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# The Natural Operators Transforming Affinors to Tensor Fields of Type (3,3)

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**Abstract**

We give a complete classification of natural operators transforming affinors to tensor fields of type (3,3).

**Key words:** Affinor, natural operator, equivariant map.

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There are few works giving a clasification of some type natural operators transforming affinors (the author known only [2], [3] and [6]) although the affinors play important roles in differential geometry (for instance complex structures, tangent structures). The reason of this is that the problem of finding all some type natural operators transforming affinors is mostly very difficult. It seems that investigation of such natural operators ought to start with determining all natural transformations of affinors to tensors of type $(r, r)$, where $r$ is a non-negative integer, because they are relatively simple to classify and can next be used for determining other type natural operators transforming affinors. For instance we hope the result obtained in this paper can be applied to finding a classification of natural operators transforming affinors to tensor fields of type (1,2) (the Nijenhuis tensor type) in a similar manner as in [3] the classification of natural operators transforming affinors to tensor fields of type (2,2) is applied to finding a classification of these transforming affinors to tensor fields of type (0,1). It is worth pointing out that the natural transformations of affinors to tensors of type $(r, r)$ can also be helpful in more complicated cases. For example if we apply the homogeneous function theorem to a natural operator lifting affinors to the cotangent bundle, there will appear natural transformations of affinors to tensors of type $(r,r)$ for $r = 1,2,3,4$.

All natural transformations of affinors to tensors of type $(r,r)$ for $r = 0$ and $r = 1$ are classified in [2], for $r = 2$ in [3]. Unfortunately, the methods used in these cases are inadequate to investigate the cases $r > 2$. In this paper we give a full characterization of natural transformations of affinors to tensors of type (3,3). Since they form a module over the ring consisting of the known natural transformations of affinors to tensors of type (0,0), we prove that this module is free and finite-dimensional, and we find a basis of it.

The *affinor* on a smooth manifold $M$ is, by definition, a smooth tensor field of type (1,1) on $M$. We will use the symbol $X^p_qM$ to denote the vector space of all smooth tensor fields of type $(p,q)$ on $M$ for non-negative integers $p,q$. If $\varphi : M \longrightarrow N$ is an immersion between two smooth manifolds of the same dimension then two tensor fields $t \in X^p_qM$ and $u \in X^p_qN$ are said to be $\varphi$-related, if

$$u(\varphi(x)) = \underbrace{T_x\varphi \otimes \ldots \otimes T_x\varphi}_{p \text{ times}} \otimes \underbrace{(T_x\varphi)^{-1*} \otimes \ldots \otimes (T_x\varphi)^{-1*}}_{q \text{ times}}(t(x))$$

for every $x \in M$.

Let $n,p,q,r,s$ be non-negative integers.

**Definition 1** A family of maps $A_M : X^p_qM \longrightarrow X^r_sM$, where $M$ is an arbitrary $n$-dimensional smooth manifold, is called the *natural operator* transforming tensor fields of type $(p,q)$ to tensor fields of type $(r,s)$ if for every injective immersion $\varphi : M \longrightarrow N$ between two $n$-dimensional smooth manifolds, for every $t \in X^p_qM$ and every $u \in X^p_qN$ the tensor fields $A_M(t)$ and $A_N(u)$ are $\varphi$-related whenever $t$ and $u$ are $\varphi$-related.

Let $k$ be a non-negative integer. A natural operator $A$ transforming tensor fields of type $(p,q)$ to tensor fields of type $(r,s)$ is said to be of order $k$ if for any $n$-dimensional smooth manifold $M$, any $x \in M$ and all $t,u \in X^p_qM$ the following implication

$$j^k_xt = j^k_xu \Longrightarrow (A_Mt)(x) = (A_Mu)(x)$$

holds (here $j^k_xt$ denotes the $k$-jet of $t$ at $x$). In [2] it is proved that if $p = q$ and $r = s$ then every natural operator transforming tensor fields of type $(p,q)$ to tensor fields of type $(r,s)$ has order zero. This reduces the problem of finding natural operators to determining equivariant maps (see [4], [5]).

Write

$$T^p_qV = \underbrace{V \otimes \ldots \otimes V}_{p \text{ times}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{q \text{ times}}.$$

for any vector space $V$. The group $GL(n,K)$, where $K$ is a field, acts on $T^p_qK^n$ in the natural way: if $t \in T^p_qK^n$ and $A \in GL(n,K)$ then

$$(t \cdot A)^{i_1\ldots i_p}_{j_1\ldots j_q} = (A^{-1})^{i_1}_{k_1} \ldots (A^{-1})^{i_p}_{k_p} t^{k_1\ldots k_p}_{l_1\ldots l_q} A^{l_1}_{j_1} \ldots A^{l_q}_{j_q}$$

for all $i_1,\ldots,i_p,j_1,\ldots,j_q \in \{1,\ldots,n\}$.

**Definition 2** A map $a : T^p_q\mathbf{R}^n \longrightarrow T^r_s\mathbf{R}^n$ is called *equivariant* if

1. $a(t \cdot A) = a(t) \cdot A$ for all $t \in T^p_q\mathbf{R}^n$, $A \in GL(n, \mathbf{R})$,
2. $a \circ b$ is smooth for every smooth map $b : \mathbf{R}^n \longrightarrow T^p_q\mathbf{R}^n$.

(The latter condition forces the smoothness of $a$, but a proof of this is not simple, see [1]).

