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Chipman Pseudoinverse of Matrix, its Computation and Application in Spline Theory

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Abstract

In this paper are given definition of Chipman pseudoinverse, which is generalized Moore–Penrose matrix, some its properties and algorithms for its computation. One example for its application in spline theory is shown at the end.

Key words: Chipman pseudoinverse of matrix, rank factorization, generalized Greville algorithm, optimal property of some cubic spline.

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1 Introduction

Consider the system of linear equations

$$Ax = b, \tag{1}$$

where A is a regular matrix of size n x n, x and b are column vectors of size $n \times 1$. Then there exist just one solution $x_0 = A^{-1}b$.

Let A be a singular or non-square matrix of size $m \times n$ and b a vector of size $m \times 1$. We attempt to find a least-squares solution of the equation (1), in other words a solution, which minimize norm of the residual vector

$$||Ax - b||, \tag{2}$$

and which has minimal norm among all these vectors.

Now we consider the Euclidean norm

$$\| x \|_2 = \sqrt{x^T x}, \tag{3}$$

Theorem 1.1 *Consider the matrix equations (1) with a matrix A of size $m \times n$ and a vector b of size $m \times 1$ and Euclidian norm (3). Then there is just one least-squares solution x_0 with minimal norm. This solution is formed as*

$$x_0 = A^+ b, \tag{4}$$

where matrix A^+ is Moore–Penrose inverse of A.

Proof See [10].

Theorem 1.2 *To each matrix A there exists exactly one Moore–Penrose inverse A^+.*

Proof See [10].

Now let N be a n x n symmetric positive definite matrix (p.d.s.) and denote

$$||x||_N = \sqrt{x^T N x}. \tag{5}$$

In next section we solve matrix equation

$$Ax = b \tag{6}$$

with matrix A of size $m \times n$ and vector b of size $m \times 1$ and consider the norm (5). So we attempt to find a least squares solution of the equation (6). This least squares solution is formed by using Chipman pseudoinverse of matrix A, which is defined in next section.

2 Definition and properties

The following definition and theorem were published in [9].

Definition 2.1 Let A be a $m \times n$ matrix, M a $m \times m$ symmetric positive definite matrix and N a $n \times n$ symmetric positive definite matrix too. The matrix $A^+_{M,N}$ of size $n \times m$, which satisfies the axioms

$$AA^+_{M,N}A = A \tag{7}$$

$$A^+_{M,N}AA^+_{M,N} = A^+_{M,N} \tag{8}$$

$$\left(MAA^+_{M,N}\right)^T = MAA^+_{M,N} \tag{9}$$

$$\left(NA^+_{M,N}A\right)^T = NA^+_{M,N}A, \tag{10}$$

is called the Chipman pseudoinverse of matrix A.

Theorem 2.2 *Basic properties of* $A^+_{M,N}$.

- *If* A *is zero matrix, then* $A^+_{M,N} = A^T$
- *If* M, N *are identity matrices, then* $A^+_{M,N} = A^+$
- *If* A *is square regular matrix, then* $A^+_{M,N} = A^{-1}$
- $\left(A^+_{M,N}\right)^+_{N,M} = A$
- $\left(A^+_{M,N}\right)^T = \left(A^T\right)^+_{N^{-1},M^{-1}}$

Proof Using definition it is easy to prove this statements.

If matrix B is full column rank or full row rank, then there is a simple way for computation its Chipman pseudoinverse with just one p.d.s. matrix.

Theorem 2.3 *Let* B *be a matrix of size* $m \times r$, $\mathrm{rank}(B) = r$, M *be p.d.s. matrix of size* $m \times m$ *and denote*

$$B^+_{M,.} = \left(B^T M B\right)^{-1} B^T M. \tag{11}$$

Then $B^+_{M,.}$ *is Chipman pseudoinverse of* B *corresponding to matrix* M *and any symmetric matrix* N *of size* $r \times r$.

Proof Matrix $B^+_{M,N}$, with a p.d.s. $r \times r$ matrix N, must satisfy equations (7)–(10). Using (11) it is easy to prove, that axioms (7)–(10) hold.

Theorem 2.4 *Let* C *be a matrix of size* $r \times n$, $\mathrm{rank}(C) = r$ *and* N *be p.d.s. matrix of size* $n \times n$ *and denote*

$$C^+_{.,N} = N^{-1} C^T \left(C N^{-1} C^T\right)^{-1}. \tag{12}$$

Then $C^+_{.,N}$ *is Chipman pseudoinverse of* C *corresponding to matrix* N *and any symmetric matrix* M *of size* $r \times r$.

