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Tilings and Isoperimetrical Shapes II: Hexagonal lattice

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Abstract

In this paper, we consider a variant of the isoperimetric inequalities in lattices. We give a precise description of the polyominoes on the honeycomb lattice which maximize their area for a fixed perimeter.

From this characterization, we can give the 'explicit' values of the smallest perimeter of a polyhexe with a fixed area. This last problem corresponds to the usual isoperimetrical problem.

Moreover, we propose to apply our characterization in order to solve: *Find the minimum density of unit hexagon to be placed on the honeycomb lattice so that to exclude all polyhexes of a fixed area.* Using our characterization we solve this last problem for some values of the fixed area.

Key words: Tiling, isoperimetrical inequality, hexagonal lattice.

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1 Introduction

We denote $\mathcal{H}$ to be the well-known Hexagonal lattice (see [5]). In this paper, a *polyhexe* is a finite set (not necessarily connected) of unit hexagon placed on $\mathcal{H}$ (see [6]). The *interior boundary* $\delta_{int}(P)$ of a polyhexe P is the set of hexagons of P having a common edge with the 'exterior' of P. The *exterior boundary* $\delta_{ext}(P)$ of a polyhexe P is the interior boundary of the complement of P. The *perimeter* of a polyhexe P is $|\delta_{int}(P)|$. For a given polyhexe P, $\Delta(P)$ denotes the area of P (that is, $|P|$).

Consider the adjacency relation α, which defines what is usually called 6-connectivity in Discrete Geometry, between unit hexagons in $\mathcal{H}$: we have $H\alpha H'$ iff H and H' have a common edge.

Here, we are interested in the following question (Q): *Suppose we are given a positive integer n. What are the polyhexes of perimeter n with maximal area?* This question was solved for the square lattice in the 2-dimensional case [7].

Problem (Q) is related to an isoperimetrical problem (P): *What is at least the perimeter of a polyhexe of a given area?* This last question was first settled in $\mathbb{Z}_+^n$ for any n by D.-L. Wang and P. Wang [10]; an alternative proof is proposed by B. Bollobás and I. Leader [2] in order to generalize Harper's theorem. Nevertheless, these results do not give any information on the 'shape' of the optimal polyominoes (for instance uniqueness of the minimal shapes for specific values of the area). That is why the authors in [7] proposed the alternative question (Q) and they obtained a characterization of the 'shapes' of the optimal polyominoes in the 2-dimensional case. Here, we adopt the same approach in order to solve question (Q) in the Hexagonal lattice. L. Alonso and R. Cerf [1] solved the question (Q) in $\mathbb{Z}^n$ (for $n = 2$ and 3) for another kind of perimeter: the length (in $\mathbb{Z}^2$) or the area (in $\mathbb{Z}^3$) of the boundary.

In Section 2, we solve (Q) in $\mathcal{H}$. In Section 4, we propose an application of this result in order to solve: *Find the minimum density of unit hexagons to be placed on $\mathcal{H}$ so that to exclude all polyhexes of a fixed area.* This kind of problem was also investigated in $\mathbb{Z}^2$ in [7]. In this case, this is related to Pentomino Exclusion Problem due to Golomb [6, 3, 4, 8]. Our characterization of isoperimetrical shapes allows to solve this problem for some values of the fixed area.

2 Maximizing the area for a given perimeter

The proofs of results mentioned in this section are given in the next section. We need to introduce some preliminary definitions (see Figure 2). For a given polyhexe P, we can associate the graph $G(P) = (V, E)$ defined by $V = \{v$ center of a unit hexagon in $P\}$ and $E = \{uv | u$ and v are the centers of two unit hexagons H_u and H_v respectively of P such that $H_u \alpha H_v\}$. A vertex v of $G(P)$ can be seen as a unit hexagon of $\mathcal{H}$, denoted by H_v. Moreover $G(\mathcal{H})$ is usually called in discrete geometry language the *triangular lattice* which can be spanned by the vectors $(1, 0)$ and $(-1/2, \sqrt{3}/2)$. Then:

- The *interior shape* (*exterior shape* respectively) of a polyhexe P is the graph $G_{int}(P) = G(\delta_{int}(P))$ ($G_{ext}(P) = G(\delta_{ext}(P))$ resp.);
- An *isolable* hexagon u of a polyhexe P is a hexagon $u \in \delta_{int}(P)$ belonging to a connected component of P not reduced to u and such that all its neighbours in P belong to $\delta_{int}(P)$.

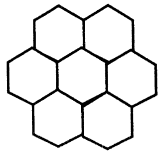

Figure 1: A flower.

- The *undress* operator applied to a polyhexe P gives the polyhexe
$$UD(P) = P - \delta_{int}(P).$$
For $q > 1$, we denote $UD^q(P) = UD(UD^{q-1}(P))$ and by convention
$$UD^1(P) = UD(P).$$
- The *flower* is the polyhexe F such that $UD(F)$ is a unit hexagon (see Figure 1).