The set of all such equivariant maps will be denoted by $E_{(p,q),(r,s),n}$. Using standard methods (see [4], [5], [2]) we can show that there is a one-to-one correspondence between natural operators of order zero transforming tensor fields of type $(p, q)$ to tensor fields of type $(r, s)$ and equivariant maps from $E_{(p,q),(r,s),n}$. Namely, if $A$ is a natural operator then the corresponding equivariant map $a$ is defied by

$$a(t(0)) = A_{\mathbf{R}^n}(t)(0)$$

for any $t \in X^p_q\mathbf{R}^n$ (since $A$ has order zero, the definition is independent of a choice of $t$). Conversely, if $a \in E_{(p,q),(r,s),n}$ then the corresponding natural operator is for every $n$-dimensional smooth manifold $M$, every $t \in X^p_qM$ and every $x \in M$ defined by

$$A_M(t)(x) = \underbrace{(T_x\varphi)^{-1} \otimes \ldots \otimes (T_x\varphi)^{-1}}_{r \text{ times}} \otimes \underbrace{(T_x\varphi)^* \otimes \ldots \otimes (T_x\varphi)^*}_{s \text{ times}}(a(u)),$$

where $\varphi$ is a chart on $M$ such that $\varphi(x) = 0$ and

$$u = \underbrace{T_x\varphi \otimes \ldots \otimes T_x\varphi}_{p \text{ times}} \otimes \underbrace{(T_x\varphi)^{-1*} \otimes \ldots \otimes (T_x\varphi)^{-1*}}_{q \text{ times}}(t(x)).$$

Since we have established the relation between all natural operators and equivariant maps for $p = q$ and $r = s$, from now on we will study equivariant maps instead of natural operators.

We first observe that $E_{(p,q),(0,0),n}$ is a ring and $E_{(p,q),(r,s),n}$ is a module over $E_{(p,q),(0,0),n}$. In the paper [2] it is given a classification of equivariant maps transforming tensors of type (1,1) to tensors of type (0,0). Namely, for every $a \in E_{(1,1),(0,0),n}$ there is a uniquely determined smooth function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ such that

$$a(t) = f(c_1(t), \ldots, c_n(t)) \tag{1}$$

for every $t \in T^1_1\mathbf{R}^n$, where $c_i : T^1_1\mathbf{R}^n \longrightarrow \mathbf{R}$ for $i \in \{1, \ldots, n\}$ are the coefficients of the characteristic polynomial of the linear endomorphism i.e.

$$\det(\lambda \operatorname{id}_{\mathbf{R}^n} - t) = \lambda^n + \sum_{i=1}^{n} c_i(t)\lambda^{n-i} \tag{2}$$

for every $\lambda \in \mathbf{R}$ and $t \in T^1_1\mathbf{R}^n$. Of course, the converse statement also is true: for every smooth map $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ formula (1) defines an equivariant map $a \in E_{(1,1),(0,0),n}$.

We can also construct examples of equivariant maps from $E_{(1,1),(r,r),n}$ for each non-negative integer $r$.

**Example 1** Suppose that $\psi : \{1, \ldots, r\} \longrightarrow \mathbf{N}$ and $\sigma \in S_r$, where $\mathbf{N}$ is the set of all non-negative integers and $S_r$ denotes the set of all permutations of the set $\{1, \ldots, r\}$. Put

$$e_{\psi,\sigma}(t)^{i_1 \ldots i_r}_{j_1 \ldots j_r} = (t^{\psi(1)})^{i_{\sigma(1)}}_{j_1} \ldots (t^{\psi(r)})^{i_{\sigma(r)}}_{j_r} \tag{3}$$

for every $t \in T^1_1\mathbf{R}^n$ and all $i_1, \ldots, i_r, j_1, \ldots, j_r \in \{1, \ldots, n\}$. Here $t^k$, where $k$ is a non-negative integer, stands for

$$\underbrace{t \circ \ldots \circ t}_{k \text{ times}}.$$

It is immediate that $e_{\psi,\sigma} \in E_{(1,1),(r,r),n}$.

We are now in a position to formulate our main result.

**Theorem 1** *The equivariant maps $e_{\psi,\sigma}$ for $\psi : \{1, 2, 3\} \longrightarrow \{0, \ldots, n-1\}$ and $\sigma \in S_3$ satisfying one out of the following four conditions:*

1. $\psi(1) = n - 1$, $\sigma = \mathrm{id}_{\{1,2,3\}}$,
2. $\psi(1) < n - 1$ *and* $\psi(2) = n - 1$,
$$\sigma \in \{\mathrm{id}_{\{1,2,3\}}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\},$$
3. $\psi(1) = n - 2$ *and* $\psi(2) < n - 1$,
$$\sigma \in \{\mathrm{id}_{\{1,2,3\}}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\},$$
4. $\psi(1) < n - 2$ *and* $\psi(2) < n - 1$

*form a basis of the module $E_{(1,1),(3,3),n}$.*

In order to prove this theorem it is convenient to show a lemma. We will denote by $M_{r,n}$ the set of all pairs $(\alpha, \beta)$ of maps $\alpha : \{1, \ldots, r\} \longrightarrow \{1, \ldots, n\}$ and $\beta : \{1, \ldots, r\} \longrightarrow \{1, \ldots, n\}$ such that for every $i \in \{1, \ldots, n\}$ the numbers of elements of the sets $\alpha^{-1}(\{i\})$ and $\beta^{-1}(\{i\})$ are equal. The number of elements of $M_{r,n}$ will be denoted by $m(r, n)$.

