Acta Mathematica et Informatica Universitatis Ostraviensis

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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 3 (1995), No. 1, 27--(36)

Persistent URL: `http://dml.cz/dmlcz/120491`

Semilinear Problems with Nonlinearities Depending Only on Derivatives

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Abstract. The existence of solutions are studied for semilinear boundary value problems of ordinary differential equations where nonlinearities depend only on derivatives.

1991 Mathematics Subject Classification: 34B15, 47H15, 47N20

1 Introduction

A. Cañada and P. Drábek have studied recently in [2] the solvability of the problem

$$u'' + g(u') = f(t), \quad t \in [0,1] \tag{1.1}$$

with Neumann or periodic boundary conditions. They have proved that if g is of class C^1 and bounded, and if we write $f = \bar{f} + \tilde{f}$ with $\bar{f}$ the mean value of f, then for each $\tilde{f}$, there exists a unique $\bar{f}$ such that the problem (1.1) is solvable. When g is neither or class C^1 nor bounded, J. Mawhin in [4] has derived existence results to the problem (1.1) by using fixed point theorems. Extensions to certain higher-order equations are also given in [4].

The purpose of this note is to continue in that direction. In Section 2, we formulate an abstract version of the problem (1.1) and give existence results based on the well-known theorems of [3]. Certain boundary value problems are studied in Section 3. The solvability of the problem (1.1) with the Dirichlet, anti-periodic and mixed boundary conditions is shown for rather general classes of g. Higher-order equations are also investigated.

2 Preliminary Results

In this section, we derive an abstract version of boundary value problems mentioned in Introduction based on a standard approach of [3]. Let X, Y, U, Z be Banach spaces such that Y is compactly embedded into U. The norm on X is denoted by $|\cdot|_X$, similarly for the other spaces. Consider the equation

$$L \circ Mx = N(Mx) + h, \quad h \in Z \tag{2.1}$$

where $L\, Y \to Z$, $M\, X \to Y$ are continuous linear Fredholm operators, $N\, U \to Z$ is continuous. We put $V = \operatorname{im} M$ and rewrite (2.1) in the form

$$Lv = N(v) + h, \quad h \in Z, v \in V\,. \tag{2.2}$$

So (2.1) is solvable if and only if (2.2) is solvable, and if v is a solution of (2.2) then $\{w + x_0 \mid x_0 \in \ker M\}$ is a family of solutions of (2.1) with $v = Mw$.

Lemma 2.1. *The operator $L/V\ V \to Z$ is a Fredholm operator of index*

$$ind\, L/V = ind\, L - codim\ im\, M\ .$$

PROOF: By putting

$$V \oplus V_1 = Y,\ V_3 \oplus \ker L \cap V = V,\ V_4 \oplus \ker L \cap V_1 = V_1\ ,$$

we have

$$\begin{gathered} V_3 \oplus V_4 \oplus \ker L \cap V \oplus \ker L \cap V_1 = Y \\ \ker L \cap V \oplus \ker L \cap V_1 = \ker L\ . \end{gathered}$$

We know that $L/V_3 \oplus V_4$ is an isomorphism on $\operatorname{im} L$. So

$$\text{codim im}\, L/V = \text{codim im}\, L + \dim V_4\ .$$

Since

$$\begin{gathered} \dim V_4 + \dim \ker L \cap V_1 = \dim V_1 = \operatorname{codim} V \\ \dim \ker L \cap V + \dim \ker L \cap V_1 = \dim \ker L\ , \end{gathered}$$

we obtain

$$\begin{aligned} \dim V_4 &= \text{codim im}\, M - \dim \ker\ L \cap V_1 = \\ &= \text{codim im}\, M + \dim \ker L \cap V - \dim \ker L\ . \end{aligned}$$

Summarizing we arrive at

$$\begin{aligned} \operatorname{ind} L/V &= \dim \ker L \cap V - \text{codim im}\, L/V = \\ &= \dim \ker L \cap V - \text{codim im}\, L - \dim V_4 = \\ &= \dim \ker L \cap V - \text{codim im}\, L - \text{codim im}\, M - \\ &\quad -\dim \ker L \cap V + \dim \ker L = \\ &= \dim \ker L - \text{codim im}\, L - \text{codim im}\, M = \operatorname{ind} L - \text{codim im}\, M\ . \end{aligned}$$

The proof is finished. □

Now we assume that the following condition holds

$$\dim \ker L \cap V = 0\ . \tag{H}$$

Lemma 2.1 and condition (H) imply that codim $\operatorname{im} L/V =$ codim $\operatorname{im} M - \operatorname{ind} L$.
Let $Q\ Z \to \operatorname{im} L/V$, $Q = \mathbb{I} - P$ be continuous projections.