Proof The proof is similary as in precedent theorem.

Theorem 2.5 (Rank factorization) *Let* A *be a non-zero matrix of size* $m \times n$, $\mathrm{rank}(A) = r$, M *be p.d.s. matrix of size* $m \times m$ *and* N *p.d.s. matrix of size* $n \times n$. *Then there exist matrices* B *of size* $m \times r$ *and* C *of size* $r \times n$ *such that*

$$A = BC, \qquad \textit{(rank factorization)}$$

$r = \mathrm{rank}(B) = \mathrm{rank}(C)$ *and*

$$A^+_{M,N} = C^+_{M,N} B^+_{M,N}. \tag{13}$$

Proof If A is $m \times n$ matrix of rank r, then there exist not unique regular square matrices D and E, such that

$$DAE = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

which give the representations

$$A = D^{-1} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} E^{-1}$$

$$A = BC = b_1 c_1^T + \ldots + b_r c_r^T, \qquad \text{(rank factorization)}$$

where B is $m \times r$ matrix of rank r consisting of the first r column vectors $b_1, \ldots, b_r$ of matrix D^{-1} and C is $r \times n$ matrix of rank r consisting of the first r row vectors $c_1, \ldots, c_r$ of E^{-1}.

We can see, that matrices B, C always exist for each matrix A, but they are not unique.

Matrices $C^+_{M,N}$ and $B^+_{M,N}$ exist and are given in two precedent theorems. Hence

$$A^+_{M,N} = N^{-1} C^T \left(C N^{-1} C^T\right)^{-1} \left(B^T M B\right)^{-1} B^T M. \tag{14}$$

Using (14) it is easy to prove that matrices A, $A^+_{M,N}$ satisfy equations (7)–(10).

Theorem 2.6 *Let A be a $m \times n$ matrix. Let M p.d.s. matrix of size $m \times m$ and N p.d.s. matrix of size $n \times n$ are fixed. Then there exist just one matrix $A^+_{M,N}$.*

Proof By Theorem 2.5. Chipman pseudoinverse $A^+_{M,N}$ really exists for each matrix A. So now we show, that if X and Y are Chipman pseudoinverses of matrix A for fixed p.d.s. matrices M and N, then $X = Y$.

Matrices X and Y satisfy equations (7)–(10). Using these equations we have

$$\begin{aligned}
X &= XAX = N^{-1} N X A X = N^{-1} (NXA)^T X = N^{-1} A^T X^T N^T X \\
&= N^{-1} A^T Y^T A^T X^T N^T X = N^{-1} A^T Y^T N N^{-1} A^T X^T N^T X \\
&= N^{-1} (NYA)^T N^{-1} (NXA)^T X = N^{-1} N Y A N^{-1} N X A X = YAX
\end{aligned}$$

$$\begin{aligned}
Y &= YAY = Y M^{-1} M A Y = Y M^{-1} (MAY)^T = Y M^{-1} Y^T A^T M^T \\
&= Y M^{-1} Y^T A^T X^T A^T M^T = Y M^{-1} Y^T A^T (MAX)^T \\
&= Y M^{-1} Y^T A^T M A X = Y M^{-1} (MAY)^T A X = Y M^{-1} M A Y A X \\
&= YAYAX = YAX
\end{aligned}$$

So $X = Y$.

Theorem 2.7 *Consider the matrix equation (1) with a matrix A of size $m \times n$ and a vector b of size $m \times 1$. Let $x_0 = A^+_{M,N}b$. Then for each vector x of size $n \times 1$, $x \neq x_0$ is*

(i) $\|Ax_0 - b\|_M < \|Ax - b\|_M$

or

(ii) $\|Ax_0 - b\|_M = \|Ax - b\|_M \qquad$ *and* $\qquad \|x_0\|_N < \|x\|_N.$

Vector x_0 is called M-least-squares solution of equation (1) with minimal norm (5).

Proof See [10].

3 Iterative algorithm

In [10] are given several computational methods for generalized inverses, but not all of which may be suitable for numerical computations. Some of them are useful in theoretical investigations. One of them, rank factorization, is described above. In next section is shown Generalized Greville algorithm, which is based on Greville algorithm, see [1]. In [5] is given universal iterative method for computing generalised inverses. So for computing Chipman pseudoinverse we get the following four theorems. Their proofs are given in [5] too.