**Lemma 1** *Let $t \in T^1_1\mathbf{R}^n$ be a linear endomorphism of $\mathbf{R}^n$ with $n$ different complex eingenvalues. Then there is a vector subspace $V \subset T^r_r\mathbf{R}^n$ such that $\dim V \leq m(r, n)$ and that for every $a \in E_{(1,1),(r,r),n}$ we have $a(t) \in V$.*

**Proof** Set

$$S = \{u \in T^1_1\mathbf{C}^n : \exists_{A \in GL(n,\mathbf{C})}\ u = t \cdot A\}.$$

We first fix an equivariant map $a : T^1_1\mathbf{R}^n \longrightarrow T^r_r\mathbf{R}^n$ and define a $GL(n, \mathbf{C})$-equivariant map $\widetilde{a} : S \longrightarrow T^r_r\mathbf{C}^n$ such that $\widetilde{a}(t) = a(t)$. For this purpose we put $\widetilde{a}(t \cdot A) = a(t) \cdot A$ for every $A \in GL(n, \mathbf{C})$. We have to show that the definition makes sense. Assume that $t\cdot B = t\cdot A$ for any $B \in GL(n, \mathbf{C})$. Thus $t\cdot(BA^{-1}) = t$. Of course, we have $a(t) \cdot B = a(t) \cdot A$ whenever $a(t) \cdot (BA^{-1}) = a(t)$. Therefore it suffices to prove that if $t \cdot C = t$ then $a(t) \cdot C = a(t)$, where $C \in GL(n, \mathbf{C})$. Obviously, there are real matrices $D, E$ such that $C = D + iE$. Treating $t$ as a matrix, we can rewrite $t \cdot C = t$ in the form $C^{-1}tC = t$, and so $tC - Ct = 0$. Hence $t(D + iE) - (D + iE)t = 0$, which gives $tD - Dt = 0$ and $tE - Et = 0$, because the matrix $t$ is real. Set $U = \{Z \in T^1_1\mathbf{R}^n : tZ - Zt = 0\}$. Consequently $D, E \in U$. Let us denote by $f$ the polynomial $T^1_1\mathbf{R}^n \longrightarrow T^r_r\mathbf{R}^n$ given by

$$f(Z)^{i_1\dots i_r}_{j_1\dots j_r} = a(t)^{i_1\dots i_r}_{k_1\dots k_r} Z^{k_1}_{j_1} \dots Z^{k_r}_{j_r} - Z^{i_1}_{l_1} \dots Z^{i_r}_{l_r} a(t)^{l_1\dots l_r}_{j_1\dots j_r}$$

for $Z \in T^1_1\mathbf{R}^n$ and $i_1, \dots, i_r, j_1 \dots, j_r \in \{1, \dots, n\}$. We have $f(Z) = 0$ for every $Z \in U \cap GL(n, \mathbf{R})$, which is due to the fact that the map $a$ is equivariant. Since $f$ is continuous and $U \cap GL(n, \mathbf{R})$ is a dense subset of $U$, we have $f(Z) = 0$ for every $Z \in U$. Let $\widetilde{U}$ be the complexification of the vector space $U$ i.e. let

$$\widetilde{U} = \{Z \in T^1_1\mathbf{C}^n : \exists_{X,Y\in U}\ Z = X + iY\}$$

and let $\widetilde{f} : T^1_1\mathbf{C}^n \longrightarrow T^r_r\mathbf{C}^n$ be the complexification of the polynomial $f$ i.e. let

$$\widetilde{f}(Z)^{i_1\dots i_r}_{j_1\dots j_r} = a(t)^{i_1\dots i_r}_{k_1\dots k_r} Z^{k_1}_{j_1} \dots Z^{k_r}_{j_r} - Z^{i_1}_{l_1} \dots Z^{i_r}_{l_r} a(t)^{l_1\dots l_r}_{j_1\dots j_r}$$

for $Z \in T^1_1\mathbf{C}^n$ and $i_1, \dots, i_r, j_1 \dots, j_r \in \{1, \dots, n\}$. It is easy to check that $\widetilde{f}(Z) = 0$ for $Z \in \widetilde{U}$, since $f(Z) = 0$ for $Z \in U$. Thus $\widetilde{f}(C) = \widetilde{f}(D + iE) = 0$, which means that $a(t) \cdot C = a(t)$ and that $\widetilde{a}$ is well defined. The map $\widetilde{a}$ is $GL(n, \mathbf{C})$-equivariant i.e. $\widetilde{a}(u \cdot Z) = \widetilde{a}(u) \cdot Z$ for every $u \in S$ and every $Z \in GL(n, \mathbf{C})$, because for $A \in GL(n, \mathbf{C})$ such that $u = t \cdot A$ we have $\widetilde{a}(u \cdot Z) = \widetilde{a}((t \cdot A) \cdot Z) = \widetilde{a}(t \cdot AZ) = a(t) \cdot AZ = (a(t) \cdot A) \cdot Z = \widetilde{a}(t \cdot A) \cdot Z = \widetilde{a}(u) \cdot Z$.

Since $t$ has $n$ different complex eingenvalues, there are a diagonal complex matrix

$$J = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

and $Q \in GL(n, \mathbf{C})$ such that $J = Q^{-1}tQ = t \cdot Q$, and so $J \in S$. Let us denote by $p_1, \dots, p_n$ $n$ different prime numbers and by $P$ the diagonal matrix

$$P = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix}.$$

Clearly, $P \in GL(n, \mathbf{C})$. As $J \cdot P = P^{-1}JP = J$ and $\widetilde{a}$ is $GL(n, \mathbf{C})$-equivariant we have $\widetilde{a}(J) \cdot P = \widetilde{a}(J)$. Therefore

$$\frac{1}{p_{i_1}} \dots \frac{1}{p_{i_r}} \widetilde{a}(J)^{i_1\dots i_r}_{j_1\dots j_r} p_{j_1} \dots p_{j_r} = \widetilde{a}(J)^{i_1\dots i_r}_{j_1\dots j_r}$$

for $i_1, \dots, i_r, j_1, \dots, j_r \in \{1, \dots, n\}$. Fix $g_1, \dots, g_r, h_1, \dots, h_r \in \{1, \dots, n\}$. If $\widetilde{a}(J)^{g_1 \dots g_r}_{h_1 \dots h_r} \neq 0$ then $p_{g_1} \dots p_{g_r} = p_{h_1} \dots p_{h_r}$, which implies that for every $k \in \{1, \dots, n\}$ the numbers of elements of the sets $\gamma^{-1}(k)$ and $\delta^{-1}(k)$ are equal, where

$$\gamma : \{1, \dots, r\} \ni l \longrightarrow g_l \in \{1, \dots, n\},$$
$$\delta : \{1, \dots, r\} \ni l \longrightarrow h_l \in \{1, \dots, n\}.$$