Theorem 2.2. *If for any $\tilde{h} \in im Q$, there is an open bounded subset $\Omega_{\tilde{h}}$ such that $0 \in \Omega_{\tilde{h}} \subset V$ and the equation*

$$Lv = \lambda\big(QN(v) + \tilde{h}\big), \quad v \in \partial\Omega_{\tilde{h}}$$

has no solution for any $\lambda \in [0,1]$*. Then for any* $\tilde{h} \in im\, Q$*, there is a* $\bar{h} \in im\,(\mathbf{I}-Q)$ *such that* (2.1) *has a solution* x *with* $h = \bar{h}+\tilde{h}$*, hence* (2.1) *has a family of solutions* $\{x + x_0 \mid x_0 \in ker\, M\}$*.*

PROOF: By using the standard approach (see [3]), we first rewrite the equation $Lv = \lambda\big(QN(v) + \tilde{h}\big)$ as $v = \lambda L^{-1}\big(QN(v) + \tilde{h}\big)$. Since $L^{-1}QN\; V \to Y$ is compact and $0 \in \Omega_{\tilde{h}}$, we have

$$\deg\Big(\mathbf{I} - \lambda L^{-1}\big(QN + \tilde{h}\big), \Omega_{\tilde{h}}, 0\Big) = \deg\big(\mathbf{I}, \Omega_{\tilde{h}}, 0\big) = 1\,,$$

where deg is the Leray-Schauder degree. So there is a $v \in \Omega_{\tilde{h}}$ such that

$$Lv = QN(v) + \tilde{h} = N(v) - PN(v) + \tilde{h}\,.$$

Consequently, we can take $\bar{h} = -PN(v)$. The proof is finished. □

Corollary 2.3. *Assume*

$$|N(v)|_Z/|v|_Y \to 0 \quad \textit{whenever} \quad V \ni v \to \infty\,,$$

then the conclusion of Theorem 2.2 *holds.*

PROOF: It is enough to take a sufficiently large ball for $\Omega_{\tilde{h}}$. □

Remark 2.4. *If (H) does not hold and* $ind\, L/V = ind\, L - codim\; im\, M = 0$*, then a coincidence degree arguments of [3] could be applied to (2.2).*

3 Boundary Value Problems

Consider the Dirichlet problem

$$\begin{aligned} u'' &= g(t,u') + h(t) \\ u(0) &= u(1) = 0\,, \end{aligned} \tag{3.1}$$

where $g \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $h \in C([0,1], \mathbb{R})$. In the framework of Section 2, we put

$$\begin{gathered} X = \Big\{x \in C^2([0,1],\mathbb{R}) \mid x(0) = x(1) = 0\Big\} \\ U = Z = C([0,1],\mathbb{R}), \quad Y = C^1([0,1],\mathbb{R}) \\ Lv = v', \quad Mu = u' \\ N(v) = g(\cdot, v), \quad h = h(\cdot) \\ V = \operatorname{im} M = \Big\{v \in C^1([0,1],\mathbb{R}) \mid \int_0^1 v(s)\,ds = 0\Big\}\,. \end{gathered}$$

It is clear that the condition (H) holds.

Theorem 3.1. *Let inf be either $+\infty$ or $-\infty$, and let $\psi\,[0,+\infty)\to(0,+\infty)$ be a continuous nondecreasing mapping such that $\int\limits_0^{+\infty}\frac{du}{\psi(u)}=+\infty$. If we assume that either*

$$\limsup_{u\to inf}|g(t,u)|<+\infty \quad \textit{uniformly with respect to } t\,,$$

or

$$\lim_{u\to inf} g(t,u)=inf \quad \textit{uniformly with respect to } t$$

and then, in addition, we suppose

$$\limsup_{u\to inf}|g(t,u)|/\psi(|u|)<+\infty \quad \textit{uniformly with respect to } t\,.$$