Theorem 3.1 *Let B be a matrix of size $m \times r$,* $\operatorname{rank}(B) = r \geq 2$, *let M be p.d.s. matrix of size $m \times m$. If $q \geq 2$ is an integer, then for iterative proces*

$$V_B = B^T M B, \quad \alpha = \frac{2}{\operatorname{tr}\left(V_B^T V_B\right)}, \quad Y_0 = \alpha V_B^T,$$

$$\begin{aligned} T_k &= I - Y_k V_B \\ Y_{k+1} &= \left(I + T_k + T_k^2 + \cdots + T^{q-1}\right) Y_k \qquad k = 1, 2, \ldots \\ X_{k+1} &= Y_{k+1} B^T M \end{aligned} \tag{15}$$

we get $\lim_{k\to\infty} X_k = B^+_{M,\cdot}$.

(*I denotes identity matrix and* $\operatorname{tr}(A)$ *denotes trace of matrix A (sum of diagonal elements.)*)

Theorem 3.2 *Let C is a matrix of size $r \times n$,* $\operatorname{rank}(C) = r \geq 2$, *let N be p.d.s. matrix of size $n \times n$. If $q \geq 2$ is an integer, then for iterative proces*

$$V_C = C N^{-1} C^T, \quad \alpha = \frac{2}{\operatorname{tr}\left(V_C^T V_C\right)}, \quad Y_0 = \alpha V_C^T,$$

$$\begin{aligned} T_k &= I - Y_k V_C \\ Y_{k+1} &= \left(I + T_k + T_k^2 + \cdots + T^{q-1}\right) Y_k \qquad k = 1, 2, \ldots \\ X_{k+1} &= N^{-1} C^T Y_{k+1} \end{aligned} \tag{16}$$

we get $\lim_{k\to\infty} X_k = C^+_{\cdot,N}$.

Theorem 3.3 *Let A be matrix of size $m \times n$, $\operatorname{rank}(A) = r \geq 2$, M be p.d.s. matrix of size $m \times m$, N be p.d.s. matrix of size $n \times n$ and let us put rank factorization $A = BC$, where matrix B is of size $m \times r$, C of size $r \times n$ and $\operatorname{rank}(A) = \operatorname{rank}(B) = \operatorname{rank}(C)$. If $q \geq 2$ is an integer, then for iterative proces*

$$V_A = B^T M A N^{-1} C^T, \quad \alpha = \frac{2}{\operatorname{tr}\left(V_A^T V_A\right)}, \quad Y_0 = \alpha V_A^T$$

$$\begin{aligned} T_k &= I - Y_k V_A \\ Y_{k+1} &= \left(I + T_k + T_k^2 + \cdots + T^{q-1}\right) Y_k \qquad k = 1, 2, \ldots \\ X_{k+1} &= N^{-1} C^T Y_{k+1} B^T M \end{aligned} \tag{17}$$

we get $\lim_{k\to\infty} X_k = A^+_{M,N}$.

Theorem 3.4 *Let A be matrix of size $m \times n$, $\operatorname{rank}(A) = 1$ and $A = BC$ its rank factorization. Then using notation from the preceding theorems there is*

$$A^+_{M,N} = \frac{1}{tr\left(V_A^T V_A\right)} N^{-1} C^T V_A^T B^T M,$$

with p.d.s. matrices M, N of corresponding size.

Proof See [5].

4 Partition of matrix

In this section we give definition of operation * and partition of matrix, which are used in the Generalized Greville algorithm for computation Chipman pseudoinverse. This algorithm we give in next section. Let $\mathcal{M}_{m,n}$ be a set of all $m \times n$ matrices.

Definition 4.1 Let $A \in \mathcal{M}_{m,n}$, $N_m \in \mathcal{M}_{m,m}$ be p.d.s. matrix and $N_n \in \mathcal{M}_{n,n}$ be p.d.s. matrix too. Define

$$A^* = N_n^{-1} A^T N_m. \tag{18}$$

Lemma 4.2 *Let $A, B \in \mathcal{M}_{m,n}$ and N_m, N_n be p.d.s. matrices of corresponding size. Then*

$$\begin{aligned} (A^*)^* &= A \\ (AB)^* &= B^* A^* \\ (A + B)^* &= A^* + B^* \end{aligned}$$

Proof Using definition it is easy to prove this statements.