The undress operator in $\mathbb{Z}^2$ was first introduced in [7].

Observe that if u is an isolable hexagon of a polyhexe P then for any $v \in \mathcal{H} - (P \cup \delta_{ext}(P))$, the polyhexe $Q = (P - u) \cup \{v\}$ satisfies $|\delta_{int}(Q)| = |\delta_{int}(P)|$ and $\Delta(Q) = \Delta(P)$; moreover v is not an isolable hexagon of Q. For a convenience, let P_n be a polyhexe with perimeter n which maximizes the area.

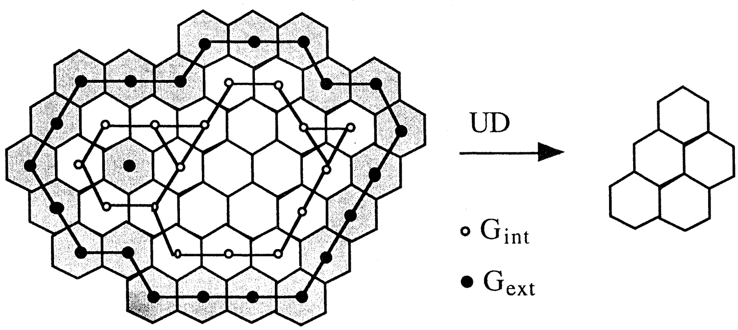

Figure 2: A polyhexe P, the graphs $G_{int}(P)$ and $G_{ext}(P)$, and the polyhexe $UD(P)$. Each unit hexagon of $UD(P)$ is an isolable hexagon.

Theorem 1 *A P_n consists of n distinct unit hexagons whenever $n \in \{1, \ldots, 5\}$. P_6 is a flower (see Figure 1). P_7 consists of a disjoint union of a flower and a unit hexagon.*

If $n = 6q + r \geq 8$ and $0 \leq r \leq 5$, then P_n is unique and is obtained by $UD^q(P_n) = H_r$ where H_i is the polyhexe described in Figure 3.

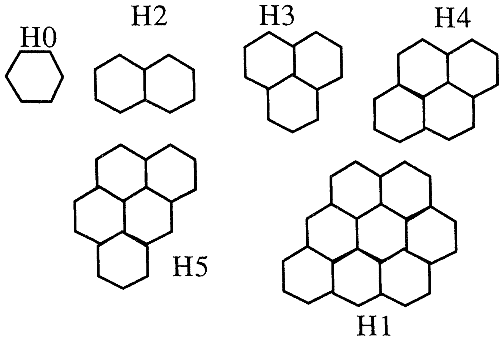

Figure 3: Extremal configurations.

Theorem 2 *The area of P_n with $n = 6q + r > 0$ and $0 \leq r \leq 5$, is given by the following function:*

$$\Delta(P_n) = \begin{cases} 3q^2 + 3q + 1 & \textit{if } r = 0, \\ 3q^2 + 4q + 1 & \textit{if } r = 1, \\ 3q^2 + 5q + 2 & \textit{if } r = 2, \\ 3q^2 + 6q + 3 & \textit{if } r = 3, \\ 3q^2 + 7q + 4 & \textit{if } r = 4, \\ 3q^2 + 8q + 5 & \textit{if } r = 5. \end{cases}$$

In order to describe the shape of an optimal polyhexe, we need some additional definitions.

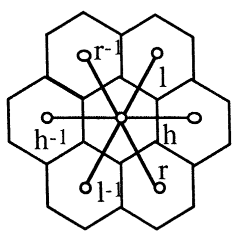

Figure 4: Type of edges in G.

A *cycle* in a graph G is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$ such that $v_i v_{i+1}$ (the indices have to be read modulo k) is an edge of G for all $i = 1, \ldots, k$. A *chord* in a cycle $v_0, \ldots, v_k$ is an edge $v_i v_j$ with $j \neq i + 1$.

Let $X = (1, 0), Y = (1/2, \sqrt{3}/2)$ and $\bar{Y} = (-1/2, \sqrt{3}/2)$. Observe that there are six type of edges in $G(\mathcal{H})$ (see Figure 4):

- A *left* edge of $G(\mathcal{H})$, denoted by l, is an edge uv where $v = u + Y$;
- An *anti-left* edge of $G(\mathcal{H})$, denoted by l^{-1}, is an edge uv where $v = u - Y$;

- A *right* edge of $G(\mathcal{H})$, denoted by r, is an edge uv where $v = u + \bar{Y}$;
- An *anti-right* edge of $G(\mathcal{H})$, denoted by r^{-1}, is an edge uv where $v = u - \bar{Y}$;
- A *horizontal* edge of $G(\mathcal{H})$, denoted by h, is an edge uv where $v = u + X$;
- An *anti-horizontal* edge of $G(\mathcal{H})$, denoted by h^{-1}, is an edge uv where $v = u - X$.

