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Some remarks on semilinear problems at resonance where the nonlinearity depends only on the derivatives

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Abstract.

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(Dedicated to the memory of Svatopluk Fučík)

1 Introduction

There has been a wide literature devoted to the study of the Neumann and the periodic boundary value problems for differentiel equations of the form

$$u'' + g(u) = f(t),$$

and S. Fučik, after several pioneering contributions to this class of problems, has given an outstanding survey of the state of the art in the late seventies in the famous monograph [2] which constitutes his mathematical legacy.

In a recent paper [1], Cañada and Drábek have considered the solvability of scalar problems of the form

$$u'' + g(u') = f(t),\ t \in [a,\ b],$$

with Neumann or periodic boundary conditions, a situation which has been less studied that the previous one (see the references in [1]). Using a shooting argument and the implicit function theorem in Banach spaces, Cañada and Drábek have proved that if g is of class C^1 and bounded, and if we write $f = \overline{f} + \widetilde{f}$ with $\overline{f}$ the mean value of f, then, for each $\widetilde{f}$, there exists a unique $\overline{f}$ such that the above problem has a solution. They also observe that it should be interesting to study related problems for higher order equations and systems of equations.

The aim of this short note is to show that a very simple approach based upon some fixed point theory applied to one of the equations of the alternative method provides such extensions and generalizations to situations where g is neither or class C^1 nor bounded. In particular, for the equation above, those two conditions upon g can simply be dropped without loosing the conclusion.

2 The Neumann problem

Let $I = [a, b]$, $g : I \times \mathbb{R}^n \to \mathbb{R}^n$, $(t, v) \mapsto g(t, v)$ a Carathéodory function and $f \in L^1(I, \mathbb{R}^n)$. We consider the Neumann problem

$$u''(t) = g(t, u'(t)) + f(t), \ (t \in I), \ u'(a) = u'(b) = 0. \tag{1}$$

If we set $v = u'$, then (1) is equivalent to the problem

$$v'(t) = g(t, v(t)) + f(t), \ (t \in I), \ v(a) = v(b) = 0, \tag{2}$$

and if v is a solution of (2) and

$$w(t) = \int_a^t v(s)\,\mathrm{d}s,$$

then, for each $c \in \mathbb{R}^n$, $u = c + w$ will be a solution of (1). For each $h \in L^1(I, \mathbb{R}^n)$, we set

$$\overline{h} = \frac{1}{b-a}\int_a^b h(s)\,\mathrm{d}s, \ \widetilde{h}(t) = h(t) - \overline{h}.$$

Concerning problem (2), we first have the following simple lemma. Let $C(I, \mathbb{R}^n)$ be the space of continuous mappings from I into $\mathbb{R}^n$ with the usual supremum norm $||.||$.

Lemma 1. *If v is a solution of the fixed point problem in $C(I, \mathbb{R}^n)$,*

$$v(t) = \int_a^t [g(s, v(s)) - \overline{g(., v(.))} + \widetilde{f}(s)]\,\mathrm{d}s, \tag{3}$$

then v is a solution of (2) with

$$f = -\overline{g(., v(.))} + \widetilde{f}.$$

PROOF: If v is a solution of (3), then $v(a) = 0$ and

$$v(b) = \int_a^b [g(s, v(s)) - \overline{g(., v(.))} + \widetilde{f}(s)]\,\mathrm{d}s = 0.$$

Moreover, for a. e. $t \in I$, we have

$$v'(t) = g(t, v(t)) - \overline{g(., v(.))} + \widetilde{f}(t),$$

and the proof is complete. □

Lemma 1 has the following immediate consequence. For each $p \geq 1$, let $\widetilde{L^p}(I, \mathbb{R}^n)$ be the subspace of $h \in L^p(I, \mathbb{R}^n)$ such that $\overline{h} = 0$.

Corollary 1. *To each $\tilde{f} \in \widetilde{L^1}(I, \mathbb{R}^n)$ and to each solution v of the corresponding fixed point equation (3), there corresponds some $\overline{f} \in \mathbb{R}^n$ such that the problem (2) with $f = \overline{f} + \tilde{f}$ has a solution.*

We can now use fixed point theory to prove the existence of solutions of (3), and hence the existence of solutions for (2), each of one providing a family of solutions for (1).