Writing

$$W = \{Z \in T^r_r \mathbf{C}^n : \forall_{\alpha, \beta \in \{1, \dots, n\}^{\{1, \dots, r\}}} \; Z^{\alpha(1) \dots \alpha(r)}_{\beta(1) \dots \beta(r)} \neq 0 \Longrightarrow (\alpha, \beta) \in M_{r,n}\}$$

we see that the dimension of the complex vector space $W$ equals $m(r, n)$ and that $\widetilde{a}(J) \in W$. Let the $\mathbf{C}$-linear isomorphism $\varphi : T^r_r \mathbf{C}^n \longrightarrow T^r_r \mathbf{C}^n$ be given by $\varphi(Z) = Z \cdot Q^{-1}$ for $Z \in T^r_r \mathbf{C}^n$ and let $V = \varphi(W) \cap T^r_r \mathbf{R}^n$. Obviously, $W$ as well as $Q$ and $\varphi$ are defined independently of $a$, then so is $V$. A trivial verification shows that $\dim_{\mathbf{R}} V \leq \dim_{\mathbf{C}} W = m(r, n)$. Moreover, $a(t) = \widetilde{a}(t) = \widetilde{a}(J \cdot Q^{-1}) = \widetilde{a}(J) \cdot Q^{-1} = \varphi(\widetilde{a}(J)) \in V$, which completes the proof. □

**Proof of theorem** We will assume that the set $K_{r,n} = \{0, \dots, n-1\}^{\{1, \dots, r\}} \times S_r$ is equiped with the following order: for $\psi, \omega : \{1, \dots, r\} \longrightarrow \{0, \dots, n-1\}$ and $\sigma, \tau \in S_r$ we have $(\psi, \sigma) < (\omega, \tau)$ if and only if

$$(\psi(1), \sigma(1), \dots, \psi(r), \sigma(r)) < (\omega(1), \tau(1), \dots, \omega(r), \tau(r))$$

with respect to the lexicographic order i.e.

$$\begin{aligned}(\psi, \sigma) < (\omega, \tau) \Longleftrightarrow \exists_{k \in \{1, \dots, r\}} \; (\forall_{l \in \{1, \dots, k-1\}} \; \psi(l) = \omega(l) \wedge \sigma(l) = \tau(l)) \\ \wedge(\psi(k) < \omega(k) \vee (\psi(k) = \omega(k) \wedge \sigma(k) < \tau(k))).\end{aligned}$$

The standard system of coordinates on $T^r_r \mathbf{R}^n$ will be denoted by $\pi$. Thus

$$\pi^{i_1 \dots i_r}_{j_1 \dots j_r} : T^r_r \mathbf{R}^n \ni Z \longrightarrow Z^{i_1 \dots i_r}_{j_1 \dots j_r} \in \mathbf{R}$$

for all $i_1, \dots, i_r, j_1, \dots, j_r \in \{1, \dots, n\}$. For $z \in \mathbf{R}^n$ the linear endomorphism $t_z : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is defined by imposing the following conditions: $t_z(e_i) = e_{i+1}$ for $i \in \{1, \dots, n-1\}$ and $t_z(e_n) = -z^n e_1 - \dots - z^1 e_n$, where $e_1, \dots, e_n$ denotes the canonical basis of $\mathbf{R}^n$, i.e.

$$t_z = \begin{bmatrix} 0 & & & -z^n \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -z^2 \\ & & 1 & -z^1 \end{bmatrix} \tag{4}$$

in the canonical basis. Finally, let us denote by $L_{r,n}$ the set consisting of all pairs $(\psi, \sigma) \in K_{r,n}$ with the property that there exists any coordinate such that this coordinate of $e_{\psi,\sigma}(t_z)$ equals 1 for every $z \in \mathbf{R}^n$ and that $(\psi, \sigma)$ is the minimal element of $K_{r,n}$ for which this coordinate of $e_{\psi,\sigma}(t_z)$ does not vanish for some $z \in \mathbf{R}^n$ i.e.

$$L_{r,n} = \{(\psi,\sigma) \in K_{r,n} : \exists_{i_1,\dots,i_r,j_1,\dots,j_r \in\{1,\dots,n\}} (\forall_{z\in\mathbf{R}^n}\ \pi^{i_1\dots,i_r}_{j_1\dots j_r}(e_{\psi,\sigma}(t_z)) = 1$$
$$\wedge \forall_{(\omega,\tau)\in K_{r,n}}\ (\omega,\tau) < (\psi,\sigma) \implies (\forall_{z\in\mathbf{R}^n}\ \pi^{i_1\dots,i_r}_{j_1\dots j_r}(e_{\omega,\tau}(t_z)) = 0))\}.$$

By (4),

$$(k \le n - j \wedge k \ne i - j) \implies (t^k_z)^i_j = 0,$$
$$k = i - j \implies (t^k_z)^i_j = 1$$

for every $z \in \mathbf{R}^n$, every non-negative integer $k$ and all $i, j \in \{1, \dots, n\}$. Therefore from (3) we see that

$$(\exists_{k\in\{1,\dots,r\}}\ \psi(k) \le n - j_k \wedge \psi(k) \ne i_{\sigma(k)} - j_k) \implies \pi^{i_1,\dots,i_r}_{j_1,\dots,j_r}(e_{\psi,\sigma}(t_z)) = 0, \quad (5)$$
$$(\forall_{k\in\{1,\dots,r\}}\ \psi(k) = i_{\sigma(k)} - j_k) \implies \pi^{i_1,\dots,i_r}_{j_1,\dots,j_r}(e_{\psi,\sigma}(t_z)) = 1 \quad (6)$$

for every $(\sigma,\tau) \in K_{r,n}$, every $z \in \mathbf{R}^n$ and all $i_1, \dots, i_r, j_1, \dots, j_r \in \{1, \dots, n\}$.

