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A Note to the Rationality of Infinite Series I*

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Abstract: The paper deals with the irrationality and rationality of infinite series. These series consist of special rational numbers of Oppenheim type. Several criteria are included too.

1. Introduction

Some problems concerning the irrationality of infinite series $\sum_{n=1}^{\infty} b_n/(\prod_{i=1}^{n} a_i)$, where b_n and a_n are integers we can find in [1]-[9]. Oppenheim [9] proved

Theorem 1.1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of integers such that $a_n > 1$ and*

$$(1) \qquad | b_n | \leq a_n - 1$$

hold for every large n. Let the sharp inequality hold infinitely often in (1) and for every positive integer q there is a positive integer N such that q divides $\prod_{i=1}^{N} a_i$. Then the number $\sum_{n=1}^{\infty} b_n/\prod_{i=1}^{n} a_i$ is rational iff $b_n = 0$ for every large n.

Theorem 1.2. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of integers such that $a_n > 1$ and $| b_n | < a_n$ hold for every large n. Let $\liminf_{n\to\infty}(| b_n | +1)/a_n = 0$. Then the number $\sum_{n=1}^{\infty} b_n/\prod_{i=1}^{n} a_i$ is rational iff $b_n = 0$ for every large n.*

Later Erdös and Strauss [5] proved the following criterion.

Theorem 1.3. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of integers such that $a_n > 1$ and $\lim_{n\to\infty} b_n/(a_{n-1}a_n) = 0$. Then the series $\sum_{n=1}^{\infty} b_n/(\prod_{i=1}^{n} a_i)$ is rational iff there exist a positive integer B and a sequence of integers $\{c_n\}_{n=1}^{\infty}$ such that*

$$Bb_n = c_n a_n - c_{n+1}$$

$$| c_{n+1} | < a_n/2$$

hold for all large n.

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This paper deals with the similar problems. Theorems 2.1 and 2.2 generalize Theorem 1.1. We will describe the relationship between the rationality of infinite series of the above type and the solution of an infinite number of certain equations.

2. Main Theorems

Theorem 2.1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of integers and k be a nonnegativ integer such that $a_n > 2$,*

$$|b_n| < a_n\Big(\prod_{i=n-k}^{n-1} a_i - 1 - 2\sum_{j=2}^{k}\prod_{i=n-k+j-1}^{n-1} a_i\Big)/2, (k \neq 0) \tag{2}$$

$$|b_n| < a_n, (k = 0) \tag{3}$$

hold for every large n. Let

$$\liminf_{n\to\infty}\Big(\frac{|b_n|}{\prod_{i=n-k}^{n} a_i} + \frac{1}{a_{n-k}}\Big) < \frac{1}{3^k} \tag{4}$$

and for every positive integer q there is a positive integer N such that q divides $\prod_{i=1}^{N} a_i$. Then the series $x = \sum_{n=1}^{\infty} b_n/\prod_{i=1}^{n} a_i$ is rational iff there exist sequences $\{c_{n,m}\}_{n=1}^{\infty}$ $m = 0, 1, ..., k$ of integers satisfying

$$b_n = \sum_{m=0}^{k} c_{n,m} \prod_{i=n-m+1}^{n} a_i \tag{5}$$

$$\sum_{m=0}^{k} c_{n+m,m} = 0 \tag{6}$$

$$|c_{n,m}| \leq a_{n-m} \tag{7}$$

$$|c_{n,0}| \leq \frac{a_n}{2} \tag{8}$$

for all large n.

Consequence 1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of integers and k be a nonnegative integer such that $a_n > 2$ and $|b_n| < (\prod_{i=n-k}^{n}(a_i - 2))/2$ hold for every large n. Let*

$$\liminf_{n\to\infty}\Big(\frac{|b_n|}{\prod_{i=n-k}^{n} a_i} + \frac{1}{a_{n-k}}\Big) < \frac{1}{3^k} \tag{9}$$

and for every positive integer q there is a positive integer N such that q divides $\prod_{i=1}^{N} a_i$. Then the series $x = \sum_{n=1}^{\infty} b_n/\prod_{i=1}^{n} a_i$ is rational iff there exist sequences $\{c_{n,m}\}_{n=1}^{\infty}$ $m = 0, 1, ..., k$ of integers satisfying (5)-(8) for all large n.

Theorem 2.2. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of integers and k be a nonnegative integer such that $a_n > 4$,*

$$(10)\qquad | b_n |< a_n((\prod_{i=n-k}^{n-1} a_i - 4\prod_{i=n-k+1}^{n-1} a_i - 1 - 2\sum_{j=3}^{k}\prod_{i=n-k+j-1}^{n-1} a_i))/2, (k > 1)$$

$$(11)\qquad | b_n |< a_n(a_{n-1} - 3)/2, (k = 1)$$

$$(12)\qquad | b_n |< a_n - 1, (k = 0)$$

hold for every large n and for every positive integer q there is a positive integer N such that q divides $\prod_{i=1}^{N} a_i$. Then the series $x = \sum_{n=1}^{\infty} b_n/\prod_{i=1}^{n} a_i$ is rational iff there exist sequences $\{c_{n,m}\}_{n=1}^{\infty}$ $m = 0, 1, ..., k$ of integers satisfying (5)-(8) for all large n.