Observe that a cycle of G can be completely characterized by a circular word (in a clockwise orientation of the cycle) on the alphabet $\{l, r, h, l^{-1}, r^{-1}, h^{-1}\}$.

Theorem 3 *If $n = 6q + r \geq 8$ and $0 \leq r \leq 5$, then $G_{int}(P_n)$ is a cycle which can be described by the words:*

$$\begin{gathered}
h^q l^q r^{-q} h^{-q} l^{-q} r^q \textit{ if } r = 0 \\
h^{q+1} l^q r^{-q} h^{-q} l^{-q-1} r^{q-1} \textit{ if } r = 1 \\
h^{q+1} l^q r^{-q} h^{-q-1} l^{-q} r^q \textit{ if } r = 2 \\
h^q l^{q+1} r^{-q} h^{-q-1} l^{-q} r^{q-1} \textit{ if } r = 3 \\
h^{q+1} l^{q+1} r^{-q} h^{-q-1} l^{-q-1} r^q \textit{ if } r = 4 \\
h^q l^{q+2} r^{-q} h^{-q-1} l^{-q-1} r^{q+1} \textit{ if } r = 5.
\end{gathered}$$

3 The proofs

Among all polyhexes P_n, choose a polyhexe P which has a maximum number of connected components. Let $\mathbf{G} = G(P), G_{int} = G_{int}(P)$ and $G_{ext} = G_{ext}(P)$.

A *vertex cutset* in a graph G is a vertex v such that $G - v$ is not connected.

Lemma 1 *Any connected component of* $\mathbf{G}$ *has neither isolable hexagon nor vertex cutset.*

Proof First assume that a connected component of $\mathbf{G}$ has an isolable hexagon v, and let u be a unit hexagon of $\mathcal{H} - (P \cup \delta_{ext}(P))$. Then the polyhexe $Q = (P - u) \cup \{v\}$ has the same area, the same perimeter and one more connected component than P, contradicting the maximality of P.

Let C be a connected component of $\mathbf{G}$. So, we may assume that C has no isolable hexagon. Suppose now that v is a vertex cutset of C. Remark that $v \in \delta_{int}(C)$. So we may assume, up to rotation, that $v + X \notin C$. Since by assumption v is not isolable, we have that there exists a vertex u in the neighborhood of v which is not in the interior boundary of C. Since v is a vertex cutset and since $v + X \notin C$, we may assume without loss of generality that $u = v - \bar{Y}$. And so $v + Y$ and $v - X$ belong to C and $v - Y \notin C$.

Let C_1 and C_2 be two connected components of $C - v$ such that $v + Y, v - X$ and $u \in C_2$ and $v + \bar{Y} \in C_1$. Remark that $\delta_{ext}(C_1) \cap \delta_{ext}(C_2) = \{v\}$. Consider now the polyhexe Q obtained from P by removing v from C, translated C_1 by the vector $-\bar{Y}$ and adding some possible unit polyhexes (at least one to replace u) in $\mathcal{H} - (P \cup \delta_{ext}(P))$ in order to have $|\delta_{int}(Q)| = |\delta_{int}(P)|$. The resulting polyhexe has the same perimeter, the area greater or equal to P and more connected components than P, contradicting the maximality of P. □

Lemma 1 shows us that any connected component C of $\mathbf{G}$ has no isolable unit hexagon and no vertex cutset. Using this assumption, we will study the geometrical shape of a connected component of $\mathbf{G}$.

We consider two types of configuration described in Figure 5:

- A *oblique* 3-*path* is a path $(a, u, b) \in G_{int}$ such that, up to rotation, $a = u + \bar{Y}$ and $b = u + Y$ and $u + X \notin P$;
- A *triangle* is a complete graph on 3 vertices $\{a, u, b\}$ belonging to G_{int}.

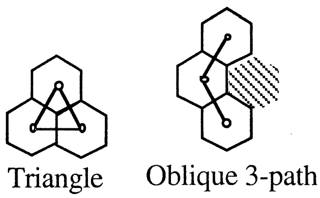

Figure 5: Forbiden configurations.

Observe that if there is no oblique 3-path in G_{int} and if P has no hole then G_{int} is a 'convex' cycle (that is, a convex polygon).

Lemma 2 *Any connected component of G has no hole and δ_{int} has neither oblique 3-path nor triangle.*

Proof Let C be any connected component of G. Then C has no isolable unit hexagon (by Lemma 1). We claim that:

$$C \text{ has no hole.} \tag{1}$$

Indeed, if C has a hole H then we construct a polyhexe Q by filling this hole and adding possible unit hexagons of $\mathcal{H} - (P \cup \delta_{ext}(P) \cup H)$ in order to have the same perimeter as P. So we obtain a contradiction with the maximality of P, since $\Delta(Q) \geq \Delta(P) + \Delta(H) > \Delta(P)$.