Lemma 4.3 *Let a be a non-zero vector of size $1 \times n$ and N_1, N_n are p.d.s. matrices of corresponding size, which define a^*. Then*

$$a^{+}_{N_1,N_n} = a^* \left(aa^*\right)^{-1} \tag{19}$$

is Chipman pseudoinverse of a.

Proof Using (19) we have

$$\begin{aligned} a^{+}_{N_1,N_n} &= N_1^{-1}a^T N_1 \left(aN_n^{-1}a^T N_1\right)^{-1} = N_n^{-1}a^T N_1 N_1^{-1} \left(aN_n^{-1}a^T\right)^{-1} \\ &= N_n^{-1}a^T \left(aN_n^{-1}a^T\right)^{-1}, \end{aligned}$$

and this is owing to (13) Chipman pseudoinverse of vector a.

Definition 4.4 Let $A \in \mathcal{M}_{m,n}$, N_m, N_{m_1}, N_{m_2}, be p.d.s. matrices of corresponding size, $m_1 + m_2 = m$, $E_{m_1} \in \mathcal{M}_{m_1,m}$, $E_{m_2} \in \mathcal{M}_{m_2,m}$, such that

$$E_{m_1}E^*_{m_1} = I_{m_1}, \qquad E_{m_2}E^*_{m_2} = I_{m_2}, \tag{20}$$

$$E^*_{m_1}E_{m_1} + E^*_{m_2}E_{m_2} = I_m. \tag{21}$$

Let the matrices

$$\begin{array}{ll} B = E_{m_1}A & (m_1 \times n), \\ C = E_{m_2}A & (m_2 \times n). \end{array} \tag{22}$$

Then the $m \times n$ matrix

$$\begin{pmatrix} B \\ C \end{pmatrix}, \tag{23}$$

is called $(m_1 + m_2, E_{m_1}, E_{m_2})$-partition of matrix A.

Remark 4.5 For each matrix $A \in \mathcal{M}_{m,n}$ and fixed p.d.s. matrices N_m, N_{m_1}, N_{m_2}, m_1, m_2, such that $m_1 + m_2 = m$, matrices E_{m_1}, E_{m_2} always exist. In Theorem 4.8 will be given instruction for finding these matrices in general case. For fixed m_1, m_2, E_{m_1}, E_{m_2} is $(m_1, m_2, E_{m_1}, E_{m_2})$-partition of matrix A unique.

Now we show one example of $(m_1 + m_2, E_{m_1}, E_{m_2})$-partition of matrix A, when p.d.s. matrices N_m, N_{m_1} and N_{m_2} are identity matrices of corresponding size. Let

$$E_{m_1} = (I_{m_1}\ O_{m_1,m_2}), \qquad E_{m_2} = (O_{m_2,m_1}\ I_{m_2}).$$

Then

$$E^*_{m_1} = \begin{pmatrix} I_{m_1} \\ 0_{m_2,m_1} \end{pmatrix} \quad \text{and} \quad E^*_{m_2} = \begin{pmatrix} 0_{m_1,m_2} \\ I_{m_2} \end{pmatrix}.$$

It can be easily seen that the relations (21), (22) hold.

Let

$$B = E_{m_1}A, \qquad C = E_{m_2}A.$$

In this case matrix B is formed from the first m_1 rows of matrix A and matrix C is formed from the $m_1 + 1, \ldots, m$ rows of matrix A.

Owing to Definition 4.4 $m \times n$ matrix

$$\begin{pmatrix} B \\ C \end{pmatrix}$$

is $(m_1, m_2, E_{m_1}, E_{m_2})$-partition of matrix A.

Theorem 4.6 (Singular value decomposition) *Let A be a matrix of size $m \times n$, $\operatorname{rank}(A) = r$. Then there are exist unitary matrices U of size $m \times m$ and V of size $n \times n$ such that*

$$A = UDV^T, \tag{24}$$

where $D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}$ is matrix of size $m \times n$, $D_1 = \operatorname{diag}(d_1, \ldots, d_r)$ is regular matrix of size $r \times r$ and $d_1, \ldots, d_r$ are singular values of matrix A.

Proof See in [3].

Remark 4.7 If A is symmetric positive definite matrix of size $n \times n$, then its singular value decomposition is

$$A = UDU^T$$

and singular values $d_1 > 0, \ldots, d_n > 0$.