Consequence 2. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of integers and k be a nonnegative integer such that $a_n > 4$ and $| b_n |< (\prod_{i=n-k}^{n}(a_i - 4))/2$ hold for every large n and for every positive integer q there is a positive integer N such that q divides $\prod_{i=1}^{N} a_i$. Then the series $x = \sum_{n=1}^{\infty} b_n/\prod_{i=1}^{n} a_i$ is rational iff there exist sequences $\{c_{n,m}\}_{n=1}^{\infty}$ $m = 0, 1, ..., k$ of integers satisfying (5)-(8) for all large n.*

Theorem 2.3. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of integers such that $a_n > 2$ holds for every n, k be a nonnegative integer and*

$$(13)\qquad \limsup_{n\to\infty} \frac{| b_n |}{\prod_{i=n-k}^{n} a_i} < \frac{1}{3^k}.$$

Suppose that for every positive integer q there is a positive integer N such that q divides $\prod_{i=1}^{N} a_i$. Then the series $x = \sum_{n=1}^{\infty} b_n/\prod_{i=1}^{n} a_i$ is rational iff there exist sequences $\{c_{n,m}\}_{n=1}^{\infty}$ $m = 0, 1, ..., k$ of integers satisfying (5)-(8) for every large n.

3. Proofs of Main Theorems

Proof of Theorem 2.1,2.2 and 2.3: We will prove Theorem 2.1,2.2 and 2.3 together. These proofs consist of two parts.

1.Sufficient condition. Let us suppose (5)-(8) hold for all $n \geq N$, where N is sufficiently large. Then we have

$$\sum_{n=1}^{\infty} \frac{b_n}{\prod_{i=1}^{n} a_i} = \sum_{n=N}^{\infty} \frac{b_n}{\prod_{i=1}^{n} a_i} + P_1 = \sum_{n=N}^{\infty} \frac{(\sum_{m=0}^{k} c_{n,m} \prod_{i=n-m+1}^{n} a_i)}{\prod_{i=1}^{n} a_i} + P_1 =$$

$$= \sum_{n=N}^{\infty}\sum_{m=0}^{k} c_{n,m}\frac{1}{\prod_{i=1}^{n-m} a_i} + P_1 = \sum_{s=N}^{\infty}\frac{(\sum_{m=0}^{k} c_{s+m,m})}{\prod_{i=1}^{s} a_i} + P_2 = P_2,$$

where P_1 and P_2 are rational numbers.

2.Necessary condition. If $b_n = 0$ for all large n, then we put $c_{n,m} = 0$ and the proof is finished. If not, then let us suppose that $x = y/z = \sum_{n=1}^{\infty} b_n/\prod_{i=1}^{n} a_n$, where $z > 0$, y are integers. We find $c_{n,m}$ satisfying (5)-(8) by mathematical induction with respect to k.

A) Let $k = 0$. Then Theorem 1.1 implies that $b_n = 0$ for all large n in Theorem 2.1,2.2 and 2.3.

B) Let us assume that Theorem 2.1, 2.2 and 2.3 hold with $k-1$ instead by k. If $b_n = 0$ for all large n, then we put $c_{n,m} = 0$ and the proof is finished. If not, then let us define $c_{n,m}$ such that

$$b_n = c_{n,0} + d_n a_n,$$

where $c_{n,0}$ and d_n are integers such that

$$| c_{n,0} | \leq a_n/2. \tag{14}$$

Now we can write

$$x = \sum_{n=1}^{\infty}\frac{b_n}{\prod_{i=1}^{n} a_i} = d_1 + \sum_{n=1}^{\infty}\frac{c_{n,0} + d_{n+1}}{\prod_{i=1}^{n} a_i}.$$

We will prove that the sequence $\{b_n\}_{n=1}^{\infty}$ replaced by $\{c_{n,0} + d_{n+1}\}_{n=1}^{\infty}$ satisfies the induction assumptions with k replaced by $k-1$ for all Theorems. From (14) we have

$$| c_{n,0} + d_{n+1} | \leq | c_{n,0} | + \frac{| c_{n+1,0} |}{a_{n+1}} + \frac{| b_{n+1} |}{a_{n+1}} \leq \frac{a_n + 1}{2} + \frac{| b_{n+1} |}{a_{n+1}}. \tag{15}$$

Theorem 2.1: From (4), (15) and $a_n > 2$ we obtain

$$\liminf_{n\to\infty}\frac{| c_{n,0} + d_{n+1} | + \prod_{i=n-k+2}^{n} a_i}{\prod_{i=n-k+1}^{n} a_i} \leq$$

$$\liminf_{n\to\infty}\frac{a_n a_{n+1}/2 + a_{n+1}/2 + | b_{n+1} | + \prod_{i=n-k+2}^{n+1} a_i}{\prod_{i=n-k+1}^{n+1} a_i} <$$

$$\frac{1}{2.3^{k-1}} + \frac{1}{2.3^{k}} + \frac{1}{3^{k}} = \frac{1}{3^{k-1}}.$$