Then (3.1) *has a solution for any $h\in C([0,1],\mathbb{R})$.*

PROOF: We apply Theorem 2.2. It is clear that now $\operatorname{im} L/V=Z$, so $Q=\mathbb{I}$. For a fixed $h\in C([0,1],\mathbb{R})$, we take

$$\Omega_h=\{v\in V \mid \quad |v(0)|<c,\ |v|_V<K\}\,,$$

where c, K are positive constants specified below. Consider the equation

$$\begin{aligned} v'&=\lambda\big(g(t,v)+h(t)\big), \quad \lambda\in[0,1]\\ v(0)&=v_0\,. \end{aligned} \tag{3.2}$$

The assumptions on g imply the existence of a continuous nondecreasing mapping $\phi\,[0,+\infty)\to(0,+\infty)$ such that

$$|g(t,u)|\le\phi(|u|) \quad \forall\,(t,u)\in[0,1]\times\mathbb{R}$$
$$\int\limits_0^{+\infty}\frac{du}{\phi(u)}=+\infty, \qquad \lim_{x\to+\infty}\phi(x)=+\infty\,.$$

Since

$$\big|\lambda\big(g(t,v)+h(t)\big)\big|\le|g(t,v)|+|h|_C\le\phi(|v|)+|h|_C\,,$$

and $\int\limits_0^{+\infty}\frac{du}{\phi(u)+|h|_C}=+\infty$, we have by the Bihari lemma (see [1]) the existence of a nondecreasing mapping $B\,[0,+\infty)\to(0,+\infty)$ such that any solution of (3.2) satisfies $|v|_C\le B(|v_0|)$. So for a fixed $c>0$, we take $K>B(c)$ in Ω_h. Let $v\in\partial\Omega_h$ be a solution of the equation

$$\begin{aligned} v'&=\lambda\big(g(t,v)+h(t)\big), \quad \lambda\in[0,1]\\ &\int\limits_0^1 v(s)\,ds=0\,. \end{aligned} \tag{3.3}$$

According to the choice of K, $v\in\partial\Omega_h$ implies $|v(0)|=c$. If $v(0)=c$ and g is bounded on $[0,1]\times[0,+\infty)$, we have $v(t)>0$ on $[0,1]$ for a fixed, sufficiently large c. The same holds when $\lim_{u\to+\infty} g(t,u) = +\infty$ uniformly with respect to t. If $v(0) = -c$, then similarly the assumptions on g imply $v(t) < 0$ on $[0,1]$ for a fixed, sufficiently large c. Hence (3.3) has no solution on $\partial\Omega_h$. Consequently, Theorem 2.2 can be applied. The proof is finished. □

Similarly we have the following results.

Theorem 3.2. *Consider the problem*

$$\begin{aligned} u'' &= g(t,u') + h(t) \\ u'(0) &= u(1) = 0\,, \end{aligned} \tag{3.4}$$

where $g \in C([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$, $h \in C([0,1],\mathbb{R}^n)$. *Let* $\psi\ [0,+\infty) \to (0,+\infty)$ *be a continuous nondecreasing mapping such that* $\int_0^{+\infty} \frac{du}{\psi(u)} = +\infty$. *If*

$$|g(t,u)| \le \psi(|u|) \quad \forall\,(t,u) \in [0,1]\times\mathbb{R}^n$$

then (3.4) *has a solution for any* $h \in C([0,1],\mathbb{R}^n)$.

PROOF: We apply Theorem 2.2 by putting

$$\begin{gathered} X = \left\{x \in C^2([0,1],\mathbb{R}^n) \mid x'(0) = x(1) = 0\right\} \\ U = Z = C([0,1],\mathbb{R}^n), \quad Y = C^1([0,1],\mathbb{R}^n) \\ Lv = v', \quad Mu = u' \\ N(v) = g(\cdot,v), \quad h = h(\cdot) \\ V = \operatorname{im} M = \left\{v \in C^1([0,1],\mathbb{R}^n) \mid v(0) = 0\right\}. \end{gathered}$$

It is clear that the condition (H) holds as well as $\operatorname{im} L/V = Z$, so $Q = \mathbb{I}$. For a fixed $h \in C([0,1],\mathbb{R}^n)$, we take

$$\Omega_h = \{v \in V \mid \quad |v|_V < K\}\,,$$

where K is a positive constant specified below. Consider the equation

$$v' = \lambda\big(g(t,v) + h(t)\big), \quad v(0) = 0,\ \lambda \in [0,1]\,. \tag{3.5}$$

