 0_p)$$

and

$$X^{\mathbf{v}}(w) = (\pi_{*w} \times \mathbf{K}_w^*)^{-1}(0_p, \mathbf{K}^*(X(w)))$$

Hence the mappings e_i, e_{n+i} let us view X as the function ${}^{\nabla}\!X = (x^1, \ldots, x^{2n}) : \mathbf{N} \to \mathbb{R}^{2n}$ where $x^\ell : \mathbf{N} \to \mathbb{R}$ are defined, for $w = \psi(p, u^*, \xi)$, by

$$x^i(p, u^*, \xi) = u^i(\pi_{*w}(X(w)) \tag{4.1}$$

and

$$x^{n+i}(p, u^*, \xi) = \mathbf{K}_w^*(X(w))(u_i) \tag{4.2}$$

As in [2], for any (0,2)-tensor field G on T^*M we define the differentiable function

$${}^{\nabla}\!G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix} : \mathbf{N} \to \mathbb{R}^{2n \times 2n}$$

as follows: if $(p, u^*, \xi) \in \mathbf{N}$ and $w = \psi(p, u^*, \xi)$, let ${}^{\nabla}\!G(p, u^*, \xi)$ be the matrix of the bilinear form $G_w = (T^*M)_w \times (T^*M)_w \to \mathbb{R}$ induced by G on $(T^*M)_w$ with respect to the basis $\{\mathbf{e}_1(p, u^*, \xi), \ldots, \mathbf{e}_{2n}(p, u^*, \xi)\}$. Hence, each $A_i : \mathbf{N} \to \mathbb{R}^{n \times n}$ satisfies the following $\mathcal{O}_\nu(n)$-invariance properties

$$A_1 \circ \mathbf{R}_a = a \,.\, A_1 \,.\, a^t \tag{4.3}$$

$$A_2 \circ \mathbf{R}_a = a \,.\, A_2 \,.\, a^{-1} \tag{4.4}$$

$$A_3 \circ \mathbf{R}_a = (a^{-1})^t \,.\, A_3 \,.\, a^{-1} \tag{4.5}$$

$$A_4 \circ \mathbf{R}_a = (a^{-1})^t \,.\, A_4 \,.\, a^t \tag{4.6}$$

and for any pair of vector fields X, Y on T^*M one gets

$$G(X, Y) \circ \psi = {}^{\nabla}\!X \,.\, {}^{\nabla}\!G \,.\, ({}^{\nabla}Y)^t \tag{4.7}$$

Just as we did in [2], we define G to be natural with respect to g if ${}^{\nabla}\!G$ only depends on the variable ξ. Here also we shall refer to ${}^{\nabla}\!G$ as the matrix of G with respect to g. Since in this case each matrix function A_i can be viewed as a function $A_i : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ satisfying

$$A_1(x \,.\, a) = a^t \,.\, A_1(x) \,.\, a \qquad A_2(x \,.\, a) = a^t \,.\, A_2(x) \,.\, (a^t)^{-1}$$
$$A_3(x \,.\, a) = a^{-1} \,.\, A_3(x) \,.\, (a^t)^{-1} \qquad A_4(x \,.\, a) = a^{-1} \,.\, A_4(x) \,.\, a$$

for any $x \in \mathbb{R}^n$ and $a \in \mathcal{O}_\nu(n)$, Theorem 2.1 implies:

Proposition 4.1 *Let G be a (0,2) tensor field on T^*M and*

$$^{\nabla}G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}$$

the matrix of G with respect to g. Then G is natural with respect to g if there exist differentiable functions $\alpha_i, \beta_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3, 4)$ such that

$$A_1(\xi) = \alpha_1(\mathbf{q}(\xi)) \,.\, I_\nu + \beta_1(\mathbf{q}(\xi)) \,.\, \xi^t \,.\, \xi \tag{4.8}$$

$$A_2(\xi) = \alpha_2(\mathbf{q}(\xi)) \,.\, I + \beta_2(\mathbf{q}(\xi)) \,.\, \xi^t \,.\, \xi \,.\, I_\nu \tag{4.9}$$

$$A_3(\xi) = \alpha_3(\mathbf{q}(\xi)) \,.\, I_\nu + \beta_3(\mathbf{q}(\xi)) \,.\, (\xi . I_\nu)^t \,.\, (\xi . I_\nu) \tag{4.10}$$

$$A_4(\xi) = \alpha_4(\mathbf{q}(\xi)) \,.\, I + \beta_4(\mathbf{q}(\xi)) \,.\, (\xi \,.\, I_\nu)^t \,.\, \xi \tag{4.11}$$