Assume that there exists an oblique 3-path (a, u, b) in $G_{int}(C)$. By Lemma 1, there exists another path $(v_1 = a, v_2, \ldots, v_k = b)$ from a to b where $v_i \neq u$ for all i in $G(C)$. Now by (1), the unit hexagons $u - Y, u - X$ and $u - \bar{Y}$ belong to C.

Let $v = u + X$. We construct the polyhexe Q obtained from $P \cup \{v\}$ by adding possible unit hexagons of $\mathcal{H} - (P \cup \{v\})$ in order to have the same perimeter as P. By the previous remarks the polyhexe Q has the same perimeter as P and $\Delta(Q) \geq \Delta(P) + 1$, which contradicts the maximality of P.

Assume now that $\{a, u, b\}$ with $a = u + X$ and $b = u + \bar{Y}$ induces a triangle in $G_{int}(C)$. Since u is not an isolable hexagon then there exists a neighbour v of u belonging to $UD(C)$.

If $v = u - \bar{Y}$ then since $u \in \delta_{int}(C)$, we have $u - Y \notin C$. So, $a, u, u - X$ induce an oblique 3-path in $G_{int}(C)$, which yields a contradiction.

If $v = u + Y$ then since a is not an isolable hexagon, we must have a vertex $a' \in UD(C)$. Since u and b belong to $\delta_{int}(C)$, a' is different from $a - Y$ and $a + Y$. Thus, by symmetry, we may assume that $a' = a + \bar{Y}$. But now, since $a \in \delta_{int}(C)$, we must have $a - Y \notin C$. Hence $a - Y, a, u$ induce an oblique 3-path in $G_{int}(C)$, a contradiction. The cases when $v = u - X$ and $v = u - Y$ holds similarly by symmetry, finishing the proof of the Lemma. □

Let C be a connected component of **G** with no isolable unit hexagon. Checking for the small values of $|\delta_{int}(C)|$, we can verify that if $|\delta_{int}(C)| > 1$ then $|\delta_{int}(C)| \geq 6$. Moreover from Lemma 2, the interior boundary of C is a convex cycle. Observe that by Lemma 2, the only sequences of length 2, with respect to the clockwise orientation of the cycle, allow in a word describing $G_{int}(C)$, are $rr, rh, rl^{-1}, ll, lh, lr^{-1}, hh, hr$ and hl and the reverses. Thus, up to rotation, there exist six positive integers p, q, s, t, u, v such that (see Figure 6):

$$G_{int}(C) \equiv (S)\ h^u l^p r^{-t} h^{-v} l^{-s} r^q.$$

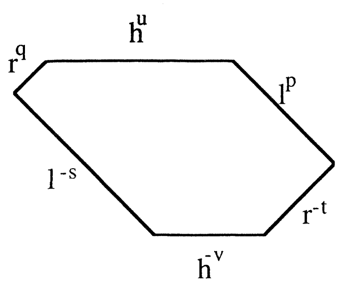

Figure 6: Optimal shapes.

Lemma 3 *For any polyhexe of boundary C described by a word of a type (S) where p, q, s and t are integers greater than 1, we have*

$$|\delta_{int}(C)| = |\delta_{int}(UD(C))| + 6.$$

Proof Assume that C is of type (S), and let p, q, s, t, u and v be positive integers of the word associated to (S). We claim that:

$$UD(C) \text{ is of type } (S) \text{ with } p-1,\ q-1,\ s-1,\ t-1,\ u-1 \text{ and } v-1. \quad (2)$$

Indeed, observe that a l^z or r^z or h^z or l^{-z} or r^{-z} or h^{-z} path becomes a l^{z-1} or a r^{z-1} h^{z-1} or l^{-z+1} or r^{-z+1} or h^{-z+1} path, respectively, in $UD(C)$.

If p, q, s, t, u and v are greater than 1 then the paths of $UD(C)$ described in (2) are distinct, so the lemma follows from (2). □

A *nice shape* (S) is one of the following shapes for some $a > 1$:

$$(S) \equiv \begin{cases} (S_0)\ h^a l^a r^{-a} h^{-a} l^{-a} r^a, \text{ or} \\ (S_1)\ h^{a+1} l^a r^{-a} h^{-a} l^{-a-1} r^{a-1}, \text{ or} \\ (S_2)\ h^{a+1} l^a r^{-a} h^{-a-1} l^{-a} r^a, \text{ or} \\ (S_3)\ h^a l^{a+1} r^{-a} h^{-a-1} l^{-a} r^{a-1}, \text{ or} \\ (S_4)\ h^{a+1} l^{a+1} r^{-a} h^{-a-1} l^{-a-1} r^a, \text{ or} \\ (S_5)\ h^a l^{a+2} r^{-a} h^{-a-1} l^{-a-1} r^{a+1} \end{cases}$$