Theorem 4.8 *Let $A \in \mathcal{M}_{m,n}$ and N_m, N_{m_1}, N_{m_2} are p.d.s. matrices, $m_1 + m_2 = m$. Moreover, let us put singular value decomposition*

$$N_i = U_i D_i U_i^T \quad \text{for } i = m_1, m_2, m.$$

Let the matrix $D_m = \operatorname{diag}(d_1, \ldots, d_m)$, then denote

$$D_m^{(1)} = (\operatorname{diag}(d_1, \ldots, d_{m_1}))^{-1} \quad \text{and} \quad D_m^{(2)} = (diag\,(d_{m_1+1}, \ldots, d_m))^{-1}$$

and

$$\begin{aligned} E_{m_1} &= U_{m_1} \left(\left[D_{m_1} D_m^{(1)} \right]^{-1/2} \quad 0_{m_1,m_2} \right) U_m^T \\ E_{m_2} &= U_{m_2} \left(0_{m_2,m_1} \quad \left[D_{m_2} D_m^{(2)} \right]^{-1/2} \right) U_m^T. \end{aligned} \tag{25}$$

Then matrices E_{m_1}, E_{m_2} satisfy (21), (22), and for

$$B = E_{m_1} A, \qquad C = E_{m_2} A,$$

matrix

$$\begin{pmatrix} B \\ C \end{pmatrix}, \tag{26}$$

is $(m_1 + m_2, E_{m_1}, E_{m_2})$-partition of matrix A.

Proof We must prove that matrices E_{m_1}, E_{m_2} satisfy statements (21), (22). Because

$$\begin{aligned} E^*_{m_1} &= N_m^{-1} E^T_{m_1} N_{m_1} \\ &= U_m D_m^{-1} U_m^T U_m \begin{pmatrix} \left[D_{m_1} D_m^{(1)}\right]^{-1/2} \\ 0_{m_2,m_1} \end{pmatrix} U^T_{m_1} U_{m_1} D_{m_1} U^T_{m_1} \qquad (27) \\ &= U_m \begin{pmatrix} [D_{m_1} D_m^{(1)}]^{1/2} \\ 0_{m_2,m_1} \end{pmatrix} U^T_{m_1} \end{aligned}$$

and similary

$$E^*_{m_2} = N_m^{-1} E^T_{m_2} N_{m_2} = U_m \begin{pmatrix} 0_{m_1,m_2} \\ \left[D_{m_2} D_m^{(2)}\right]^{1/2} \end{pmatrix} U^T_{m_2}, \qquad (28)$$

we have

$$\begin{aligned} & E_{m_1} E^*_{m_1} = \\ &= U_{m_1} \left(\left[D_{m_1} D_m^{(1)}\right]^{-1/2} 0_{m_1,m_2} \right) U_m^T U_m \begin{pmatrix} \left[D_{m_1} D_m^{(1)}\right]^{1/2} \\ 0_{m_2,m_1} \end{pmatrix} U^T_{m_1} = I_{m_1}. \end{aligned}$$

Similary for E_{m_2}.

5 Generalized Greville algorithm

In this section we give the Generalized Greville algorithm for computing Chipman pseudoinverse $A^+_{M,N}$ of matrix $A \in \mathcal{M}_{m,n}$ for fixed p.d.s. matrices $M \in \mathcal{M}_{m,m}$, $N \in \mathcal{M}_{n,n}$.

Greville algorithm (see [1]) for computing Moore–Penrose inverse A^+ of matrix A is based on computing Moore–Penrose inverse A^+_k of matrix A_k, where A_k is formed from the first k-rows of matrix A, $k = 1, \ldots, m$.

Computing Chipman pseudoinverse $A^+_{M,N}$ of matrix A is based on computing Chipman pseudoinverse for matrices A_k, where matrix A_k will be defined in Theorem 5.1 as partition of matrix A_{k+1}, $k = 1, \ldots, m-1$, $A_m = A$. We use informations from Theorem 4.8, in which are given instruction for finding partition of $m \times n$ matrix A by using singular value decompositions of p.d.s. matrices N_{m_1}, N_{m_2}, N_m, $m = m_1 + m_2$. Also for finding matrix A_k we need p.d.s. matrices N_{k+1}, N_k, N_1. Altogether in algorithm we will need, except p.d.s. matrices M, N, sequence $\{N_i\}_{i=1}^{m-1}$ of p.d.s. matrices, $N_i \in \mathcal{M}_{i,i}$. In k-th step of *Generalized Greville algorithm* we compute for matrix A_k its Chipman pseudoinverse $A^+_k = A^+_{N_k,N}$.