Fix $i_1, \dots, i_r, j_1, \dots, j_r \in \{1, \dots, n\}$. We now describe an algorithm for finding the minimal pair $(\psi,\sigma) \in K_{r,n}$ such that $\pi^{i_1,\dots,i_r}_{j_1,\dots,j_r}(e_{\psi,\sigma}(t_z)) \ne 0$ for some $z \in \mathbf{R}^n$. The construction of $(\psi,\sigma)$ is by induction. Our algorithm does not work for arbitrary $i_1, \dots, i_r, j_1, \dots, j_r \in \{1, \dots, r\}$. Necessary conditions will be formulated in the course of the construction. Suppose that $\psi(1), \sigma(1), \dots, \psi(k-1), \sigma(k-1)$ are defined, where $k \in \{1, \dots, r\}$. We will define $\psi(k)$ and $\sigma(k)$. Let

$$g = \min\{i \in \{1, \dots, n\} : \exists_{u\in\{1,\dots,r\}} (\forall_{v\in\{1,\dots,k-1\}}\ u \ne \sigma(v)) \wedge i = i_u\}.$$

If we put $\psi(k) = l$, where $l < g - j_k$, then, by (5), we would obtain

$$\pi^{i_1,\dots,i_r}_{j_1,\dots,j_r}(e_{\psi,\sigma}(t_z)) = 0$$

for every $z \in \mathbf{R}^n$ (irrespective of a choice of $\sigma(k), \psi(k+1), \sigma(k+1), \dots, \psi(r), \sigma(r)$), because it have to be $\sigma(k) \notin \{\sigma(1), \dots, \sigma(k-1)\}$. Thus, whenever $g - j_k \ge 0$, we put $\psi(k) = g - j_k$. If $g - j_k < 0$, then the algorithm breaks down. Let

$$h = \min\{u \in \{1, \dots, r\} : (\forall_{v\in\{1,\dots,k-1\}}\ u \ne \sigma(v)) \wedge i_u = g\}.$$

If we put $\sigma(k) = m$, where $m < h$ and $m \notin \{\sigma(1), \dots, \sigma(k-1)\}$, then, by (5), we would obtain $\pi^{i_1,\dots,i_r}_{j_1,\dots,j_r}(e_{\psi,\sigma}(t_z)) = 0$ for every $z \in \mathbf{R}^n$ (irrespective of a choice of $\psi(k+1), \sigma(k+1), \dots, \psi(r), \sigma(r)$). Thus we put $\sigma(k) = h$. If it is possible to continue the construction to the very end, then, by (6), we obtain $(\psi,\sigma)$ such that $\pi^{i_1,\dots,i_r}_{j_1,\dots,j_r}(e_{\psi,\sigma}(t_z)) = 1$ for every $z \in \mathbf{R}^n$ and it is clear that $(\psi,\sigma)$ is the minimal element of $K_{r,n}$ such that $\pi^{i_1,\dots,i_r}_{j_1,\dots,j_r}(e_{\psi,\sigma}(t_z)) \ne 0$ for some $z \in \mathbf{R}^n$. In fact, we see that $(\psi,\sigma) \in L_{r,n}$, which is worth pointing out.

In order to estimate the number of elements of $L_{3,r}$, we divide the set $L_{3,n}$ into four disjoint subsets:

$$\begin{aligned} A &= \{(\psi,\sigma) \in L_{3,n} : \psi(1) = n-1\}, \\ B &= \{(\psi,\sigma) \in L_{3,n} : \psi(1) < n-1 \wedge \psi(2) = n-1\}, \\ C &= \{(\psi,\sigma) \in L_{3,n} : \psi(1) = n-2 \wedge \psi(2) < n-1\}, \\ D &= \{(\psi,\sigma) \in L_{3,n} : \psi(1) < n-2 \wedge \psi(2) < n-1\}. \end{aligned}$$

If $\psi : \{1,2,3\} \longrightarrow \{0,\dots,n-1\}$ is such that $\psi(1) = n-1$ than $(\psi, \mathrm{id}_{\{1,2,3\}}) \in$ *$K_{3,n}$. It is easily seen that our algorithm can by successfully applied to the coordinate $\pi^{n,n,n}_{1,n-\psi(2),n-\psi(3)}$. As the result we get just $(\psi, \mathrm{id}_{\{1,2,3\}})$, hence $(\psi, id_{\{1,2,3\}}) \in L_{3,n}$, and so $(\psi, id_{\{1,2,3\}}) \in A$. Since there are $n^2$ maps $\psi$ : $\{1,2,3\} \longrightarrow \{0,\dots,n-1\}$ such that $\psi(1) = n-1$, there are at least $n^2$ elements of $A$.*

*If $\psi : \{1,2,3\} \longrightarrow \{0,\dots,n-1\}$ is such that* $\psi(1) < n-1$ and $\psi(2) = n-1$ than we see that:

1. Our algorithm applied to the coordinate $\pi^{n-1,n,n}_{n-1-\psi(1),1,n-\psi(3)}$ yields the pair $(\psi, \mathrm{id}_{\{1,2,3\}})$. Hence $(\psi, \mathrm{id}_{\{1,2,3\}}) \in B$.

2. Our algorithm applied to the coordinate $\pi^{n,n-1,n}_{n-1-\psi(1),1,n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}).$$

Hence this pair is an element of $B$.

3. Our algorithm applied to the coordinate $\pi^{n,n,n-1}_{n-1-\psi(1),1,n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}).$$

Hence this pair is an element of $B$.