By repeating the arguments to (3.2), we see that there is a constant K such that any solution v of (3.5) satisfies $|v|_V < K$. So we take this constant K in Ω_h. Consequently the equation $v' = \lambda\big(g(t,v) + h(t)\big)$ has no solution in $\partial\Omega_h$. The proof is finished by Theorem 2.2. □

Theorem 3.3. *Consider the problem*

$$u'' = g(t,u') + h(t)u'(0) = 0,\ u(0) = u(1)\,, \tag{3.6}$$

where $g \in C([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$, $h \in C([0,1],\mathbb{R}^n)$. *If*

$$|g(t,u)|/|u| \to 0 \quad \textit{whenever} \quad |u| \to +\infty$$

uniformly with respect to $t \in [0,1]$, *then for any* $\tilde{h} \in ([0,1], \mathbb{R}^n)$ *satisfying*

$$\int_0^1 (1-s)\tilde{h}(s)\,ds = 0\,,$$

there is a $\bar{h} \in \mathbb{R}^n$ *such that the problem* (3.6) *has a solution* u *with* $h = \bar{h} + \tilde{h}$, *and hence a family of solutions* $c + u$ *where* $c \in \mathbb{R}^n$ *is arbitrary.*

PROOF: We apply Corollary 2.3 by putting

$$\begin{gathered} X = \left\{x \in C^2([0,1], \mathbb{R}^n) \mid x'(0) = 0,\, x(0) = x(1)\right\} \\ U = Z = C([0,1], \mathbb{R}^n), \quad Y = C^1([0,1], \mathbb{R}^n) \\ Lv = v', \quad Mu = u' \\ N(v) = g(\cdot, v), \quad h = h(\cdot)\,. \end{gathered}$$

It is not hard to see that

$$\begin{aligned} \ker M &= \{x(t) \text{ is constant on } [0,1]\} \\ V = \operatorname{im} M &= \left\{v \in C^1([0,1], \mathbb{R}^n) \mid v(0) = 0,\, \int_0^1 v(s)\,ds = 0\right\} \\ \operatorname{im} L/V &= \left\{v \in C([0,1], \mathbb{R}^n) \mid \int_0^1 (1-s)v(s)\,ds = 0\right\}. \end{aligned}$$

So the condition (H) holds as well as $Z = \operatorname{im} L/V \oplus \text{Const}$, where Const is the space of constant functions on $[0,1]$. So we take the projection $Q\, Z \to Z$ such that

$$\operatorname{im} Q = \operatorname{im} L/V, \quad \operatorname{im}(\mathbb{I} - Q) = \text{Const}\,.$$

This projection is defined by

$$(Qy)(s) = y(s) - 2\int_0^1 (1-s)y(s)\,ds\,.$$

Of course, the assumptions on g implies the condition on N of Corollary 2.3. The proof is finished. □

Theorem 3.4. *If in addition to the assumptions of Theorem* 3.2, *respectively Theorem* 3.3, *the mapping* g *is locally Lipschitz continuous in the variable* u. *Then* (3.4) *has a unique solution for any* $h \in C([0,1], \mathbb{R})$, *respectively the derived solution of* (3.6) *in Theorem* 3.3 *is unique up to a constant additive.*

PROOF: By taking $v = u'$ in the both equations (3.4) and (3.6), we arrive at the Cauchy problem

$$v' = g(t, v) + h(t), \quad v(0) = 0\,.$$

Since g is locally Lipschitz continuous in u, the classical result (see [3]) about the local uniqueness of the Cauchy problem implies that any solution of the above Cauchy problem on $[0,1]$ is unique. The proof is finished. □

Theorem 3.5. *Consider the problem*

$$\begin{aligned} u'' &= g(t,u') + h(t) \\ u'(0) &= -u'(1),\, u(0) = -u(1)\,, \end{aligned} \tag{3.7}$$

where $g \in C([0,1]\times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C([0,1],\mathbb{R}^n)$. *Let* $\langle\cdot,\cdot\rangle$ *be an inner product on* $\mathbb{R}^n$. *If*

$$\langle g(t,u),u\rangle/|u| \to \pm\infty \quad \textit{whenever} \quad |u| \to +\infty$$

uniformly with respect to $t \in [0,1]$, *then for any* $\tilde{h} \in C([0,1],\mathbb{R}^n)$ *the problem* (3.7) *has a solution.*