On sequence $\{N_i\}_{i=1}^{m-1}$ of p.d.s. matrices is from theoretical investigations no requirement, but for numerical computation, because this algorithm use inverse of N_k, is in place to N_k be good conditioned.

At first we give algorithm for general sequence of p.d.s. matrices $\{N_i\}_{i=1}^{m-1}$ and this algorithm we prove. In case, when we use sequence of identity matrices $\{I_i\}_{i=1}^{m-1}$, this algorithm is more simply. In k-th step we compute for matrix A_k its Chipman pseudoinverse $A_k^+ = A_{I_k,N}^+$. This is showed in Remark 5.3.

Theorem 5.1 (Generalized Greville algorithm) *Let $A \in \mathcal{M}_{m,n}$, p.d.s. matrices $M \in \mathcal{M}_{m,m}$, $N \in \mathcal{M}_{n,n}$ are given and $\{N_i\}_{i=1}^{m-1}$ be a sequence of p.d.s. matrices, where $N_i \in \mathcal{M}_{i,i}$.*

1. *Denote $A_m = A$, $N_m = M$.*

   Let us compute for $k = m, m-1, \ldots, 2$ matrices

$$\begin{aligned} A_{k-1} &= E_{k-1}^{(k)} A_k & \quad & (k-1) \times n \\ a_k &= E_1^{(k)} A_k & & 1 \times n \end{aligned}$$

   where matrices $E_{k-1}^{(k)}$, $E_1^{(k)}$ are computed by using singular value decompositions of p.d.s. matrices $N_k = U_k D_k U_k^T$, $N_{k-1} = U_{k-1} D_{k-1} U_{k-1}^T$, $N_1 = U_1 D_1 U_1^T$.

   Let the matrix $D_k = \operatorname{diag}(d_1, \ldots, d_k)$, then denote

$$D_k^{(1)} = (\operatorname{diag}(d_1, \ldots, d_{k-1}))^{-1} \qquad \text{and} \qquad D_k^{(2)} = 1/d_k$$

   and

$$\begin{aligned} E_{k-1}^{(k)} &= U_{k-1} \left(\left[D_{k-1} D_k^{(1)} \right]^{-1/2} \quad 0_{k-1,1} \right) U_k^T \\ E_1^{(k)} &= U_1 \left(0_{1,k-1} \quad \left[D_1 D_k^{(2)} \right]^{-1/2} \right) U_k^T . \end{aligned}$$

2. *Put*

$$A_1^+ = N^{-1} A_1^T \left(A_1 N^{-1} A_1^T \right)^{-1} \tag{29}$$

3. *Let us for $k = 2, 3, \ldots, m$ compute $n \times k$ matrices*

$$A_k^+ = \left[\begin{pmatrix} E_{k-1}^{(k)*} & E_1^{(k)*} \end{pmatrix} \begin{pmatrix} \left(A_{k-1}^+\right)^* - d_k^* b_k \\ b_k \end{pmatrix} \right]^* , \tag{30}$$

   where

$$\begin{aligned} d_k &= a_k A_{k-1}^+ , & (31) \\ c_k &= a_k - d_k A_{k-1} , & (32) \end{aligned}$$

$$b_k = \begin{cases} \left(\left(c_k^*\right)^T N c_k^* \right)^{-1} \left(c_k^*\right)^T N & \text{if } c_k \neq 0 \\ \left(1 + d_k d_k^*\right)^{-1} d_k \left(A_{k-1}^+\right)^* & \text{if } c_k = 0. \end{cases} \tag{33}$$

Then matrix $A^+_{N_k,N} = A^+_k$ is for $k = 1, 2, \ldots, m$ Chipman pseudoinverse of matrix A_k. For given matrix A and p.d.s. matrices M, N is matrix $A^+_{M,N} = A^+_m$ its Chipman pseudoinverse.

Proof Proof is based on the mathematical induction proving axioms (7)–(10) for Chipman pseudoinverse of matrix A_k.

Remark 5.2 Chipman pseudoinverse $A^+_{M,N}$ is independent on sequence of p.d.s. matrices $\{N_i\}_{i=1}^{m-1}$, this sequence is given only for computing partitions of matrices A_k, $k = m, \ldots, 2$, $A_m = A$ and their Chipman pseudoinverse $A^+_{N_k,N}$.