Theorem 1. *Assume that*

$$\frac{g(t,\ v)}{|v|} \to 0 \text{ whenever } |v| \to \infty, \tag{4}$$

uniformly a. e. in $t \in I$. Then, for each $\tilde{f} \in \widetilde{L^1}(I, \mathbb{R}^n)$ there corresponds some $\overline{f} \in \mathbb{R}^n$ such that problem (1) with $f = \overline{f} + \tilde{f}$ has a solution u, and hence a family of solutions $c + u$ where $c \in \mathbb{R}^n$ is arbitrary.

PROOF: If

$$\mathcal{T} : C(I,\ \mathbb{R}^n) \to C(I,\ \mathbb{R}^n)$$

is defined by

$$\mathcal{T}(v)[t] = \int_a^t [g(s,\ v(s)) - \overline{g(.,\ v(.))} + \tilde{f}(s)]\,\mathrm{d}s,$$

then it is standard to check that $\mathcal{T}$ is a completely continuous mapping and the assumption easily implies that

$$\frac{||\mathcal{T}(v)||}{||v||} \to 0 \text{ whenever } ||v|| \to \infty.$$

The result follows easily from the Schauder fixed point theorem (cf. [2] or [4]). □

Remark 1. Condition (4) is in particular satisfied when $|g(t,\ u)|$ is bounded by an L^1-function of t, so that Theorem 1 extends to systems the existence part of Theorem 3.3 in [1].

Let now $G : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function and ∇G its gradient, $f \in L^2(I, \mathbb{R}^n)$ and consider the Neumann problem

$$u''(t) = \nabla G(u'(t)) + f(t), \quad (t \in I), \quad u'(a) = u'(b) = 0. \tag{5}$$

Theorem 2. *For each $\tilde{f} \in \widetilde{L^2}(I, \mathbb{R}^n)$ there corresponds some $\overline{f} \in \mathbb{R}^n$ such that problem (5) with $f = \overline{f} + \tilde{f}$ has a solution u, and hence a family of solutions $c + u$ where $c \in \mathbb{R}^n$ is arbitrary.*

PROOF: By Lemma 1, we must prove the existence of a solution of the fixed point problem

$$v(t) = \mathcal{S}(v)[t] := \int_a^t [\nabla G(v(s)) - \overline{\nabla G(v(.))} + \tilde{f}(s)]\,\mathrm{d}s,$$

in $C(I, \mathbb{R}^n)$. Of course, $\mathcal{S}$ is completely continuous and the existence of a fixed point will follow from the Leray-Schauder continuation theorem (see e. g. [2], [3] or [4]) if we find an a priori bound independent of $\lambda \in [0, 1]$ for the possible solution of the family of equation

$$v = \lambda \mathcal{S}(v), \quad \lambda \in [0, 1].$$

If v is such a solution for some $\lambda \in [0, 1]$, then

$$v'(t) = \lambda \nabla G(v(t)) - \lambda \overline{\nabla G[v(.)]} + \lambda \tilde{f}(t), \; v(a) = v(b) = 0,$$

and hence, taking the inner product with $v'(t)$, integrating over I and using the boundary conditions and Schwarz inequality, we get

$$\int_a^b |v'(t)|^2 \, dt = \lambda \int_a^b (\tilde{f}(t)|v'(t)) \, \mathrm{d}t \leq \left[\int_a^b |\tilde{f}(t)|^2 \, \mathrm{d}t\right]^{1/2} \left[\int_a^b |v'(t)|^2 \, \mathrm{d}t\right]^{1/2},$$

and hence

$$||v'||_{L^2} \leq ||\tilde{f}||_{L^2}.$$

From the first boundary condition we deduce immediately that, for each $t \in I$,

$$|v(t)| = |\int_a^t v'(s) \, \mathrm{d}s| \leq (b-a)^{1/2} ||v'||_{L^2} \leq (b-a)^{1/2} ||\tilde{f}||_{L^2}, \; (t \in I),$$

which provides the a priori bound for v. □

Corollary 2. *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then for each $\tilde{f} \in \widetilde{L^2}(I)$, there corresponds some $\overline{f} \in \mathbb{R}$ such that the Neumann problem*

$$u''(t) = g(u'(t)) + f(t), \; u'(a) = u'(b) = 0, \tag{6}$$

with $f = \overline{f} + \tilde{f}$ has a solution u, and hence a family of solutions $c + u$ where $c \in \mathbb{R}$ is arbitrary.

