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Composition of Shape Generators *)

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Abstract. The concept of generators of fuzzy quantities can simplify the practical application of some algorithms, and also explain some phenomena, practically observed in real situations and connected with the vagueness but not modelled by the recent theory of fuzzy numbers. Further development of that concept demands investigation of formal properties of the generators. In this brief note we deal with so called shape generators (cf. [3], [4]) and their basic properties. Main attention is paid to the operation of composition of shape generators connected with different fuzzy quantities entering algebraic operations.

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1 Fuzzy Quantity

In the following paragraphs we denote by R the set of real numbers. *Fuzzy quantity* is any fuzzy subset a of R with membership function $\mu_a : R \longrightarrow [0,1]$ such that

$$\exists\, x_a \in R : \mu_a(x_a) = 1. \tag{1}$$

By $I\!R$ we denote the set of all fuzzy quantities and for any $a \in I\!R$ the value $x_a \in R$ fulfilling (1) is called the *modal value* of a.
Our attention will be focused to the sources of fuzzy quantities which can be described by generating procedures. They are formally represented by *generating functions* $f : R \to R$ such that

$$f \text{ is continuous and strictly increasing,} \tag{2}$$

$$f(0) = 0, \tag{3}$$

and *shape generators* $\varphi : R \longrightarrow [0,1]$ such that $\varphi(\varphi_L, \varphi_R)$, $\varphi_L : R_- \longrightarrow [0,1]$, $\varphi_R : R_+ \longrightarrow [0,1]$, where

$$R_- = \{x \in R : x \le 0\}, \qquad R_+ = \{x \in R : x \ge 0\}$$

and

$$\varphi_L(0) = \varphi_R(0) = 1, \tag{4}$$

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$$\varphi_L \text{ and } \varphi_R \text{ are non-negative,} \tag{5}$$
$$\varphi_L \text{ is increasing,} \tag{6}$$
$$\varphi_R \text{ is decreasing.} \tag{7}$$

By $\mathcal{F}$ we denote the class of all generating functions, and by Φ we denote the class of all shape generators.

2 Interpretations

The previous concepts can be interpreted as follows. Each fuzzy quantity $a \in \mathbb{R}$ is considered for some generalization (extension) of the crisp number x_a which is its modal value (or one of modal values). The formal outlook of this extension is derived from a generating function f by means of some shape generator φ.

The generating functions reflect the character of vagueness typical for the considered source of numerical data. Typical examples of such functions are linear or piece-wise linear ones fulfilling (2) and (3), or

$$\begin{aligned} f(x) &= \ln(x+1) \quad \text{for } x \geq 0, \\ &= -\ln(|x-1|) \quad \text{for } x < 0, \end{aligned} \tag{8}$$

and also for $r > 1$

$$f(x) = \operatorname{sign}(x) \cdot r \cdot \left(1 - 1/2^{|x|}\right), \tag{9}$$

as well as their combinations fulfilling (2), (3).

The shape generators determine the procedure transforming the general structure of vagueness being represented by $f \in \mathcal{F}$ into actual fuzzy quantities with membership functions. If $a \in \mathbb{R}$ then

$$\mu_a(x) = \varphi(f(x) - f(x_a)) \tag{10}$$

where the notation used above is preserved, and the equation in (10) means

$$\begin{aligned} \mu_a(x) &= \varphi_L(f(x) - f(x_a)) \quad \text{for } x \leq x_a, \\ &= \varphi_R(f(x) - f(x_a)) \quad \text{for } x \geq x_a. \end{aligned} \tag{11}$$

Condition (3) is consistent with (11) as well as with (1).