Remark 5.3 Now we describe in more details the *Generalized Greville algorithm* for matrix $A \in \mathcal{M}_{m,n}$, p.d.s. matrices $M \in \mathcal{M}_{m,m}$, $N \in \mathcal{M}_{n,n}$ and sequence p.d.s. matrices $\{I_i\}_{i=1}^{m-1}$. Construction of matrices E_{k-1}, E_k using singular value decomposition of p.d.s. matrices I_k, I_{k-1}, I_1 are given in Remark 4.5.

Algorithm:

1. Let $M = UDU^T$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ is singular value decomposition of matrix M.

   Then owing to theorem 5.8 is $(m-1) \times m$ matrix

$$A_{m-1} = E_{m-1}A = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \ldots & 0 & 0 \\ 0 & \sqrt{\lambda_2} & \ldots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & \sqrt{\lambda_{m-1}} & 0 \end{pmatrix} U^T A$$

   and vector $1 \times m$

$$a_m = E_1 A = \begin{pmatrix} 0_{1,m-1} & \sqrt{\lambda_m} \end{pmatrix} U^T A.$$

   For $k = 1, 2, \ldots, m-1$ denote a_k the k-th row of matrix A_{m-1} and

$$A_k = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$$

   the submatrix formed from the first k rows of matrix A_{m-1}.

2. Put

$$A_1^+ = N^{-1}A_1^T \left(A_1 N^{-1} A_1^T\right)^{-1}$$

3. Let for $k = 2, 3, \ldots, m$ is

$$\begin{aligned} d_k &= a_k A^+_{k-1} \\ c_k &= a_k - d_k A_{k-1} \\ b_k &= \begin{cases} \left(c_k N^{-1} c_k^T\right)^{-1} c_k & \text{if } c_k \neq 0 \\ \left(1 + d_k d_k^T\right)^{-1} d_k \left(A^+_{k-1}\right)^T N & \text{if } c_k = 0. \end{cases} \end{aligned}$$

Then for $k = 2, \dots, m-1$ is

$$A_k^+ = \left(A_{k-1}^+ - N^{-1}b_k^T d_k \; N^{-1}b_k^T\right)$$

and

$$A_m^+ = \left(A_{m-1}^+ - N^{-1}b_m^T d_m \; N^{-1}b_m^T\right) D^{1/2}U^T.$$

Matrix A_m^+ is Chipman pseudoinverse of matrix A for given p.d.s. matrices M, N.

6 Optimal properties of some cubic splines

Let us have given function values g_i, $i = 0, 1, \dots, n+1$ in spline knots

$$(\Delta x): \qquad x_0 < x_1 < x_2 < \dots < x_n < x_{n+1}.$$

The cubic splines $S_{31}(x) \in \mathcal{C}^2$ interpolating prescribed values have two free parametres which can be used for some boundary condition. In spline theory is known, that the minimum of the functional

$$J(s) = \|s''\|_2^2 = \int\limits_{x_0}^{x_{n+1}} \left[s''(x)\right]^2 \mathrm{d}x \tag{34}$$

is attained by interpolatory natural cubic spline on the class of interpolants from W_2^2 (see [6]).

With local parametres $g_i = s(x_i)$ and $M_i = s''(x_i)$, we can state the continuity conditions as recurrences (see in [6])

$$h_{i-1}M_{i-1} + 2\left(h_{i-1} + h_i\right) M_i + h_i M_{i+1} = f_i, \qquad i = 1\,(1)\,n \tag{35}$$

where

$$h_i = x_{i+1} - x_i, \quad f_i = 3\left[\frac{g_{i+1} - g_i}{h_i} - \frac{g_i - g_{i-1}}{h_{i-1}}\right].$$

Recurrences (42) we can write in vector notation as

$$AM = f \tag{36}$$

with tridiagonal matrix A of size $n \times (n+2)$, $\mathrm{rank}(A) = n$,

$$A = \begin{pmatrix} h_0 & 2(h_0+h_1) & h_1 & & & & \\ & h_1 & 2(h_1+h_2) & h_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} & \\ & & & & h_{n-1} & 2(h_{n-1}+h_n) & h_n \end{pmatrix}$$

and vectors

$$M = \left(M_0, M_1, M_2, \dots, M_{n+1}\right)^T, \qquad f = \left(f_1, f_2, \dots, f_n\right)^T$$

Functional (41) we can rewrite

$$J(s) = \sum_{i=0}^{n} \frac{1}{h_i} \left(M_i^2 + M_i M_{i+1} + M_{i+1}^2 \right) = \frac{1}{6} M^T R M, \tag{37}$$

with p.d.s. matrix R of size $(n+2) \times (n+2)$.