Remark 2. The above result in Corollary 2 is sharp, as shown by the linear problem

$$u''(t) = Au'(t) + \overline{f} + \tilde{f}(t), \; u'(a) = u'(b) = 0.$$

Elementary computations show that, for $A \neq 0$, the unique $\overline{f}$ for which a solution exists is given by

$$\overline{f} = \frac{A}{1 - e^{A(b-a)}} \int_a^b e^{A(b-s)} \tilde{f}(s) \, \mathrm{d}s,$$

and, for $A = 0$, the corresponding unique $\overline{f}$ is zero.

In this linear example, for each $\tilde{f}$, the existence holds for one $\overline{f}$ only. This is always the case for scalar equations, as shown by the following result which generalizes in various directions the 'uniqueness' conclusion of Theorem 3.3 of [1].

Theorem 3. *Assume that $n = 1$. Then, for each $\widetilde{f} \in \widetilde{L^1}(I)$, there exists at most one $\overline{f} \in \mathbb{R}$ such that problem (1) with $f = \overline{f} + \widetilde{f}$ has a solution.*

PROOF: Problem (3) can be written in the equivalent form

$$v(t) = \int_a^t [g(s,\ v(s)) - r + \widetilde{f}(s)]\,\mathrm{d}s,$$
$$r = \overline{g(.,\ v(.))},$$

or

$$v'(t) = g(t,\ v(t)) - r + \widetilde{f}(t),\ v(a) = 0,$$
$$r = \overline{g(.,\ v(.))}.$$

Assume that there are two different values $r_1 < r_2$ for which the problem (3) has solutions, respectively $v_1(t)$ and $v_2(t)$. Then,

$$v_1(a) = v_2(a) = 0,$$
$$v_1'(t) = g(t,\ v_1(t)) - r_1 + \widetilde{f}(t),$$
$$v_2'(t) = g(t,\ v_2(t)) - r_2 + \widetilde{f}(t) < g(t,\ v_2(t)) - r_1 + \widetilde{f}(t),$$

for $t \in I$. Notice that, as $v_1(a) = v_2(a) = 0$ and

$$v_1'(a) = g(a, 0) - r_1 + \widetilde{f}(0) > g(a, 0) - r_2 + \widetilde{f}(0) = v_2'(a),$$

we shall have $v_1(t) > v_2(t)$ for all $t \in]a,\ a + \epsilon[$ and some $\epsilon > 0$. We shall get a contradiction by showing that $v_1(t) > v_2(t)$ for each $t \in]a,\ b]$, and in particular that $v_1(b) > v_2(b)$. If it is not the case, there will some $c \in]a,\ b[$ such that

$$v_1(t) > v_2(t),\ (t \in]a,\ c[),\ v_1(c) = v_2(c).$$

Hence, for $t \in]a,\ c[$, we have

$$\frac{v_2(t) - v_2(c)}{t - c} > \frac{v_1(t) - v_1(c)}{t - c},$$

and thus $v_2'(c) \geq v_1'(c)$, a contradiction with

$$v_2'(c) = g(c,\ v_2(c)) - r_2 + \widetilde{f}(c) = g(c,\ v_1(c)) - r_2 + \widetilde{f}(c) <$$
$$< g(c,\ v_1(c)) - r_1 + \widetilde{f}(c) = v_1'(c).$$

□

When $n > 1$, the corresponding uniqueness appears to be much harder to get, as there is no ordering of the solutions of the Cauchy problem.

3 The periodic problem

Let g and f be like in the beginning of Section 2. We consider the periodic problem

$$u''(t) = g(t,\ u'(t)) + f(t),\ (t \in I),\ u(a) - u(b) = u'(a) - u'(b) = 0. \tag{7}$$

If we set $v = u'$, then (7) is equivalent to the problem

$$v'(t) = g(t,\ v(t)) + f(t),\ (t \in I),\ v(a) - v(b) = 0, \int_a^b v(s)\,\mathrm{d}s\ = 0, \tag{8}$$

and if v is a solution of (8) and

$$w(t) = \int_a^t v(s)\,\mathrm{d}s,$$

then, for each $c \in \mathbb{R}^n$, $u\ = c + w$ will be a solution of (7). Using the notations of Section 2, we know (see e.g. [3]) that for each $\widetilde{h} \in \widetilde{L^1}(I,\ \mathbb{R}^n)$, there exists a unique absolutely continuous $H(\widetilde{h})$ such that

$$\overline{H(\widetilde{h})} = 0,\ H(\widetilde{h})(a) - H(\widetilde{h})(b) = 0,\ \text{and}\ [H(\widetilde{h})]'(t) = \widetilde{h}(t)$$

for a.e. $t \in I$. Explicitely,

$$H(\widetilde{h})(t) = \int_a^t \widetilde{h}(s)\,\mathrm{d}s\ - \frac{1}{b-a}\int_a^b \int_a^t \widetilde{h}(s)\,\mathrm{d}s\,\mathrm{d}t.$$