If $k > 1$ then (2) and (15) imply

$$| c_{n,0} + d_{n+1} | \leq (a_n + 1)/2 + | b_{n+1} | /a_{n+1} <$$

$$(a_n+1)/2+(\prod_{i=n-k+1}^{n} a_i - 1 - 2\sum_{j=2}^{k}\prod_{i=n-k+j}^{n} a_i)/2 =$$

$$a_n(\prod_{i=n-k+1}^{n-1} a_i - 1 - 2\sum_{j=2}^{k-1}\prod_{i=n-k+j}^{n-1} a_i)/2.$$

Similary if $k = 1$ then (2) and (15) imply

$$| c_{n,0} + d_{n+1} | \leq (a_n+1)/2 + | b_{n+1} | /a_{n+1} < (a_n+1)/2 + (a_n-1)/2 = a_n.$$

Theorem 2.2: If $k > 2$ then (10) and (15) imply

$$| c_{n,0} + d_{n+1} | < (a_n+1)/2 + | b_{n+1} | /a_{n+1} <$$

$$a_n/2 + 1/2 + (\prod_{i=n-k+1}^{n} a_i - 4\prod_{i=n-k+2}^{n} a_i - 1 - 2\sum_{j=3}^{k}\prod_{i=n-k+j}^{n} a_i)/2 =$$

$$a_n(\prod_{i=n-k+1}^{n-1} a_i - 4\prod_{i=n-k+2}^{n-1} a_i - 1 - 2\sum_{j=3}^{k-1}\prod_{i=n-k+j}^{n-1} a_i)/2.$$

If $k = 2$ then (10) and (15) imply

$$| c_{n,0} + d_{n+1} | < (a_n+1)/2 + | b_{n+1} | /a_{n+1} <$$

$$(a_n+1)/2 + (a_n a_{n-1} - 4a_n - 1)/2 = a_n(a_{n-1}-3)/2.$$

If $k = 1$ then (11) and (15) imply

$$| c_{n,0} + d_{n+1} | < (a_n+1)/2 + | b_{n+1} | /a_{n+1} < (a_n+1)/2 + (a_n-3)/2 = a_n - 1.$$

Theorem 2.3: From (13), (15) and $a_n > 2$ we receive

$$\limsup_{n\to\infty} \frac{| c_{n,0} + d_{n+1} |}{\prod_{i=n-k+1}^{n} a_i} \leq \frac{1}{2.3^{k-1}} + \frac{1}{2.3^{k}} + \frac{1}{3^k} = \frac{1}{3^{k-1}}.$$

We proved (2)-(4), (9)-(13) for b_n replaced by $c_{n,0} + d_{n+1}$ and for k replaced by $k-1$. Using the induction hypothesis there exists sequences $\{q_{n,m}\}_{n=1}^{\infty}$ $(m = 0, 1, ..., k-1)$ of integers such that

$$c_{n,0} + d_{n+1} = \sum_{m=0}^{k-1} q_{n,m} \prod_{i=n-m+1}^{n} a_i \tag{16}$$

$$\sum_{m=0}^{k-1} q_{n+m,m} = 0 \tag{17}$$

(18) $$| q_{n,m} | \leq a_{n-m}$$

(19) $$| q_{n,0} | \leq a_n/2$$

hold for all large n. From (16) we have

(20) $$\begin{aligned} b_{n+1} &= c_{n+1,0} + a_{n+1}d_{n+1} \\ &= c_{n+1,0} + (q_{n,0} - c_{n,0})a_{n+1} + \sum_{m=1}^{k-1} q_{n,m} \prod_{i=n-m+1}^{n+1} a_i. \end{aligned}$$

Let us put

(21) $$c_{n+1,1} = q_{n,0} - c_{n,0}$$

(22) $$c_{n+1,m} = q_{n,m-1}, (m = 2, 3..., k).$$

To finish the proof we will prove (5)-(8) for sequences $\{c_{n,m}\}_{n=1}^{\infty}$ $(m = 0, 1, ..., k)$ for all large n. From (14) we obtain (8). From (14) and (19) we receive

$$| c_{n+1,1} |=| q_{n,0} - c_{n,0} |\leq a_n$$

and this is (7) for $m = 1$. From (22) and (18) we have (7) for $m = 2, 3, ..., k$. From (17), (20), (21) and (22) we obtain

$$\sum_{m=0}^{k} c_{n+1+m,m} = c_{n+1,0} + (q_{n+1,0} - c_{n+1,0}) + \sum_{m=2}^{k} q_{n+m,m-1} = 0.$$

Thus (6) holds. Finally from (20), (21) and (22) we receive (5) and proofs of Theorems 2.1,2.2 and 2.3 are finished.

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