Lemma 4 *The area of a polyhexe P with nice shape (S) and for some $a \geq 1$ is given by:*

$$\Delta(P) = \begin{cases} 3a^2 + 3a + 1 \text{ if } (S) \equiv (S_0) \\ 3a^2 + 4a + 1 \text{ if } (S) \equiv (S_1) \\ 3a^2 + 5a + 2 \text{ if } (S) \equiv (S_2) \\ 3a^2 + 6a + 3 \text{ if } (S) \equiv (S_3) \\ 3a^2 + 7a + 4 \text{ if } (S) \equiv (S_4) \\ 3a^2 + 8a + 5 \text{ if } (S) \equiv (S_5) \end{cases}$$

Proof First observe that as in the proof of Lemma 3, a polyhexe of nice shape is uniquely obtained by undressing from one of the polyhexes described in Figure 3. Lemma 4 follows easily from Lemma 3. □

Now, we are ready to prove the following crucial lemma.

Lemma 5 *For a connected component C of G with $|C| > 1$, the polyhexe $C_i = G_{int}(C)$ has a nice shape for some $a \geq 1$.*

Proof Let (S) with p, q, s, t, u and v greater or equal to 1, be the shape of C.

We may assume that $\min\{p, q, s, t, u, v\} = 1$. For otherwise, we prove Lemma 5 for $UD^{k-1}(C_i)$ where $k = \min\{p, q, s, t, u, v\}$, and conclude by Lemma 3. Up to rotation, we may assume that $u = 1$. Now, using elementary geometrical arguments we can see that:

$$\begin{array}{c} \Delta(C) = (t + v + 1)(s + v + 1) - \frac{u(u+1)}{2} - \frac{v(v+1)}{2}, \\ |\delta_{int}(C)| = p + u + q + s + v + t, \\ t + v = q + u \text{ and } s + v = p + u. \end{array} \tag{3}$$

We proceed by induction on $|\delta_{int}(C)|$.

If $p = 1$ then by (3), $s = v = 1$ and $q = t$. Moreover, if $t \leq 2$ then C up to rotation, has nice shape (S_0) or (S_2). So, we may assume that $t = q \geq 3$. By (3), we have $\Delta(C) = 3(p + 2) - 2$ and $|\delta_{int}(C)| = 2p + 4$. Let $p = 6i + r$ with $0 \leq r \leq 5$.

If $r = 0$ or $r = 3$ then the polyhexe of nice shape (S_4) with $a = 2i + 1$ for $r = 0$ and $a = 2i + 2$ for $r = 3$ has same perimeter as C but larger area, contradicting the maximality of P.

If $r = 1$ or $r = 4$ then the polyhexe of nice shape (S_0) with $a = 2i + 1$ for $r = 1$ and $a = 2i + 2$ for $r = 4$ has same perimeter as C but larger area, contradicting the maximality of P.

Similarly, if $r = 2$ or $r = 5$ then the polyhexe of nice shape (S_2) with $a = 2i+1$ for $r = 2$ and $a = 2i+2$ for $r = 5$ has same perimeter as C but larger area, contradicting the maximality of P.

If $p = q = 2$ then by (3), C has, up to rotation, a shape (S_3) or (S_4).

So, we may assume that $p > 2$ and $q \geq 2$. Let $|\delta_{int}(C)| = 6a + r$ with $0 \leq r \leq 5$. By (3) and since $p > 2$ and $q \geq 2$, we have $6a + r \geq 12$ (and so $a > 1$).

Consider the polyhexe $C' = C - \{h_1, h_2\}$ where h_1h_2 is the horizontal-edge in the description of (S). Observe that C' has the shape $h^2l^{p-1}r^{-t}h^{-v}l^{-s}r^{q-1}$. Hence,

$$\Delta(C') = \Delta(C) - 2 \text{ and } |\delta_{int}(C')| = 6a + r - 1.$$

We apply now the induction hypothesis to C' and using Lemma 4 we obtain a contradiction with the maximality of P.

If $r = 0$ then by Lemma 4 and by induction hypothesis, we have $\Delta(C') \leq 3(a-1)^2 + 8(a-1) + 5 \leq 3a^2 + 2a$. So since $a > 1$ we have $\Delta(C) \leq 3a^2 + 2a + 2 < 3a^2 + 3a + 1$.

If $r = 1$ then by Lemma 4 and by induction hypothesis, we have $\Delta(C') \leq 3a^2 + 3a + 1$. So since $a > 1$ we have $\Delta(C) \leq 3a^2 + 3a + 3 \leq 3a^2 + 4a + 1$. If $\Delta(C) = 3a^2 + 4a + 1$ then $a = 2$. It is straightforward to check that in this case, we have $(S) \equiv (S_1)$.

If $r = 2$ then by Lemma 4 and by induction hypothesis, we have $\Delta(C') \leq 3a^2 + 4a + 1$. So since $a > 1$ we have $\Delta(C) \leq 3a^2 + 4a + 3 < 3a^2 + 5a + 2$.