We first have the following analog of Lemma 1.

Lemma 2. *If v is a solution of the fixed point problem in $C(I,\ \mathbb{R}^n)$,*

$$v\ = H[g(.,\ v(.)) - \overline{g(.,\ v(.))} + \widetilde{f}], \tag{9}$$

then v is a solution of (8) with

$$f = -\overline{g(.,\ v(.))} + \widetilde{f}.$$

PROOF: If v is a solution of (9), then $\overline{v} = 0$ and $v(b) - v(a) = 0$. Moreover, for a.e. $t \in I$, we have

$$v'(t) = g(t,\ v(t)) - \overline{g(.,\ v(.))} + \widetilde{f}(t),$$

and the proof is complete. □

Lemma 2 has the following immediate consequence.

Corollary 3. *To each $\widetilde{f} \in \widetilde{L^1}(I,\ \mathbb{R}^n)$ and to each solution v of the corresponding fixed point equation (9), there corresponds some $\overline{f} \in \mathbb{R}^n$ such that the problem (8) with $f = \overline{f} + \widetilde{f}$ has a solution.*

We can now use fixed point theory to prove the existence of solutions of (9), and hence the existence of solutions for (8), each of one providing a family of solutions for (7). We do not repeat the proof of the first result, which is completely analogous to that of Theorem 1.

Theorem 4. *Assume that*

$$\frac{g(t,\ v)}{|v|} \to 0 \textit{ whenever } |v| \to \infty, \tag{10}$$

uniformly a. e. in $t \in I$. *Then, for each* $\tilde{f} \in \widetilde{L^1}(I,\ \mathbb{R}^n)$ *there corresponds some* $\overline{f} \in \mathbb{R}^n$ *such that problem (7) with* $f = \overline{f} + \tilde{f}$ *has a solution* u, *and hence a family of solutions* $c + u$ *where* $c \in \mathbb{R}^n$ *is arbitrary.*

Remark 3. Condition (10) is in particular satisfied when $|g(t,\ u)|$ is bounded by an L^1-function of t, so that Theorem 3 extends to systems the existence part of Theorem 3.4 in [1].

Let now $G : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function and ∇G its gradient, $f \in L^2(I,\ \mathbb{R}^n)$ and consider the periodic problem

$$u''(t) = \nabla G(u'(t)) + f(t), \quad (t \in I), \quad u(a) - u(b) = u'(a) - u'(b) = 0. \tag{11}$$

Again, the proof of the following Theorem 5 is entirely similar to that of Theorem 2, except that, at the end, we use the Sobolev type embedding identity given on p. 208 of [3] instead of the fundamental theorem of the calculus and Schwarz inequality.

Theorem 5. *For each* $\tilde{f} \in \widetilde{L^2}(I,\ \mathbb{R}^n)$ *there corresponds some* $\overline{f} \in \mathbb{R}^n$ *such that problem (11) with* $f = \overline{f} + \tilde{f}$ *has a solution* u, *and hence a family of solutions* $c + u$ *where* $c \in \mathbb{R}^n$ *is arbitrary.*

Corollary 4. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be continuous. Then for each* $\tilde{f} \in \widetilde{L^2}(I)$, *there corresponds some* $\overline{f} \in \mathbb{R}$ *such that the periodic problem*

$$u''(t) = g(u'(t)) + f(t), \ u(a) - u(b) = u'(a) - u'(b) = 0, \tag{12}$$

with $f = \overline{f} + \tilde{f}$ *has a solution* u, *and hence a family of solutions* $c + u$ *where* $c \in \mathbb{R}$ *is arbitrary.*

Theorem 3 can also be repeated in the present setting. In the special case of the linear problem

$$u''(t) = Au'(t) + \overline{f} + \tilde{f}(t), \quad u(a) - u(b) = u'(a) - u'(b) = 0,$$

elementary computations show that, for each $\tilde{f}$, the unique $\overline{f}$ for which a solution exists is $\overline{f} = 0$.

