Ivan Chajda; Josef Zedník

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Distributivity and Modularity of Lattices of Quasiorders *

IVAN CHAJDA [1], JOSEF ZEDNÍK [2]

[1] *Department of Algebra and Geometry, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic*
e-mail: chajda@risc.upol.cz

[2] *Department of Mathematics, University of T. Baťa, 762 72 Zlín, Czech Republic*

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Abstract

We introduce a triangular scheme which characterizes distributivity of the lattice Quord $\mathcal{A}$ of all quasiorders of an algebra $\mathcal{A}$. Moreover, we can apply a similar scheme that was used by H.-P. Gumm [7] for congruence modularity and relate it with modularity of Quord $\mathcal{A}$.

Key words: Quasiorder, quasiorder lattice, modularity, distributivity, triangular scheme.

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Congruence distributivity for varieties of algebras was characterized by B. Jónsson [8]. For single algebras a certain scheme characterizing congruence distributivity was introduced in [2]. Since the congruence lattice Con $\mathcal{A}$ is a sublattice of Quord $\mathcal{A}$ (see e.g. [1]), the lattice of all compatible quasiorders on $\mathcal{A}$, we can try to use a similar method for checking whether Quord $\mathcal{A}$ is distributive. Let us mention that distributivity of Quord $\mathcal{A}$ for all $\mathcal{A}$ of a given variety was characterized by a Maltsev type condition either in [3] or by G. Czédli and A. Lenkehegyi in [6].

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Congruence modularity was treated by H.-P. Gumm in [7]. He introduced the so called Rectangular Scheme and Shifting Principle and showed that if $\operatorname{Con}\mathcal{A}$ is modular then $\mathcal{A}$ satisfies the Rectangular Scheme; morever, Shifting Principle yields modularity of $\operatorname{Con}\mathcal{A}$ and, in varieties of algebras, the both conditions are equivalent to congruence modularity.

Let us recall the fundamental concepts. Let $\mathcal{A} = (A, F)$ be an algebra. By a *quasiorder* on $\mathcal{A}$ is meant any reflexive and transitive binary relation on A having the substitution property with respect to all operations of F. It is well-known (see e.g. [1], [5]) that the set $\operatorname{Quord}\mathcal{A}$ of all quasiorders on $\mathcal{A}$ forms an algebraic lattice with respect to set inclusion. For $\alpha, \beta \in \operatorname{Quord}\mathcal{A}$, $\alpha \wedge \beta = \alpha \cap \beta$ (the set theoretical intersection) and

$$\alpha \vee \beta = (\alpha \cdot \beta) \cup (\alpha \cdot \beta \cdot \alpha) \cup (\alpha \cdot \beta \cdot \alpha \cdot \beta) \cup \ldots$$

Definition 1 An algebra $\mathcal{A}$ is said to satisfy the *Quasiorder Principle* if for every $\alpha, \beta, \gamma \in \operatorname{Quord}\mathcal{A}$ such that $\alpha \cap \beta \subseteq \gamma$ and each λ_n (where $\lambda_0 = \gamma$, $\lambda_{k+1} = \lambda_k \cdot \alpha \cdot \gamma$) the following implication is satisfied:

$$\langle z, x\rangle \in \alpha,\ \langle z, y\rangle \in \beta,\ \langle x, y\rangle \in \lambda_n \Longrightarrow \langle z, y\rangle \in \gamma\,.$$

Remark 1 The Quasiorder Principle can be vizualized as shown in Fig. 1:

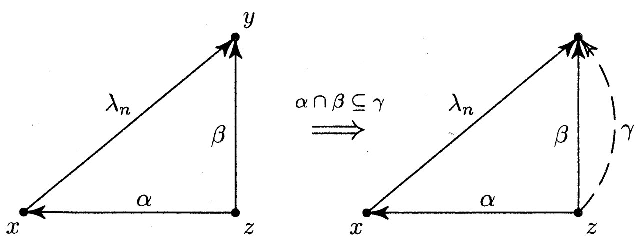

Fig. 1

Theorem 1 *An algebra $\mathcal{A}$ satisfies the Quasiorder Principle if and only if* $\operatorname{Quord}\mathcal{A}$ *is distributive.*

Proof Suppose that $\operatorname{Quord}\mathcal{A}$ is distributive and $\alpha, \beta, \gamma \in \operatorname{Quord}\mathcal{A}$ with $\alpha \cap \beta \subseteq \gamma$. Let $\langle z, x\rangle \in \alpha$, $\langle z, y\rangle \in \beta$ and $\langle x, y\rangle \in \lambda_n$ for some $n \in \mathbb{N}$. Then $\langle z, y\rangle \in \beta \cap (\alpha \cdot \lambda_n) \subseteq \beta \cap (\alpha \vee \gamma) = (\beta \cap \alpha) \vee (\beta \cap \gamma) \subseteq \gamma \vee (\beta \cap \gamma) = \gamma$, i.e. $\mathcal{A}$ satisfies the Quasiorder Principle.

