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Extremal problems about additive bases

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Abstract: Let $A \subset \mathbb{N}_0$ be an asymptotic basis of order 3. (That is, every sufficiently large integer is sum of three integers belonging to A.) We prove that for almost all (all but finitely many) a belonging to A, the set $B = A \setminus \{a\}$ is an asymptotic basis of order at most 6. Several open problems are discussed.

Résumé: Soit A un ensemble d' entiers naturels. Nous supposons que A est une base asymptotique d' ordre 3, c' est-à-dire que tout entier suffisamment grand est somme de trois entiers appartenant à A. Nous démontrons que, à un nombre fini d' exceptions près, tout élément a de A est tel que l' ensemble $B = A \setminus \{a\}$ soit une base asymptotique d' ordre inférieur ou égal à 6. L' article se termine par quelques problèmes ouverts.

Key Words: Asymptotic additive bases.

Mathematics Subject Classification: 11B13

Definitions and notations.

All sets $A, B, \ldots$ in this article, are strictly increasing sequences of integers. Thus a set

$$A = \{a_0 < a_1 < \ldots\}$$

may contain a finite number of negative terms.

The (equivalence) relation $A \sim B$ means that the symetric difference $A \triangle B$ is finite.

Let h be an integer, $h \geq 2$. We note hA the set of all sums of h, not necessarily distinct, elements of A. That is,

$$hA = \{x_1 + \ldots + x_h : x_j \in A, 1 \leq j \leq h\}.$$

If there is an integer h such that $hA \sim \mathbb{N}$, then A is called a **basis.** In other terms we call here "(additive) basis" what is usually called "asymptotic basis" in the literature. The smallest integer k such that $kA \sim \mathbb{N}$ is called the **order** $G(A)$ of the basis A.

We discuss questions related to the following problem.

Problem:. Let A be a set such that $hA \sim \mathbb{N}$ and let $a \in A$. Put $B = A \setminus \{a\}$. Is B a basis? And, if this is the case, what can be said about its order $G(B)$?

The following Lemma is very useful.

Lemma. *Let k be a positive integer, n an integer, X a set of integers and F a subset of X. Then*

$$k(X+n) = kX + kn$$

and

$$k\big((X+n) \setminus (F+n)\big) = k(X \setminus F) + kn.$$

Furthemore, we have that $kX \sim \mathbb{N}$ if and only if $k(X+n) \sim \mathbb{N}$. Also, $(X+n) \setminus (F+n)$ is a basis of order k if and only if $X \setminus F$ is a basis of order k.

The proof is straightforward.

A result for $h = 3$.

In this section we prove a result for $h = 3$. The corresponding problem for $h = 2$ was completely solved in [2]. The reader can find a more general context for this kind of problems in section 3.

Theorem. *Let A be a set such that $3A \sim \mathbb{N}$. Then, for all sufficiently large a belonging to A, the set $B = A \setminus \{a\}$ verifies $6B \sim \mathbb{N}$; that is, B is a basis and $G(B) \leq 6$.*

Remark:. Note that $G(A) \leq 3$. If $G(A) = 2$ we already know [2] that $G(B) \leq 3$ for all a large enough. So the theorem proved here is useful when $G(A) = 3$. In any case, this theorem gives $G(A) \leq G(B) \leq 6$ for almost all a belonging to A.

Proof of the theorem.

STEP 1. In view of the Lemma in section 1, it is sufficient to prove the theorem under the assumption that $a_0 = 0$. We assume in the sequel that

$$A = \{0 = a_0 < a_1 < a_2 < \dots\} \subset \mathbb{N}_0.$$

As $a \in A$ will be large enough, the set $B = A \setminus \{a\}$ contains 0 and satisfies $kB \subset (k+1)B$ for any k.

STEP 2. We note n_0 an integer greater or equal to 1, such that any integer $n \geq n_0$ belongs to $3A$. We also put $r_1 = \min(\mathbb{N} \setminus A)$. We shall prove that, for all $a \in A$,

$$a \geq 3(n_0 + r_1) + 4, \tag{1}$$

the set $B = A \setminus \{a\}$ satisfies $6B \sim \mathbb{N}$.