Since there are $n(n-1)$ maps $\psi : \{1,2,3\} \longrightarrow \{0,\dots,n-1\}$ such that $\psi(1) < n-1$ and $\psi(2) = n-1$, there are at least $3n(n-1)$ elements of $B$.

If $\psi : \{1,2,3\} \longrightarrow \{0,\dots,n-1\}$ is such that $\psi(1) = n-2$ and $\psi(2) < n-1$ than we see that:

1. Our algorithm applied to the coordinate $\pi^{n-1,n-1,n}_{1,n-1-\psi(2),n-\psi(3)}$ yields the pair $(\psi, \mathrm{id}_{\{1,2,3\}})$. Hence $(\psi, \mathrm{id}_{\{1,2,3\}}) \in C$.

2. Our algorithm applied to the coordinate $\pi^{n-1,n,n-1}_{1,n-1-\psi(2),n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}).$$

Hence this pair is an element of $C$.

3. Our algorithm applied to the coordinate $\pi^{n,n-1,n-1}_{1,n-1-\psi(2),n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}).$$

Hence this pair is an element of $C$.

4. Our algorithm applied to the coordinate $\pi^{n,n-1,n}_{1,n-\psi(2),n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}).$$

Hence this pair is an element of $C$.

5. Our algorithm applied to the coordinate $\pi^{n,n,n-1}_{1,n-\psi(2),n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}).$$

Hence this pair is an element of $C$.

Since there are $n(n-1)$ maps $\psi : \{1,2,3\} \longrightarrow \{0,\dots,n-1\}$ such that $\psi(1) = n-2$ and $\psi(2) < n-1$, there are at least $5n(n-1)$ elements of $C$.

If $\psi : \{1,2,3\} \longrightarrow \{0,\dots,n-1\}$ is such that $\psi(1) < n-2$ and $\psi(2) < n-1$ than we see that:

1. Our algorithm applied to the coordinate $\pi^{n-2,n-1,n}_{n-2-\psi(1),n-1-\psi(2),n-\psi(3)}$ yields the pair $(\psi, \mathrm{id}_{\{1,2,3\}})$. Hence $(\psi, \mathrm{id}_{\{1,2,3\}}) \in D$.

2. Our algorithm applied to the coordinate $\pi^{n-2,n,n-1}_{n-2-\psi(1),n-1-\psi(2),n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}).$$

Hence this pair is an element of $D$.

3. Our algorithm applied to the coordinate $\pi^{n-1,n-2,n}_{n-2-\psi(1),n-1-\psi(2),n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}).$$

Hence this pair is an element of $D$.

4. Our algorithm applied to the coordinate $\pi^{n-1,n,n-2}_{n-2-\psi(1),n-1-\psi(2),n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}).$$

Hence this pair is an element of $D$.

5. Our algorithm applied to the coordinate $\pi^{n,n-2,n-1}_{n-2-\psi(1),n-1-\psi(2),n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}).$$

Hence this pair is an element of $D$.

6. Our algorithm applied to the coordinate $\pi^{n,n-1,n-2}_{n-2-\psi(1),n-1-\psi(2),n-\psi(3)}$ yields the pair

$$(\psi, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}).$$

Hence this pair is an element of $D$.

Since there are $n(n-1)(n-2)$ maps $\psi : \{1,2,3\} \longrightarrow \{0,\ldots,n-1\}$ such that $\psi(1) < n-2$ and $\psi(2) < n-1$, there are at least $6n(n-1)(n-2)$ elements of $D$.

Of course, the elements of $A \cup B \cup C \cup D$ found above coincide with those specified in our theorem. Let $P_n$ denote the set consisting of these elements. Therefore $P_n \subset L_{3,n}$ and the number of elements of $P_n$ is equal to $n^2 + 3n(n-1) + 5n(n-1) + 6n(n-1)(n-2) = 6n^3 - 9n^2 + 4n$. The set of the sequences of positive integers $(i_1, i_2, i_3, j_1, j_2, j_3)$, which we used above to obtain the elements of $P_n$, will be denoted by $Q_n$. We will write $\gamma(i_1, i_2, i_3, j_1, j_2, j_3)$ for the result of our algorithm applied to $(i_1, i_2, i_3, j_1, j_2, j_3) \in Q_n$. Therefore $\gamma : Q_n \longrightarrow P_n$ is a bijection.

Let $z \in \mathbf{R}^n$. We now show that the vectors $e_{\psi,\sigma}(t_z) \in T_3^3\mathbf{R}^n$ for $(\sigma, \psi) \in P_n$ are linearly independent. Suppose that

$$\sum_{(\psi,\sigma)\in P_n} \lambda_{\psi,\sigma} e_{\psi,\sigma}(t_z) = 0,$$

where $\lambda_{\psi,\sigma} \in \mathbf{R}$ for $(\psi, \sigma) \in P_n$. We have to prove that $\lambda_{\psi,\sigma} = 0$ for $(\psi, \sigma) \in P_n$. The proof is by induction on $(\psi, \sigma)$. Fix $(\psi, \sigma) \in P_n$ and assume $\lambda_{\omega,\tau} = 0$ for $(\omega, \tau) \in P_n$ such that $(\omega, \tau) > (\psi, \sigma)$. Taking $(i_1, i_2, i_3, j_1, j_2, j_3) = \gamma^{-1}(\psi, \sigma)$ we get

$$\lambda_{\psi,\sigma} = \pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}\Big(\sum_{(\psi,\sigma)\in P_n} \lambda_{\psi,\sigma} e_{\psi,\sigma}(t_z)\Big) = 0,$$

which is our claim.