If $r = 3$ then by Lemma 4 and by induction hypothesis, we have $\Delta(C') \leq 3a^2 + 5a + 2$. So since $a > 1$ we have $\Delta(C) \leq 3a^2 + 5a + 4 < 3a^2 + 6a + 3$.

If $r = 4$ then by Lemma 4 and by induction hypothesis, we have $\Delta(C') \leq 3a^2 + 6a + 3$. So since $a > 1$ we have $\Delta(C) \leq 3a^2 + 6a + 5 < 3a^2 + 7a + 4$.

If $r = 5$ then by Lemma 4 and by induction hypothesis, we have $\Delta(C') \leq 3a^2 + 7a + 4$. So since $a > 1$ we have $\Delta(C) \leq 3a^2 + 7a + 6 < 3a^2 + 8a + 5$.

In all cases, we obtain a polyhexe with the same perimeter but with larger area, which contradicts the maximality of P. □

Lemma 6 *For any $n \geq 8$, $G(P_n)$ is connected.*

Proof Let n be the smallest integer such that there exists a non-connected P_n. Assume that $n \geq 8$. Let $C_1, \ldots, C_t$ be $t \geq 2$ connected components of $G(P_n)$. It is easy to see that since $n \geq 8$ there exists some C_i which is not a square. We may assume now that C_i has no isolable unit hexagon for $i = 1, \ldots, a$ for some $a > 0$ (possibly $a = t$), and C_j is a square for $j = a+1, \ldots, t$. Thus, by Lemma 5, C_i's for $i \leq a$ can be described by a nice shape (S). Let $p_i = |\delta_{int}(C_i)|$. Remark that $p_i \geq 6$ for all $i \leq a$. Without loss of generality, we may assume that $p_1 + p_2 + p_3 \geq 8$ (with convention $p_3 = 0$ if $t = 2$). Then by Lemma 4, $\Delta(P_{p_1+p_2+p_3}) \geq 3(p_1+p_2+p_3)^2 + 3(p_1+p_2+p_3) + 1$ and $\Delta(P_{p_1}) + \Delta(P_{p_2}) + \Delta(P_{p_3}) \leq 3(p_1^2 + p_2^2 + p_3^2) + 8(p_1 + p_2 + p_3) + 5$. Hence $\Delta(P_{p_1+p_2+p_3}) > \Delta(P_{p_1}) + \Delta(P_{p_2}) + \Delta(P_{p_3})$, which contradicts the maximality of P_n. □

Theorem 1 follows from Lemmas 3 and 6. Theorem 2 follows from Lemmas 4 and 6. Theorem 3 follows from Lemmas 5 and 6. □

4 Applications

We will use here notations and definitions given in Introduction and in Section 2.

4.1 Isoperimetrical problem (P)

From the results of Section 2, we can solve the problem (P) in $\mathcal{H}$ with sharped values of perimeter. Denote P_n a polyhexe of perimeter n which maximizes its area.

Theorem 4 *Any polyhexe P of area Δ satisfies $|\delta_{int}(P)| \geq n$ where n is defined by $\Delta(P_{n-1}) < \Delta \leq \Delta(P_n)$.*

Observe that by Lemma 3, we can establish an analogue of Theorem 4 for the exterior boundary:

Theorem 5 *Any polyhexe P of area Δ satisfies $|\delta_{ext}(P)| \geq n+6$ where n is defined by $\Delta(P_{n-1}) < \Delta \leq \Delta(P_n)$.*

4.2 Golomb type problem

We denote by (G_Δ) the problem: *Find the minimal density of unit hexagon to be placed on $\mathcal{H}$ so that to exclude all polyhexes of area greater than Δ.* An *admissible* solution of (G_Δ) problem is a set $\mathcal{S}$ of hexagons centered on the triangular lattice $\mathcal{T}$ such that any connected component in α adjacency of $\mathcal{H}-\mathcal{S}$ has area less than or equal to Δ. We color 'black' the hexagons belonging to an admissible solution $\mathcal{S}$ and 'white' the others.

Now we need a measure called 'density' of an admissible solution of (G_Δ) in order to compare two admissible solutions.

If T is a finite subset of $\mathcal{T}$ then a natural way to define the density of $\mathcal{S}$ relatively to T is $\frac{|\mathcal{S}\cap T|}{|T-\mathcal{S}|}$. We show how this definition can be extended for infinite case:

For an admissible solution $\mathcal{S}$ of (G_Δ), observe that the **plane** graph $G(\mathcal{S})$ defines a tiling of $\mathbb{R}^2$ (see Figure 7) where tiles are the faces of $G(\mathcal{S})$. To a face (or a tile) $\langle C\rangle$ of $G(\mathcal{S})$ there corresponds a unique polyhexe C with $\delta_{ext}(C) \subset \mathcal{S}$.

Some of these tiles correspond to some connected components of $\mathcal{T}-\mathcal{S}$. Some others are triangles corresponding to 3 mutually adjacent elements of $\mathcal{S}$ (in this case $C=\emptyset$).