4 Remarks on some higher order problems

If $k \geq 2$ is an integer, the same approach leads to similar results for 'Neumann problems' on I of the form

$$u^{(k)}(t) = g(t,\ u^{(k-1)}(t)) + f(t),\quad u^{(j)}(a) = 0,\ (1 \leq j \leq k-1),\ u^{(k-1)}(b) = 0, \tag{13}$$

and to periodic problems of the form

$$u^{(k)}(t) = g(t,\ u^{(k-1)}(t)) + f(t),\quad u^{(j)}(a) - u^{(j)}(b) = 0,\ (0 \leq j \leq k-1), \tag{14}$$

when g satisfies the conditions respectively given in Section 2 and 3. Indeed, letting $v = u^{(k-1)}$, we see that if v is a solution of the problem (2) (resp. (8)), then, for every $c \in \mathbb{R}^n$,

$$u(t) = c + \int_a^t \frac{(t-s)^{(k-2)}}{(k-2)!} v(s)\,\mathrm{d}s,$$

(resp.

$$u\ = c + H^{(k-1)}v,)$$

will be solutions of (13) (resp. 14). Hence the sufficient conditions for the existence of a solution of (2) (resp. (8)) given in Section 2 (resp. Section 3) will provide sufficient conditions for the existence of a solution of (13) (resp. (14)).

The same methodology applies to higher order problems on I of the form

$$\begin{aligned} &u^{(p+q)}(t) = g(t,\ u^{(p)}(t),\ \ldots,\ u^{(p+q-1)}(t)) + f(t), \\ &u^{(j)}(a) = 0,\ (1 \leq j \leq p+q-1),\ u^{(p+q-1)}(b) = 0, \end{aligned} \tag{15}$$

or

$$\begin{aligned} u^{(p+q)}(t) = g(t,\ u^{(p)}(t),\ \ldots,\ u^{(p+q-1)(t)}) + f(t), \\ u^{(j)}(a) - u^{(j)}(b) = 0,\ (0 \leq j \leq p+q-1), \end{aligned} \tag{16}$$

where $p \geq 1$ and $q \geq 1$ are integers. Setting $v = u^{(p)}$, we are reduced to consider the problems

$$\begin{aligned} &v^{(q)}(t) = g(t,\ v(t),\ \ldots,\ v^{(q-1)}(t)) + f(t), \\ &v^{(j)}(a) = 0,\ (1 \leq j \leq q-1),\ v^{(q-1)}(b) = 0, \end{aligned} \tag{17}$$

or

$$\begin{aligned} v^{(q)}(t) = g(t,\ v(t),\ \ldots,\ v^{(q-1)}(t)) + f(t), \\ v^{(j)}(a) - v^{(j)}(b) = 0,\ (1 \leq j \leq q-1), \overline{v} = 0. \end{aligned} \tag{18}$$

To each solution v of (17) will correspond the solutions

$$u(t) = c + \int_a^t \frac{(t-s)^{p-1}}{(p-1)!} v(s)\,\mathrm{d}s,$$

of (15), and to each solution v of (18) will correspond the solutions

$$u = c + H^p v,$$

where $c \in \mathbb{R}^n$ is arbitrary. The corresponding fixed point problems to be considered in the space $C^{(q-1)}(I, \mathbb{R}^n)$ of mappings of class $C^{(q-1)}$ from I to $\mathbb{R}^n$ will be respectively of the form

$$v(t) = \int_a^t \frac{(t-s)^{p-1}}{(p-1)!}[g(s,\ v(s),\ \ldots,\ v^{(q-1)}(s)) - \overline{g(.,\ v(.),\ \ldots,\ v^{(q-1)}(.))} + \tilde{f}(s)]\,\mathrm{d}s,$$

and

$$v = H^p[g(.,\ v(.),\ \ldots,\ v^{(q-1)}(.)) - \overline{g(.,\ v(.),\ \ldots,\ v^{(q-1)}(.))} + \tilde{f}].$$

In particular, when $|g(t,\ u_1,\ \ldots,\ u_q)|$ is bounded everywhere by a L^1-function of t, we have existence theorems of the type of Theorem 1 or Theorem 3.

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