If $z \in \mathbf{R}^n$ is such that $t_z$ has $n$ different complex eingenvalues then, by our lemma, there is a subspace $V \subset T_3^3\mathbf{R}^n$ such that $\dim V \leq m(3,n)$ and $a(t_z) \in V$ for every $a \in E_{(1,1),(3,3),n}$. An easy computation shows that $m(3,n) = 6n^3 - 9n^2 + 4n$, which is equal to the number of elements of $P_n$, and of $Q_n$ as well. Therefore $e_{\psi,\sigma}(t_z)$ for $(\psi, \sigma) \in P_n$ form a basis of $V$. We now prove that if $x \in V$ is such that $\pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}(x) = 0$ for every $(i_1, i_2, i_3, j_1, j_2, j_3) \in Q_n$, then $x = 0$. The vector $x$ is a linear combination of the vectors of our basis of $V$, i. e.

$$x = \sum_{(\psi,\sigma)\in P_n} x_{\psi,\sigma} e_{\psi,\sigma}(t_z),$$

where $x_{\psi,\sigma} \in \mathbf{R}$ for $(\psi, \sigma) \in P_n$. Thus it is sufficient to show that $x_{\psi,\sigma} = 0$ for every $(\psi, \sigma) \in P_n$. The proof is by induction on $(\psi, \sigma)$. Fix $(\psi, \sigma) \in P_n$ and assume $x_{\omega,\tau} = 0$ for $(\omega, \tau) \in P_n$ such that $(\omega, \tau) > (\psi, \sigma)$. Taking $(i_1, i_2, i_3, j_1, j_2, j_3) = \gamma^{-1}(\psi, \sigma)$ we get

$$x_{\psi,\sigma} = \pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}\Big(\sum_{(\psi,\sigma)\in P_n} x_{\psi,\sigma} e_{\psi,\sigma}(t_z)\Big) = \pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}(x) = 0,$$

which is our claim.

We next prove that if $a, b \in E_{(1,1),(3,3),n}$ are such that $\pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}(a(t_z)) = \pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}(b(t_z))$ for every $(i_1, i_2, i_3, j_1, j_2, j_3) \in Q_n$ and every $z \in \mathbf{R}^n$, then $a = b$.

Clearly, it suffices to show that if $a \in E_{(1,1),(3,3),n}$ is such that $\pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}(a(t_z)) = 0$ for every $(i_1, i_2, i_3, j_1, j_2, j_3) \in Q_n$ and every $z \in \mathbf{R}^n$, then $a = 0$. Let $u \in T^1_1\mathbf{R}^n$. We have to prove that $a(u) = 0$. We first consider the case that $u$ has $n$ different complex eingenvalues. An easy computation shows that for every $z \in \mathbf{R}^n$ the coefficients of the characteristic polynomial of $t_z$ coincide with the coordinates of $z$, i.e.

$$\det \begin{bmatrix} \lambda & & & z^n \\ -1 & \ddots & & \vdots \\ & \ddots & \lambda & z^2 \\ & & -1 & \lambda + z^1 \end{bmatrix} = \lambda^n + \sum_{i=1}^{n} z^i \lambda^{n-i} \tag{7}$$

for every $\lambda \in \mathbf{R}$, where $(z^1, \ldots, z^n) = z$. Thus, writing $c(u)$ for the vector $(c_1(u), \ldots, c_n(u)) \in \mathbf{R}^n$, where $c_1(u), \ldots, c_n(u)$ are the coefficients of the characteristic polynomial of $u$, we see that the characteristic polynomial of $u$ is the same as that of $t_{c(u)}$. Combining this with the fact that both $u$ and $t_{c(u)}$ have $n$ different complex eingenvalues, we conclude that there is $A \in GL(n, \mathbf{R})$ such that $u = t_{c(u)} \cdot A$. Since $a(t_{c(u)}) = 0$, which is due to the fact proved in the previous paragraph, we have $a(u) = a(t_{c(u)} \cdot A) = a(t_{c(u)}) \cdot A = 0 \cdot A = 0$ as desired. We now turn to the case of an arbitrary $u$. Let $v \in T^1_1\mathbf{R}^n$ be an arbitrary matrix with $n$ different complex eingenvalues and let $R$ be an $n$-dimensional affine subspace in $T^1_1\mathbf{R}^n$ such that $u \in R$ and $v \in R$. Suppose that $D(Z)$ denotes the discriminant of the characteristic polynomial of a matrix $Z \in T^1_1\mathbf{R}^n$. Then $D : T^1_1\mathbf{R}^n \longrightarrow \mathbf{R}$ is a polynomial and $D(Z) \neq 0$ if and only if $Z$ has $n$ different complex eingenvalues. Of course, $D|R \neq 0$, because $D(v) \neq 0$. Therefore $S = \{Z \in R : D(Z) \neq 0\}$ is a dense subset of $R$. We known that $a|S = 0$. Suppose that $P : \mathbf{R}^n \longrightarrow T^1_1\mathbf{R}^n$ is an affine parametrization of $R$. By the definition of equivariant maps, the composition $a \circ P$ is smooth and so is $a|R = (a \circ P) \circ P^{-1}$. Since each continous map vanishing on a dense subset vanishes, we have $a|R = 0$. In particular $a(u) = 0$ as required.

Fix $a \in E_{(1,1),(3,3),n}$. Our next goal is to determine smoth functions $f_{\psi,\sigma} : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $(\psi, \sigma) \in P_n$ such that

$$\pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}(a(t_z)) = \pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}\Big(\sum_{(\psi,\sigma)\in P_n} f_{\psi,\sigma}(z) e_{\psi,\sigma}(t_z)\Big) \tag{8}$$

for every $(i_1, i_2, i_3, j_1, j_2, j_3) \in Q_n$ and every $z \in \mathbf{R}^n$. The definition is by induction on $(\psi, \sigma) \in P_n$. Suppose that $(\psi, \sigma) \in P_n$ and that $f_{\omega,\tau}$ for $(\omega, \tau) \in P_n$ such that $(\omega, \tau) > (\psi, \sigma)$ are defined. We take $(i_1, i_2, i_3, j_1, j_2, j_3) = \gamma^{-1}(\psi, \sigma)$ and put