Let $D \subset \mathbb{R}^2$. The density of an admissible solution $\mathcal{S}$ of (G_Δ) relative to D is

$$d(\mathcal{S}, D) = \frac{\text{'black' area of } \bar{D}}{\text{'white' area of } \bar{D}},$$

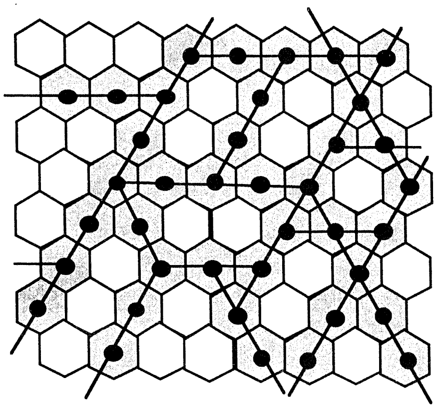

Figure 7: $\mathcal{S}$ and $G(\mathcal{S})$.

where $\bar{D}$ is the union of all faces of G($\mathcal{S}$) which intersect D.

Notice that $\bar{D}$ defines a polyhexe P with some unit hexagon in $\mathcal{S}$; moreover all the hexagons in the interior boundary of P belong to $\mathcal{S}$. Moreover, observe that $d(\mathcal{S}, D)$ is well-defined since each face of $G(\mathcal{S})$ defines a polyhexe with bounded area.

Let B_r be the ball of $\mathbb{R}^2$ of radii r. Then

$$\underline{d}(\mathcal{S}) = \liminf_{r \to \infty} d(\mathcal{S}, B_r)$$

and

$$\bar{d}(\mathcal{S}) = \limsup_{r \to \infty} d(\mathcal{S}, B_r),$$

are called the *lower* or the *upper* density, respectively.

If these two values coincide, their common value is called *density* $d(\mathcal{S}, D)$. This definition of density is more or less standard (see for example [9]). Such a density was first introduced for the lattice $\mathbb{Z}^2$ in [7]. We are ready now to state our result on Golomb-type problem:

Theorem 6 *Let $n = 6q + r \geq 8$ with $0 \leq r \leq 5$ be an integer such that $\Delta \geq \Delta(P_n)$. If $q > 1$ and $\Delta - \Delta(P_n) \leq \lceil \frac{q}{2} \rceil$ then an optimal solution $\mathcal{S}$ of (G_Δ) satisfies :*

$$\underline{d}(\mathcal{S}) \geq \frac{\frac{|\delta_{ext}(P_n)|}{2} - 1}{\Delta(P_n)}.$$

Moreover for any $\Delta \geq \Delta(P_n)$, we have:

$$\bar{d}(\mathcal{S}) \leq \frac{\frac{|\delta_{ext}(P_n)|}{2} - 1}{\Delta(P_n)}.$$

Proof Let $\mathcal{S}$ be an optimal solution of (G_Δ). Let $D \subset \mathbb{R}^2$ and let

$$\bar{D} = \cup_{\{\langle C\rangle \in G(\mathcal{S}) \ | \ \langle C\rangle \cap D \neq \emptyset\}} \langle C\rangle.$$

First we claim that:

$$\text{We may assume that any } \langle C\rangle \text{ has no hole.} \tag{4}$$

If $\langle C\rangle$ has a hole then move it close to the exterior boundary of $\langle C\rangle$ in order to obtain a new face $\langle C'\rangle$ with no hole.

If we do that for any $\langle C\rangle$ having a hole then we obtain a new admissible solution of (G_Δ) with the same density than $\mathcal{S}$.

Now we assume that any face of $G(\mathcal{S})$ has no hole. Using the structure of $\mathcal{H}$, we claim that:

$$d(\mathcal{S}, D) = \frac{\sum_{\langle C\rangle \in \bar{D}} \frac{|\delta_{ext}(C)|}{2} - 1}{\sum_{\langle C\rangle \in \bar{D}} |C|}. \tag{5}$$

Let P be the polyhexe defined by $\bar{D}$. Since the topological dual of $\mathcal{H}$ is the triangular lattice $\mathcal{T}$ and since each point of $\mathcal{T}$ can be placed in $\mathbb{Z}^2$ then Pick's theorem gives us that the area of $\bar{D}$ is $|UD(P)| + \frac{|\delta_{int}(P)|}{2} - 1$ and the area of each $\langle C\rangle$ is $|C| + \frac{|\delta_{ext}(C)|}{2} - 1$, since by the assumption each $\langle C\rangle$ has no hole. Now by additivity of the area and since $\bar{D}$ is partitionate by $\cup_{\langle C\rangle \in \bar{D}} \langle C\rangle$, we have that $\sum_{\langle C\rangle \in \bar{D}} \frac{|\delta_{ext}(C)|}{2} - 1$ is equal to the number of hexagons in $\mathcal{S} \cap P$ not contained in the interior boundary of P, plus half of the number of hexagons in the interior boundary of P which corresponds to the 'black' area of $\bar{D}$. $\sum_{\langle C\rangle \in \bar{D}} |C|$ is the number of hexagons in P not contained in $\mathcal{S}$, which corresponds to the 'white' area of $\bar{D}$.