PROOF: We consider the positive sign in the assumption on g, the negative one can be treated similarly. We apply Theorem 2.2 by putting

$$\begin{gathered} X = \left\{x \in C^2([0,1],\mathbb{R}^n) \mid x'(0) = -x'(1),\, x(0) = -x(1)\right\} \\ U = Z = C([0,1],\mathbb{R}^n), \quad Y = C^1([0,1],\mathbb{R}^n) \\ Lv = v', \quad Mu = u' \\ N(v) = g(\cdot, v), \quad h = h(\cdot) \\ V = \operatorname{im} M = \left\{v \in C^1([0,1],\mathbb{R}^n) \mid v(0) = -v(1)\right\}. \end{gathered}$$

The condition (H) holds as well as $\operatorname{im} L/V = Z$, so $Q = \mathbb{I}$. Consider the equation

$$\begin{aligned} v' &= \lambda\big(g(t,v) + h(t)\big), \quad \lambda \in [0,1] \\ v(0) &= -v(1)\,. \end{aligned} \tag{3.8}$$

The assumptions on g imply the existence of a constant $K > 0$ such that

$$\langle g(t,v),v\rangle - \langle h(t),v\rangle > 0$$

for any $t \in [0,1]$ and $|v| \ge K$. Hence (3.8) gives $\frac{1}{2}\big(|v(t)|^2\big)' = \langle v'(t), v(t)\rangle > 0$ for any $t \in [0,1]$ such that $|v(t)| \ge K$. Consequently, if $|v(t_0)| \ge K$ then $|v(t)| \ge K$ for any $1 \ge t \ge t_0$, and $|v(t)|$ is increasing on $[t_0, 1]$. We take

$$\Omega_h = \left\{v \in V \mid \quad |v|_V < K\right\}.$$

If $v \in \partial\Omega_h$ solves (3.8), then there is a $t_0 \in [0,1]$ such that $|v(t_0)| = K$, hence $|v(0)| = |v(1)| \ge K$ and $|v(t)|$ is increasing on $[0,1]$, which contradicts to $|v(0)| = |v(1)|$. So the equation (3.8) has no solution in $\partial\Omega_h$. The proof is finished. □

Finally we consider two three-order problems.

Theorem 3.6. *Consider the problem*

$$\begin{aligned} u''' &= g(t,u'') + h(t) \\ u'(0) &= u'(1) = 0,\, u(0) = u(1)\,, \end{aligned} \tag{3.9}$$

where $g \in C([0,1]\times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C([0,1],\mathbb{R}^n)$. *If*

$$|g(t,u)|/|u| \to 0 \quad \textit{whenever} \quad |u| \to +\infty$$

uniformly with respect to $t \in [0,1]$, *then for any* $\tilde{h} \in C([0,1],\mathbb{R}^n)$ *satisfying*

$$\int_0^1 s(1-s)\tilde{h}(s)\,ds = 0\,,$$

there is a $\bar{h} \in \mathbb{R}^n$ *such that the problem* (3.9) *has a solution* u *with* $h = \bar{h} + \tilde{h}$, *and hence a family of solutions* $c + u$ *where* $c \in \mathbb{R}^n$ *is arbitrary.*

PROOF: We apply Corollary 2.3 by putting

$$\begin{gathered} X = \left\{x \in C^3([0,1],\mathbb{R}^n) \mid x'(0) = x'(1) = 0,\, x(0) = x(1)\right\} \\ U = Z = C([0,1],\mathbb{R}^n), \quad Y = C^1([0,1],\mathbb{R}^n) \\ Lv = v', \quad Mu = u'' \\ N(v) = g(\cdot, v), \quad h = h(\cdot)\,. \end{gathered}$$

It is clear that the condition (H) holds. Now we derive $V = \operatorname{im} M$. If $v \in Y$ then the problem $u'' = v$, $u'(0) = u'(1) = 0$ has a solution only if $\int_0^1 v(s)\,ds = 0$, and then $u(t) = d + \int_0^t \int_0^s v(z)\,dz\,ds$, where d is a constant. So such u satisfies $u(0) = u(1)$ if and only if

$$0 = \int_0^1 \int_0^s v(z)\,dz\,ds = \int_0^1 (1-s)v(s)\,ds\,.$$

Hence

$$V = \operatorname{im} M = \left\{v \in C^1([0,1],\mathbb{R}^n) \mid \int_0^1 (1-s)v(s)\,ds = \int_0^1 v(s)\,ds = 0\right\}.$$