$$f_{\psi,\sigma}(z) = \pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}(a(t_z)) - \sum_{\substack{(\omega,\tau)\in P_n \\ (\omega,\tau)>(\psi,\sigma)}} f_{\omega,\tau}(z) \pi^{i_1 i_2 i_3}_{j_1 j_2 j_3}(e_{\omega,\tau}(t_z)) \tag{9}$$

for every $z \in \mathbf{R}^n$. It is easily seen that, by the smoothness of the map $\mathbf{R}^n \ni z \longrightarrow a(t_z) \in T^3_3\mathbf{R}^n$, we obtain smooth functions which satisfy the claimed condition (8). Write

$$\widetilde{a} : T_1^1\mathbf{R}^n \ni t \longrightarrow \sum_{(\psi,\sigma)\in P_n} f_{\psi,\sigma}(c_1(t),\ldots,c_n(t))e_{\psi,\sigma}(t) \in T_3^3\mathbf{R}^n,$$

where $c_1,\ldots,c_n$ are given by (2). Of course, $\widetilde{a} \in E_{(1,1),(3,3),n}$. According to (7) and (8), we have $\pi^{i_1i_2i_3}_{j_1j_2j_3}(\widetilde{a}(t_z)) = \pi^{i_1i_2i_3}_{j_1j_2j_3}(a(t_z))$ for every $(i_1,i_2,i_3,j_1,j_2,j_3) \in Q_n$ and every $z \in \mathbf{R}^n$. Hence $\widetilde{a} = a$, which is due to the fact proved in the previous paragraph. Therefore $e_{\psi,\sigma}$ for $(\psi,\sigma) \in P_n$ are generators of $E_{(1,1),(3,3),n}$.

It remains to prove that they are linearly independent. Assume that

$$\sum_{(\psi,\sigma)\in P_n} g_{\psi,\sigma}(c_1(t),\ldots,c_n(t))e_{\psi,\sigma}(t) = 0$$

for every $t \in T_1^1\mathbf{R}^n$, where $g_{\psi,\sigma} : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $(\psi,\sigma) \in P_n$ are smooth functions and $c_1,\ldots,c_n$ are given by (2). Hence, by (7),

$$\sum_{(\psi,\sigma)\in P_n} g_{\psi,\sigma}(z)e_{\psi,\sigma}(t_z) = 0$$

for every $z \in \mathbf{R}^n$. We have to prove that $g_{\psi,\sigma} = 0$ for $(\psi,\sigma) \in P_n$. Of course, the proof will be by induction on $(\psi,\sigma)$. Suppose that $(\psi,\sigma) \in P_n$ and that $g_{\omega,\tau} = 0$ for $(\omega,\tau) \in P_n$ such that $(\omega,\tau) > (\psi,\sigma)$. We take $(i_1,i_2,i_3,j_1,j_2,j_3) = \gamma^{-1}(\psi,\sigma)$. Then

$$0 = \pi^{i_1i_2i_3}_{j_1j_2j_3}\Big(\sum_{(\omega,\tau)\in P_n} g_{\omega,\tau}(z)e_{\omega,\tau}(t_z)\Big) = g_{\psi,\sigma}(z)$$

for every $z \in \mathbf{R}^n$, and so $g_{\psi,\sigma} = 0$. This proves the theorem. □

It is worth pointing out that the final part of the proof (formula (9)) yields a method of calculating the coordinates of an arbitrary equivariant map (for instance $e_{\psi,\sigma}$ with an arbitrary $\psi : \{1,2,3\} \longrightarrow \mathbf{N}$ and an arbitrary $\sigma \in S_3$) in our basis.

The proof of our theorem leads to the following corollary.

**Corollary 1** *$E_{(1,1),(3,3),n}$ is a free module of dimension $6n^3 - 9n^2 + 4n$.*

**Remark 1** Using similar arguments we can obtain a classification of equivariant maps from $E_{(1,1),(r,r),n}$ for $r = 1$ and $r = 2$.

In the case $r = 1$ it is sufficient to apply our algorithm to the coordinates $\pi^n_{n-\psi(1)}$, where $\psi : \{1\} \longrightarrow \{0,\ldots,n-1\}$, because $m(1,n) = n$. As a result we get the basis of $E_{(1,1),(1,1),n}$ consisting of $e_{\psi,\,\mathrm{id}_{\{1\}}}$ for $\psi : \{1\} \longrightarrow \{0,\ldots,n-1\}$, which coincides with the one found in [2].

In the case $r = 2$ it is sufficient to apply our algorithm to the coordinates $\pi^{n,n}_{1,n-\psi(2)}$, where $\psi : \{1,2\} \longrightarrow \{0,\ldots,n-1\}$ is such that $\psi(1) = n-1$, as well as to $\pi^{n-1,n}_{n-1-\psi(1),n-\psi(2)}$ and $\pi^{n,n-1}_{n-1-\psi(1),n-\psi(2)}$, where $\psi : \{1,2\} \longrightarrow \{0,\ldots,n-1\}$

is such that $\psi(1) < n-1$, because $m(2,n) = 2n^2 - n$. As a result we get the basis of $E_{(1,1),(2,2),n}$ consisting of $e_{\psi,\ \mathrm{id}_{\{1,2\}}}$ for $\psi : \{1,2\} \longrightarrow \{0,\dots,n-1\}$ such that $\psi(1) = n-1$ and of $e_{\psi,\sigma}$ for $\psi : \{1,2\} \longrightarrow \{1,\dots,n-1\}$ such that $\psi(1) < n-1$ and $\sigma \in S_2$. This basis is different from the one found in [3].

Unfortunately, in the case $r > 3$ our method probably breaks down. For instance, taking $r = 4$, $n = 2$ and applying our algorithm we can find only 66 linearly independent equivariant maps, while $m(4,2) = 70$.

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