Conversely, let $\mathcal{A}$ satisfy the Quasiorder Principle and suppose that $\operatorname{Quord}\mathcal{A}$ is not distributive. Then $\operatorname{Quord}\mathcal{A}$ contains a sublattice isomorphic to M_3 or N_5 as shown in Fig. 2:

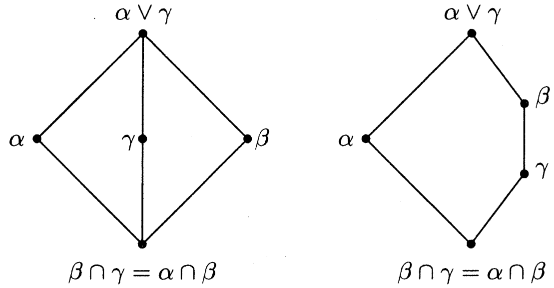

Fig. 2

In both the cases, we have $\alpha \cap \beta \subseteq \gamma$.

Let $\langle z, y\rangle \in \beta$. Then $\langle z, y\rangle \in \beta \cap (\alpha \vee \gamma)$, i.e. there exists $n \in \mathbb{N}_0$ such that $\langle z, y\rangle \in \beta \cap (\alpha \cdot \lambda_n)$. Using the Quasiorder Principle, we obtain $\langle z, y\rangle \in \gamma$, whence $\langle z, y\rangle \in \beta \cap \gamma$. We have shown $\beta \subseteq \beta \cap \gamma$, i.e. $\beta \subseteq \gamma$ which contradicts to $\beta \parallel \gamma$ in M_3 or to $\gamma \subseteq \beta$ in N_5. Hence, Quord $\mathcal{A}$ must be distributive. □

Remark 2 It is easy to prove that the Quasiorder Principle is equivalent to: For all $\alpha, \beta, \gamma \in \operatorname{Quord} \mathcal{A}$ with $\alpha \cap \beta \subseteq \gamma$, the following implication is satisfied:

$$\langle z, x\rangle \in \alpha,\ \langle z, y\rangle \in \beta,\ \langle x, y\rangle \in \gamma \vee \alpha \Longrightarrow \langle z, y\rangle \in \gamma.$$

This can be visualized as depicted in Fig. 3:

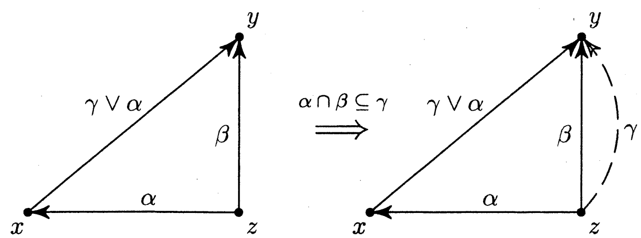

Fig. 3

A similar scheme was used in [2] as a necessary condition for distributivity of Con $\mathcal{A}$. We can repeat it here and apply for Quord $\mathcal{A}$:

Definition 2 An algebra $\mathcal{A}$ is said to satisfy the *Triangular Scheme for quasiorders* if for every $\alpha, \beta, \gamma \in \operatorname{Quord} \mathcal{A}$ such that $\alpha \cap \beta \subseteq \gamma$

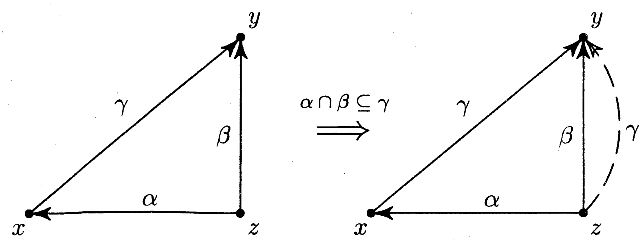

Fig. 4

It is easy to prove the following

Lemma 1 *Let $\mathcal{A}$ be an algebra. If* Quord $\mathcal{A}$ *is distributive then $\mathcal{A}$ satisfies the Triangular Scheme for quasiorders.*

Proof Suppose $\langle z,x\rangle \in \alpha$, $\langle z,y\rangle \in \beta$ and $\langle x,y\rangle \in \gamma$. Then $\langle z,y\rangle \in \alpha\cdot\gamma$ and hence

$$\langle z,y\rangle \in \beta\cap(\alpha\cdot\gamma) \subseteq \beta\cap(\alpha\vee\gamma) = (\beta\cap\alpha)\vee(\beta\cap\gamma) \subseteq \gamma\vee(\beta\cap\gamma) = \gamma$$

due to distributivity and $\alpha\cap\beta\subseteq\gamma$. □

Now, we turn our attention to the modularity of Quord $\mathcal{A}$.