Similarly we obtain

$$\operatorname{im} L/V = \left\{v \in C([0,1],\mathbb{R}^n) \mid \int_0^1 s(1-s)v(s)\,ds = 0\right\}.$$

Consequently, $Z = \operatorname{im} L/V \oplus \mathrm{Const}$, where Const is again the space of constant functions on $[0,1]$. So we take the projection $Q\, Z \to Z$ such that

$$\operatorname{im} Q = \operatorname{im} L/V, \quad \operatorname{im}(\mathbb{I} - Q) = \mathrm{Const}\,.$$

This projection is defined by

$$(Qy)(s) = y(s) - 6\int_0^1 s(1-s)y(s)\,ds\,.$$

Of course, the assumptions on g implies the condition on N of Corollary 2.3. Since $\ker M = \{x(t) \text{ is constant on } [0,1]\}$, the proof is finished. □

Similarly we have

Theorem 3.7. *Consider the problem*

$$\begin{aligned} u''' &= g(t, u') + h(t) \\ u'(0) &= u''(0) = u''(1) = 0 , \end{aligned} \tag{3.10}$$

where $g \in C([0,1] \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C([0,1], \mathbb{R}^n)$. *If*

$$|g(t,u)|/|u| \to 0 \quad \textit{whenever} \quad |u| \to +\infty$$

uniformly with respect to $t \in [0,1]$, *then for any* $\tilde{h} \in C([0,1], \mathbb{R}^n)$ *satisfying*

$$\int_0^1 \tilde{h}(s)\, ds = 0 ,$$

there is a $\bar{h} \in \mathbb{R}^n$ *such that the problem* (3.10) *has a solution* u *with* $h = \bar{h} + \tilde{h}$, *and hence a family of solutions* $c + u$ *where* $c \in \mathbb{R}^n$ *is arbitrary. Moreover, if* g *is locally Lipschitz continuous in the variable* u, *then the derived solution* u *is unique up to a scalar additive.*

Proof: We apply Corollary 2.3 by putting

$$\begin{gathered} X = \{x \in C^3([0,1], \mathbb{R}^n) \mid x'(0) = x''(0) = x''(1) = 0\} \\ U = Z = C([0,1], \mathbb{R}^n), \quad Y = C^2([0,1], \mathbb{R}^n) \\ Lv = v'', \quad Mu = u' \\ N(v) = g(\cdot, v), \quad h = h(\cdot) . \end{gathered}$$

The condition (H) holds, and

$$\begin{gathered} V = \operatorname{im} M = \{v \in C^2([0,1], \mathbb{R}^n) \mid v(0) = v'(0) = v'(1) = 0\} \\ \operatorname{im} L/V = \Big\{v \in C([0,1], \mathbb{R}^n) \mid \int_0^1 v(s)\, ds = 0\Big\} \\ \ker M = \{x(t) \text{ is constant on } [0,1]\} . \end{gathered}$$

So we take the projection Q defined by

$$(Qy)(s) = y(s) - \int_0^1 y(s)\, ds$$

such that

$$\operatorname{im} Q = \operatorname{im} L/V, \quad \operatorname{im}(\mathbb{I} - Q) = \text{Const} ,$$

where Const is the usual space.

Of course, the assumptions on g implies the condition on N of Corollary 2.3. Since $\ker M = \{x(t) \text{ is constant on } [0,1]\}$, the first part of the theorem is proved. The second one follows from the fact that the function $v = u'$ satisfies the Cauchy problem

$$v'' = g(t, v) + h(t), \quad v(0) = v'(0) = 0\,.$$

The proof is finished. □

Remark 3.8. *Similarly as in [4, p. 68], we can study higher-order equations. For instance, Theorem 3.1 is valid also for the problem*

$$\begin{gathered} u^{(k)} = g(t, u^{(k-1)}) + h(t) \\ u^{(j)}(0) = 0,\ 1 \le j \le k-2,\ u^{(k-2)}(1) = 0\ , \end{gathered}$$

where $g \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $h \in C([0,1], \mathbb{R})$ *and* $k \geq 2$ *is an integer number.*

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