Remark 4.1 If $\nu = 0$ then $A_i(\xi) = \alpha_i(\mathbf{q}(\xi)) \,.\, I + \beta_i(\mathbf{q}(\xi)) \,.\, \xi^t \,.\, \xi$ for any $i = 1, 2, 3, 4$. In this case proposition above is Theorem 5.2 of [2]. Consequently, only in the Riemannian case natural (0,2)-tensor fields on tangent and cotangent bundles have the same matrices $^{\nabla}G$.

Remark 4.2 Let θ be the canonical 1-form on T^*M which is defined by

$$\theta(X)(w) = w(\pi_*(X(w))$$

for any vector field X on T^*M and any co-vector $w \in T^*M$. If $\otimes$ denotes the tensor product, set $\theta^2 = \theta \otimes \theta$, and let $d\theta$ be the exterior derivative of θ. The corresponding $^{\nabla}G$ matrices are given by

$$^{\nabla}\theta^2 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad ^{\nabla}d\theta = \begin{pmatrix} B_1 & -I \\ I & 0 \end{pmatrix}$$

where $A_1(p, u^*, \xi) = \xi^t \,.\, \xi$ and $B_1(p, u^*, \xi) = (w(T(u_i, u_j)))$ with $w = \psi(p, u^*, \xi)$ and being T the torsion tensor. Yet T is null, we have that θ^2 and $d\theta$ are natural with respect to g.

Remark 4.3 Let g^* be the (2,0)-tensor field on M induced by g and for $w \in T^*M$ let $|w|^2 = g^*(w, w)$. If G is natural with respect to g, one sees from (4.1), (4.2) and from (4.7) to (4.11) that for any pair of vector fields X, Y on T^*M the real valued functions $G(X^{\mathbf{h}}, Y^{\mathbf{h}})$, $G(X^{\mathbf{h}}, Y^{\mathbf{v}})$, $G(X^{\mathbf{v}}, Y^{\mathbf{h}})$, $G(X^{\mathbf{v}}, Y^{\mathbf{v}})$ are given for any $w \in T^*M$ by:

$$\begin{aligned} G(X^{\mathbf{h}}, Y^{\mathbf{h}})(w) &= \alpha_1(|w|^2) \,.\, g(\pi_*(X(w)), \pi_*(Y(w))) \\ &\quad + \beta_1(|w|^2) \,.\, \theta(X)(w) \,.\, \theta(Y)(w) \end{aligned}$$

$$\begin{aligned} G(X^{\mathbf{h}}, Y^{\mathbf{v}})(w) &= \alpha_2(|w|^2) \,.\, \mathbf{K}^*(Y(w))(\pi_*(X(w))) \\ &\quad + \beta_2(|w|^2) \,.\, g^*(w, \mathbf{K}^*(Y(w))) \,.\, \theta(X)(w) \end{aligned}$$

$$G(X^{\mathbf{v}}, Y^{\mathbf{h}})(w) = \alpha_4(|w|^2) \,.\, \mathbf{K}^*(X(w))(\pi_*(Y(w))) + \beta_4(|w|^2) \,.\, g^*(w, \mathbf{K}^*(X(w))) \,.\, \theta(Y)(w)$$

$$G(X^{\mathbf{v}}, Y^{\mathbf{v}})(w) = \alpha_3(|w|^2) \,.\, g^*(\mathbf{K}^*(X(w)), \mathbf{K}^*(Y(w))) + \beta_3(|w|^2) \,.\, g^*(w, \mathbf{K}^*(X(w))) \,.\, g^*(w, \mathbf{K}^*(Y(w)))$$

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