Each shape generator represents specific properties of the generated fuzzy quantities, frequently characterized by verbal description like "approximately x_a", "about x_b or rather more", "may be x_c", etc. In the first case, the shape generator can be symmetric and linear, e. g.,

$$\varphi(f(x) - f(x_a)) = \max(0,\ 1 - |f(x) - f(x_a)|), \tag{12}$$

in the second case, it will be asymmetric, e. g.,

$$\begin{aligned} \varphi_L(f(x) - f(x_b)) &= \max(0,\ 1 - r \cdot (f(x_b) - f(x))) \\ \varphi_R(f(x) - f(x_b)) &= \max(0,\ 1 - s \cdot (f(x_b) - f(x))), \\ &\quad \text{for some } r > 1,\ 0 < s < 1, \end{aligned} \tag{13}$$

and in the third case it can be, e. g.

$$\varphi(f(x) - f(x_c)) = r \cdot \exp(-|f(x) - f(x_c)|) - r + 1 \quad \text{for some } r \geq 1. \tag{14}$$

If the generating function f is described by (9) for some $r > 1$ (this case seems to be advantageous for its reflecting the increasing absolute uncertainty for larger modal values of generated fuzzy quantities) then it is possible to generate even such vague quantities like "much", e. g. by (12) where x_a tends to $+\infty$ and

$$f(x_a) = \lim_{x \to \infty} f(x).$$

It is also very easy to construct shape generator for "crisp x_d"

$$\varphi(f(x) - f(x_d)) = \max(0, \text{int}(1 - |f(x) - f(x_d)|)), \tag{15}$$

where $\text{int}(x)$ is the integer part of $x \in R$ (i. e. the greatest integer $n \in N$ such that $n \leq x$), for "crisp interval $(x_d - \delta_1, x_d + \delta_2)$", $\delta_1, \delta_2 > 0$ by

$$\begin{aligned} \varphi_L(f(x) - f(x_d)) &= \max(0, 1 - \text{int}((f(x_d) - f(x))/\delta_1)), \\ \varphi_R(f(x) - f(x_d)) &= \max(0, 1 - \text{int}((f(x_d) - f(x))/\delta_2)). \end{aligned} \tag{16}$$

Of course, proper combination of these (and other) patterns of the shape generator is suitable for modelling of even more complicated structures of vagueness of quantitative data.

3 Composition of Sources

The main purpose of the introduction of generating functions and shape generators is to transfer the operations with particular fuzzy quantities to analogous operations with generating functions, and in this way to simplify the application of frequently used numerical algorithms. Instead of repetitive calculation of the output fuzzy quantities from the input ones it would be possible to calculate the modal value of the result from the modal values of the inputs (all of them are crisp), and then to generate the final fuzzy quantity from this modal value and from the output generating function derived from the input ones. In other words, the sources of fuzzy data can be combined to construct a single fuzzy source of the results of the algorithm.

The transition of algebraic operations with fuzzy quantities to the operations with generating functions was dealt with in [3]. Here we want to suggest a general model of composition of the shape generators. As the subject is completely new (the first – very heuristic – attempt to manage it was done only in [4]), the following paragraphs represent rather a suggestion of possible strategy of its elaboration.

Meanwhile the operations with generating functions reflect and generalize the analogous algebraic operations with fuzzy quantities, the combination of shape generators should reflect another quality of relations. Each shape generator represents some structure of possible values of fuzzy quantities. In certain sense, it is a pattern of vagueness which is reproduced in particular fuzzy quantities generated by means of it. Hence, the combinations of shape generators represent the combined simultaneous effect of the patterns, and their character is nearer to the character of logical connectives than to algebraic operations. The shape generator of result of binary numerical operation with $a,\ b \in \mathbb{R}$ does not depend on the operation but rather on the structural relation between shape generators of input quantities, e. g., if a is "not rather bigger than x_a" *and* b is "approximately x_b" or if there is another relation between them.

Let us formulate the general properties of such *composition of shape generators.*

Definition 1. A point-visely defined non-decreasing commutative binary operation $C : \Phi \times \Phi \longrightarrow \Phi$ is called a *composition of shape generators.* It is called a *homogenous composition of shape generators* iff there is a binary operation $B : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ such that

$$C(\varphi^{(1)}, \varphi^{(2)})\,(x) = B(\varphi^{(1)}(x), \varphi^{(2)}(x)) \quad \text{for all } x \in R. \tag{17}$$

B is called a *basis* for C and the notation $C = C_B$ will be used.