Now we prove the lower bounds on $\underline{d}(\mathcal{S})$. From (5), we have:

$$d(\mathcal{S}, D) \geq \min_{\langle C\rangle \in \bar{D}} \frac{\frac{|\delta_{ext}(C)|}{2} - 1}{|C|}. \tag{6}$$

Let $\langle C\rangle$ be a face of $G(\mathcal{S})$.

If $|C| \leq \Delta(P_n)$ then let $p = |\delta_{ext}(C)|$.

If $p \geq n + 6$ then

$$\frac{\frac{p}{2} - 1}{|C|} \geq \frac{\frac{n+6}{2} - 1}{\Delta(P_n)}.$$

If $p < n + 6$ then by Theorem 5, $|C| \leq \Delta(P_{p-6})$. Let $s = n + 6 - p$. Then by Theorem 2, $|C| \leq \Delta(P_{p-6}) \leq \Delta(P_n) - qs$ and $\Delta(P_n) \leq q(n+4)$ if $q > 1$. Hence, if $q > 1$ then

$$\frac{\frac{p}{2} - 1}{|C|} \geq \frac{\frac{n+6-s}{2} - 1}{\Delta(P_n) - qs} \geq \frac{\frac{n+6}{2} - 1}{\Delta(P_n)}.$$

If $\Delta \geq |C| \geq \Delta(P_n)$ then by Theorem 5 we have $|\delta_{ext}(C)| \geq n+7$, so

$$\frac{\frac{|\delta_{ext}(C)|}{2}-1}{|C|} \geq \frac{\frac{n+7}{2}-1}{\Delta}.$$

If $\Delta - \Delta(P_n) \leq \lceil \frac{q}{2} \rceil$, then by easy computation based on the values of $\Delta(P_n)$ given in Theorem 2, we get

$$\frac{\frac{n+7}{2}-1}{\Delta} \geq \frac{\frac{n+6}{2}-1}{\Delta(P_n)} = \frac{\frac{|\delta_{ext}(P_n)|}{2}-1}{\Delta(P_n)}.$$

In each case, we obtain by (6), that for $q > 1$ and $\Delta - \Delta(P_n) \leq \lceil \frac{q}{2} \rceil$

$$\underline{d}(\mathcal{S}) \geq \frac{\frac{|\delta_{ext}(P_n)|}{2}-1}{\Delta(P_n)}.$$

We prove now the upper bounds of $\bar{d}(\mathcal{S})$. By (5), we have:

$$d(\mathcal{S}, D) \leq \max_{\langle C \rangle \in \bar{D}} \frac{\frac{|\delta_{ext}(C)|}{2}-1}{|C|}. \tag{7}$$

Since all nice shapes are hexagons, then we know tilings of $\mathbb{R}^2$ by these shapes and, by (7), the density of such tilings is precisely

$$\frac{\frac{|\delta_{ext}(P_n)|}{2}-1}{\Delta(P_n)} \qquad \square$$

A direct consequence of the proof of Theorem 6 is that whenever $q > 1$ and $\Delta - \Delta(P_n) \leq \lceil \frac{q}{2} \rceil$ the density of an optimal solution of (G_Δ) exists and is equal to

$$\frac{\frac{|\delta_{ext}(P_n)|}{2}-1}{\Delta(P_n)}.$$

Moreover this density is independent on the position of the ball B_r.

5 Concluding remarks

We proved that there exists a family of 6 polyhexes in $\mathcal{H}$ such that any polyhexe in $\mathcal{H}$ of maximal area for a fixed perimeter can be obtained (whenever the perimeter is large enough) from the element of this family by an undressing procedure. Moreover in [7], the authors shown a family of 5 polyominoes which by an undressing procedure gave all optimal polyominoes for analogue in $\mathbb{Z}^2$ of problem (Q).

We can ask what happens in the triangular lattice. We mentioned that even for the small value of a perimeter, it seems that the undressing procedure is not appropriate to solve an analogue of problem (P).

Perhaps in the triangular lattice the family would be infinite.

The density defined in previous section allows to compare tilings of $\mathbb{R}^2$ when the tiles are polygons with vertices belonging to a lattice. Moreover, the density allows to compare any two tilings belonging to distinct lattices, whenever we compute this density. For example, in triangular lattice, computing the density of a tiling seems to be more complicate than in $\mathcal{H}$ or $\mathbb{Z}^2$ since for instance the topological dual of $\mathcal{T}$ is $\mathcal{H}$ for which we have no (according to my knowledge) analogue to Pick's Theorem.

It would be interesting to complete Theorem 6 in order to obtain a classification (via density) of tilings in $\mathcal{H}$.

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