So owing to (5) there is

$$J(s) = \frac{1}{6} \|s''\|_R^2$$

We want to find the minimum of the functional

$$J(s) = M^T R M \tag{38}$$

owing to

$$AM = f. \tag{39}$$

In other words, we want to find such vector M of second derivatives of interpolatory cubic spline, which minimizes functional (38).

In regard to Theorem 2.7 we can state following

Theorem 6.1 *Let us have given spline knots (Δx) and values g_i in knots x_i, $i = 0, 1, \ldots, n+1$. Then functional (38) is minimized, in the class of cubic splines on the given knotset (Δx) for data g_i, by interpolatory cubic spline, whose local parametres $\hat{M}_i$ are given as R-least squares solution of equation (39), so*

$$\hat{M} = A^{+}_{\cdot,R} f, \tag{40}$$

where matrix $A^{+}_{\cdot,R}$ is Chipman pseudoinverse of matrix A (see Theorem 2.4). This cubic spline is unique.

7 Numerical results

We compare computing Chipman pseudoinverse of random matrices by given methods:

1. *Rank factorization*
2. *Iterative method*
3. *Generalized Greville algorithm*

Note, that for computing by Rank factorization and Iterative method we must know rank of matrix, but for computing by Generalized Greville algorithm is not necessary.

Denote A_i^+ as Chipman pseudoinverse computing by i-th method for matrix A and fixed p.d.s. matrices M, N of corresponding size. Now let

$$\begin{aligned} v(1) &= \max\left(AA_i^+A - A\right) \\ v(2) &= \max\left(A_i^+AA_i^+ - A_i^+\right) \\ v(3) &= \max\left((MAA_i^+)^T - MAA_i^+\right) \\ v(4) &= \max\left((NA_i^+A)^T - NA_i^+A\right) \end{aligned}$$

and $v = \max_i v(i)$, now v_i denotes v computed by i-th method and t_i denotes time necessary for computing Chipman pseudoinverse by i-th method.

Follow numerical computing were given on computer Intel Pentium II, 333 MHz, RAM 64 MB, HDD 4.8 GB for matrices with random numbers from interval $(0, 1)$ of size $m \times n$, with variant rank $= r$. P.d.s. matrix M, N are full matrix. Computing Chipman pseudoinverse by Iterative method use $q = 15$, see section 3. For computing by Generalized Greville algorithm we use sequence of identity matrices, then we count by algorithm given in Remark 5.3.

m, n	r	v_1	v_2	v_3
$50, 100$	50	$1.87\ 10^{-10}$	$1.51\ 10^{-10}$	
$75, 100$	50	$2.10\ 10^{-10}$	$2.52\ 10^{-10}$	$3.74\ 10^{-9}$
$100, 100$	50	$5.59\ 10^{-10}$	$9.62\ 10^{-10}$	$1.92\ 10^{-8}$
$150, 100$	100	$4.48\ 10^{-9}$	$2.60\ 10^{-7}$	$2.42\ 10^{-8}$
$150, 100$	50	$1.15\ 10^{-9}$	$2.95\ 10^{-9}$	$5.89\ 10^{-8}$
$250, 100$	50	$1.69\ 10^{-9}$	$7.20\ 10^{-9}$	$7.68\ 10^{-8}$
$250, 100$	100	$5.07\ 10^{-9}$	$1.42\ 10^{-7}$	$3.79\ 10^{-9}$
$250, 250$	250	$1.26\ 10^{-6}$	$2.68\ 10^{-5}$	—
$800, 800$	600	$1.15\ 10^{-5}$	$1.24\ 10^{-5}$	—

m, n	r	t_1	t_2	t_3
$50, 100$	50	0.22	5.6	6.97
$75, 100$	50	0.22	7.69	5.38
$100, 100$	50	0.28	9.94	6.36
$150, 100$	100	0.55	16.64	28.18
$150, 100$	50	0.6	24.11	20.44
$250, 100$	50	0.99	51.96	34.11
$250, 100$	100	1.04	31.53	30.48
$800, 800$	600	$3.74\ 10^2$	$2.87\ 10^3$	—

From fhese two tables we can see, when we know rank of matrix, then computing by Rank factorization gives very good results. Generalized Greville algorithm gives good results too, but this method need more time. This algorithm is in place to use, when we do not know rank of matrix.

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