Remark 1. It is easy to see that a mapping $C : \Phi \times \Phi \longrightarrow \Phi$ is a composition of shape generators iff there is a system $\{B_x : x \in R\}$ of commutative non-decreasing binary operations on $[0, 1]$ such that

$$C(\varphi^{(1)}, \varphi^{(2)})\,(x) = B_x(\varphi^{(1)}(x), \varphi^{(2)}(x)) \quad \text{for all } x \in R. \tag{18}$$

Further, C is a homogenous composition of shape generators iff $B_x = B_y$ for any $x,\ y \in R$.

Remark 2. The class Φ of all shape generators is partially ordered, its minimal element is φ_0

$$\varphi_0(0) = 1, \quad \varphi_0(x) = 0, \quad x \neq 0, \tag{19}$$

(Dirac function, see also (15)) and its maximal element is φ_1,

$$\varphi_1(x) = 1 \quad \text{for all } x \in R, \tag{20}$$

and, generally, $\varphi^{(1)} \geq \varphi^{(2)}$ iff $\varphi^{(1)}(x) \geq \varphi^{(2)}(x)$ for all $x \in R$.

Theorem 1. *Let C be an associative homogeneous composition of shape generators with neutral element φ^*, i. e. $C(\varphi, \varphi^*) = \varphi$ for all $\varphi \in \Phi$. Then $\varphi^*(x) = c$ for all $x \neq 0$, where c is some constant from $[0, 1]$. Further, let B be the corresponding basis of C, i. e., $C = C^B$. Then*

(i) if $c = 1$, B is a triangular norm,

(ii) if $c = 0$, B is a triangular conorm,

(iii) if $c \in (0, 1)$, B is a uni-norm (see [7]).

PROOF: The homogenity of C ensures that for each $x \in R$, possibly up to $x = 0$, the element $\varphi^*(x)$ is a neutral element c of the basis B. Now, the basis B is commutative, associative, non-decreasing binary operation on $[0, 1]$ with neutral element c. The result follows. □

Now, we are able to show some results for the compositions of shapes.

Theorem 2. *Let C be an associative composition of shape generators with neutral element φ^*. Let C fulfil the stability property*

$$C(\varphi, \varphi) = \varphi \quad \textit{for all } \varphi \in \Phi. \tag{21}$$

Then

(i) $\varphi^* = \varphi_0$ if and only if C is a homogeneous composition of shape generators with basis $B = \max$ (see also (19), (22)),

(ii) $\varphi^* = \varphi_1$ if and only if C is a homogeneous composition of shape generators with basis $B = \min$ (see also (20), (23)).

PROOF: Following Remarks 1 and 2, let B_x, $x \neq 0$, be the corresponding binary operation in $[0, 1]$ describing C in the point x. Then B_x is a commutative non-decreasing associative independent binary operation which is max if and only if its neutral element $c = 0$, and which is min if and only if $c = 1$. Further, for $x = 0$ the only requirement on B_0 exists, namely, $B_0(1, 1) = 1$ which is fulfilled both, by max and min. The result follows. □

In general, each basis B of a homogenous composition of shape generators $C^B \subset B$ is a commutative aggregation operator on $[0, 1]$, including t-norms, t-conorms, uninorms, averaging operators (arithmetical mean, geometrical mean, e. g.). We can introduce several notions for the basis and, consequently, for the homogenous compositions of shape generators, e. g., the continuity, Archimedian property, cancelativity, etc.

4 Examples of Compositions

The following simple examples of the operation of composition of shape generators are probably the most useful ones for practical applications. In all examples we use the notations of (17).

Maximalisation $C^{\max}$: for all $x \in R$

$$C^{\max}(\varphi^{(1)}, \varphi^{(2)})\,(x) = \max(\varphi^{(1)}(x), \varphi^{(2)}(x)). \tag{22}$$

This composition is commutative, associative and stable (21).

Minimalisation $C^{\min}$: for all $x \in R$

$$C^{\min}(\varphi^{(1)}, \varphi^{(2)})(x) = \min(\varphi^{(1)}(x), \varphi^{(2)}(x)), \tag{23}$$

analogously to the previous example, the properties of commutativity, associativity and stability are fulfilled.