Definition 3 An algebra $\mathcal{A}$ is said to satisfy the *Rectangular Quasiorder Scheme* if for every $\alpha,\beta,\gamma \in$ Quord $\mathcal{A}$ such that $\alpha\cap\beta\subseteq\gamma$

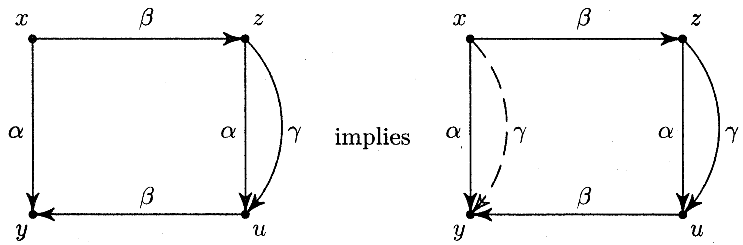

Fig. 5

Lemma 2 *Let $\mathcal{A}$ be an algebra. If* Quord $\mathcal{A}$ *is modular then $\mathcal{A}$ satisfies the Rectangular Quasiorder Scheme.*

Proof Suppose $\alpha,\beta,\gamma \in$ Quord $\mathcal{A}$ and $\langle x,y\rangle \in \alpha$, $\langle x,z\rangle \in \beta$, $\langle z,u\rangle \in \alpha$, $\langle z,u\rangle \in \gamma$ and $\langle u,y\rangle \in \beta$. Then

$$\langle x,y\rangle \in \alpha\cap(\beta\cdot(\alpha\cap\gamma)\cdot\beta) \subseteq \alpha\cap(\beta\vee(\alpha\cap\gamma)).$$

Due to modularity, we have

$$\langle x,y\rangle \in (\alpha\cap\beta)\vee(\alpha\cap\gamma) \subseteq \gamma\vee(\alpha\cap\gamma) = \gamma$$

proving the Rectangular Quasiorder Scheme. □

To set up a sufficient condition for quasiorder modularity, we can use a principle similar to that of Gumm [7] for congruence modularity:

Definition 4 An algebra $\mathcal{A}$ is said to satisfy the *Rectangular Quasiorder Principle* if for every $\alpha,\gamma \in$ Quord $\mathcal{A}$ and each λ_n (where $\lambda_1 = \beta$, $\lambda_{k+1} = \lambda_k\cdot\gamma\cdot\beta$) such that $\alpha\cap\lambda_n\subseteq\gamma\subseteq\alpha$ the following is satisfied

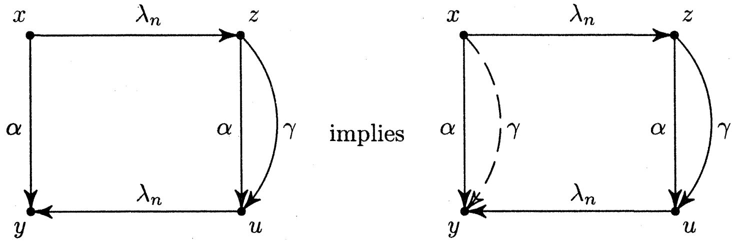

Fig. 6

Theorem 2 *If an algebra $\mathcal{A}$ satisfies the Rectangular Quasiorder Principle then* Quord $\mathcal{A}$ *is modular.*

Proof Suppose $\alpha, \beta, \gamma \in \operatorname{Quord} \mathcal{A}$ with $\gamma \subseteq \alpha$ and define $\lambda_1 = \beta$, $\lambda_{k+1} = \lambda_k \cdot \gamma \cdot \beta$ for $k = 1, 2, \ldots$ Evidently,

$$\alpha \cap (\beta \vee \gamma) = \cup\{\alpha \cap \lambda_n;\ n \in \mathbb{N}\}.$$

To prove modularity of Quord $\mathcal{A}$, we need only to show

$$\alpha \cap \lambda_n \subseteq (\alpha \cap \beta) \vee \gamma$$

for all $n \in \mathbb{N}$. We proceed by induction. For $n = 1$ the proof is trivial. Suppose $\alpha \cap \lambda_k \subseteq (\alpha \cap \beta) \vee \gamma$ and let $\langle x, y\rangle \in \alpha \cap \lambda_{k+1}$. Then $\langle x, y\rangle \in \alpha$ and $\langle x, y\rangle \in \lambda_k \cdot \gamma \cdot \beta \subseteq \lambda_k \cdot \gamma \cdot \lambda_k$, i.e. there are $u, v \in A$ such that $\langle x, u\rangle \in \lambda_k$, $\langle u, v\rangle \in \gamma$, $\langle v, y\rangle \in \lambda_k$. Hence, we have

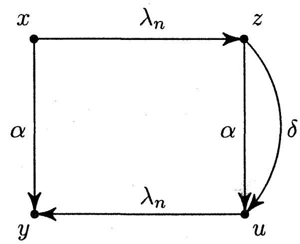

Fig. 7

for $\delta = (\alpha \cap \beta) \vee \gamma$. Clearly $\gamma \subseteq \delta$ and, using $\gamma \subseteq \alpha$ and the induction hypothesis, $\alpha \cap \lambda_k \subseteq \delta \subseteq \alpha$. We can apply the Rectangular Quasiorder Principle to obtain $\langle x, y\rangle \in \delta = (\alpha \cap \beta) \vee \gamma$. Hence, the inclusion holds for each $n \in \mathbb{N}$ and Quord $\mathcal{A}$ is modular. □

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