Product C^{prod}: for all $x \in R$

$$C^{\text{prod}}(\varphi^{(1)}, \varphi^{(2)})(x) = \varphi^{(1)}(x) \cdot \varphi^{(2)}(x), \tag{24}$$

which also fulfills commutativity and associativity but not the stability.

Combined Sum C^{sum}: for all $x \in R$

$$C^{\text{sum}}(\varphi^{(1)}, \varphi^{(2)})(x) = \varphi^{(1)}(x) + \varphi^{(2)}(x) - \varphi^{(1)}(x) \cdot \varphi^{(2)}(x) \tag{25}$$

which fulfills commutativity and associativity but not the stability.

Mean C^{mean}: for all $x \in R$

$$C^{\text{mean}}(\varphi^{(1)}, \varphi^{(2)})(x) = (\varphi^{(1)}(x) + \varphi^{(2)}(x))/2 \tag{26}$$

which fulfills commutativity and stability but not associativity.

Of course, further modifications as well as combinations where each of C_L and C_R are defined in different way, can be constructed.

5 Repetitive Compositions

The generators of fuzzy quantities were introduced and investigated to offer a tool for simplification of algebraic processing of fuzzy data. This approach can be effective, namely, if the realized algorithm consists of many arithmetic operations. The properties of generating functions constructed in such situations follow, e. g., from [3] or [4]. Here, we briefly mention some results concerning the composition of the shape generators.

Definition 2. We say that a composition of shape generators $C : \Phi \times \Phi \to \Phi$ is *contractive* if $C \leq C^{\min}$, i. e., for any $x \in R$ and any $\varphi^{(1)}, \varphi^{(2)} \in \Phi$

$$C(\varphi^{(1)}, \varphi^{(2)})(x) \leq \min(\varphi^{(1)}(x), \varphi^{(2)}(x)).$$

Remark 3. Evidently, each C^T with T a t-norm as a basis is contractive, see, e. g., (23), (24). On the other hand, no C^S, S a t-conorm, is contractive, see, e. g. (22), (25).

Remark 4. Evidently, the minimum and product compositions (23), (24) are contractive, meanwhile the maximum and combined sum (22), (25) are not.

The following statements are useful especially for the defuzzification or generally stabilization and concentration of possible values of the outputs of operations over fuzzy numbers.

Theorem 3. *If the composition C is contractive, $\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}, \ldots, \varphi^{(n)}, \ldots$ are shape generators, if $\psi^{(1)}, \psi^{(2)}, \ldots \in \Phi$ are such that*

$$\psi^{(1)} = C(\varphi^{(1)}, \varphi^{(2)}), \quad \psi^{(k)} = C(\psi^{(k-1)}, \varphi^{(k+1)}), \quad k = 2, 3, \ldots$$

then the sequence $\{\psi^{(k)}\}_{k=1}^{\infty}$ is convergent.

PROOF: The statement immediately follows from
the fact that $\{\psi^{(k)}\}_{k=1}^{\infty}$ is monotonous (decreasing) and limited. □

Theorem 4. *If C is contractive minimum composition (23) then, under the assumption of Theorem 1, for any $x \in R$*

$$\lim_{k \to \infty} \psi^{(k)}(x) = \inf(\varphi^{(k)}(x) : k = 1, 2, \ldots).$$

PROOF: The statement follows from Theorem 1 and from (23) immediately. □

Definition 3. We say that $\varphi \in \Phi$ is unimodal iff $\varphi(x) < 1$ for all $x \neq 0$.

Theorem 5. *Let C be contractive product composition (24), and let the notation of Theorem 1 be preserved. If all shape generators $\varphi^{(k)}$, $k = 1, 2, \ldots$, fulfill $\varphi^{(k)} = \varphi$ for some $\varphi \in \Phi$, φ is unimodal and $\psi \in \Phi$ is such that for all $x \in R$*

$$\psi(x) = \lim_{k \to \infty} \psi^{(k)}(x) \tag{27}$$

then $\psi(0) = 1$, $\psi(x) = 0$ for $x \neq 0$.

PROOF: The statement follows from Theorem 3 and from (24) immediately. □

Corollary. *The shape generator $\psi \in \Phi$ defined by (27) in Theorem 5 generates "crisp" fuzzy quantities $\langle x_a \rangle$ for any $x_a \in R$ such that*

$$\mu_{\langle x_a \rangle}(x_a) = 1, \quad \mu_{\langle x_a \rangle}(y) = 0 \quad \text{for } y \neq x_a.$$

The previous corollary reflects a more general fact implicitly following also from Theorems 1 and 2. The contractive composition of shape generators leads to their "narrowing", their support sets do not increase, as well as their functional values. It means that the shape generators which result from the contractive compositions also generate "less vague" fuzzy quantities with more stabilized and concentrated possible values. This conclusion can be, after more detailed investigation, used as an effective tool for defuzzification and also limitation of vagueness in computation with fuzzy data.

6 Conclusive Comments

After having introduced and briefly discussed the exact formalism and some of basic properties of the shape generators, it is possible to recall the motivation of the research summarized above, together with [3] and partly also in [1]. The fuzzy numbers theory has already developed effective tools for particular algebraic operations on fuzzy quantities in the sense that the fuzzy data can be individually processed and their (fuzzy) values can be manipulated. Anyhow, the progress in this branch leads to several methodogical questions.

(a) It is possible to formalize the model of sources of fuzzy data? Namely, to distinguish the differences in the quality (exactness or vagueness) of particular sources and also to model variable quality of any source for different ranges of values.

(b) It is possible to transmit the operations with particular fuzzy numbers (which operations are usually represented by rather complex manipulations with the membership functions) to analogous "global" operations with entire sources? This transmission offers a possibility to realize even complex algorithms only once on the general level of sources, to derive a final "composed" source of results and to compute the resulting fuzzy quantities by means of simple processing of modal values and deriving the corresponding fuzzy component of the result from the "terminal source".

(c) Is it possible to formalize the generation of each particular fuzzy quantity as a fuzzy extension of its modal value where the membership function respects the quality (vagueness) of the source in the neighborhood of the modal value and parallely, specific pattern reflecting the structure of vagueness of the data expressed, e. g., by its verbal description?

(d) What are the general principles of combination and composition of such generators of specific shapes of fuzzy data?

Questions (a) and (b) were treated in [1] and [3] where the concept of generating functions was suggested. Here we have focused our attention to the shape generators (briefly introduced in [3] and dealt also in [4]) to find the answer on (c) and (d) questions.

The wide structure of the model developed in the referred papers is as follows. The generating function $f \in \mathcal{F}$ represent one particular source of fuzzy data (expert opinion, unreliable measurement, uncertain and incomplete notices, etc.), and its gradient is proportional to the exactness of the obtained values. Each numerical datum is formed by a modal value (in certain sense the "crisp core" of the value under consideration) and its fuzzy extension. The shape generators represent basic patterns of vagueness, frequently expressed by verbal phrases. In combination with the properties of the source of data they generate for each crisp modal value the specific shape of the membership functions of each actual fuzzy datum. Processing fuzzy quantities means, on the general level, combination of their generators – the generating functions and shape generators. Their general properties are investigated mainly in [3] and above.

The main purpose of this investigation is to formulate theoretical and sufficiently general model supporting the practical methods of processing fuzzy data. Its formulation is motivated by the will to investigate some questions still existing in the fuzzy quantities theory. For example, the recent theory of fuzzy quantities shows that the extent of uncertainty of fuzzy quantities increases with their algebraic manipulation, with every algebraic operation being used. Hence, after application of a sufficiently complex algorithm the extent of possible values of the output fuzzy quantity is enormously large. Nevertheless, everyday experience with using vague data shows something else – the results are surprisingly stable. Proper choice of the shape generators of the input quantities and of the operation of their combination can, perhaps, be the way to the adequacy of the theoretical model to the practical experience. The suggested model also offers theoretical tools for choosing the shape generators of input quantities (or some of them) to reach a desirable shapes of outputs of some algebraic operations. These problems keep open for further research based on the suggested model.

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