STEP 3. Note that

$$3A = 3B \cup (a + 2B) \cup (2a + B) \cup \{3a\} \tag{2}$$

and $3A \sim \mathbb{N}$ so that

$$3B \cup (a + 2B) \cup (2a + B) \sim \mathbb{N}. \tag{3}$$

Also $3B \subset 6B$. It is a consequence of (1) that $a - a_1 \geq n_0$. Therefore

$$a - a_1 = x_1 + x_2 + x_3\,,\ x_i \in A\,,\ i = 1,2,3\,.$$

But no x_i can be equal to a. So $x_i \in B$, $i = 1,2,3$,

$$a \in 4B \tag{4}$$

and $a + 2B \subset 6B$. In view of these remarks and of (3), in order to prove that $6B \sim \mathbb{N}$, it suffices to show that *every sufficiently large* n *such that* $n = 2a + b$, $b \in B$, *belongs to* $6B$. This is true if $2a \in 5B$. So we assume in the sequel that

$$2a \notin 5B. \tag{5}$$

STEP 4. The condition (5) implies that

$$[a + 1, 2a - n_0] \cap 2B = \emptyset. \tag{6}$$

For suppose that this set were not empty. Let x be an element of it. We would have $a < x \leq 2a - n_0$, so that $n_0 \leq 2a - x < a$, which would give that $2a - x \in 3B$ and

$$2a = (2a - x) + x \in 3B + 2B = 5B.$$

Another consequence is that

$$[\frac{a}{2}\,,\, a - \frac{n_0}{2}] \cap B = \emptyset. \tag{7}$$

For if $\frac{a}{2} \in B$, then $2a = 4\frac{a}{2} \in 4B \subset 5B$. If $\frac{a}{2} < x \leq a - \frac{n_0}{2}$ and $x \in B$, then $2x$ would belong to the set in (6).

STEP 5. There is an integer

$$z_1 \in B \cap [\,\frac{a-1}{3}\,,\, \frac{a}{2}\,[.$$

The reason is that $a - 1$ belongs to $3B$, and so $a - 1 = z_1 + z_2 + z_3$ with, say, $z_1 \geq z_2 \geq z_3$ and $z_i \in B$, $i = 1,2,3$.

STEP 6. We have that

$$]a - \frac{n_0}{2}\,,\, a - 1] \cap B = \emptyset. \tag{9}$$

Because if x was an element of this set, then $x + z_1$ would belong to $2B$ and verify

$$a - \frac{n_0}{2} + \frac{a-1}{3} < x + z_1 < a - 1 + \frac{a}{2}.$$

Easy consequences of (1) are the inequalities

$$a - 1 + \frac{a}{2} \leq 2a - n_0$$

and

$$a+1 \leq a-\frac{n_0}{2}+\frac{a-1}{3}.$$

Thus $x+z_1$ would belong to

$$2B \cap [a+1, 2a-n_0].$$

Or by step 4, this set is empty.

STEP 7. We conclude that

$$B \cap [\frac{a}{2}\,,\, 2a-n_0] = \emptyset. \tag{10}$$

This is a consequence of (7), (9), (6) and the observation that B is a subset of $2B$.

STEP 8. We shall prove that each integer $n \geq 6a$ of the form $n = 2a+b$, $b \in B$, with $2a \notin 5B$, belongs to $6B$. Since $[n_0, a-1] \subset 3B$, it suffices to prove that

$$[n-a+1, n-n_0] \cap 3B \neq \emptyset. \tag{11}$$

For if x is an integer in the above set, then $n = x+y$ where $y = n-x \in [n_0, a-1] \subset 3B$ and therefore $n \in 3B+3B = 6B$.

STEP 9. Let $n_1 = n - \frac{a-1}{3} + 1$. We easily verify, using (1), that $n_0 \leq 3a < n_1 \leq n-n_0$. Each integer m in $[n_1, n-n_0]$ satisfies at least one of the following conditions:

- $m \in 3B$,
- $m \in 2B+a$,
- $m \in B+2a$.

If $m \in 3B$, then we write $n = m+y$ and as $n_1 \leq m \leq n-n_0$ we verify that $y = n-m \geq n-(n-n_0) = n_0$ and $y = n-m \leq n-n_1 = \frac{a-1}{3} - 1 < a$. Therefore $y \in 3B$ and $n = m+y \in 3B+3B = 6B$.

If $m \in 2B+a$, then $m-a \in 2B$ and by step 5, $m-a+z_1 \in 2B+B = 3B$. We also have

$$n-a+1 = n_1-a+\frac{a-1}{3} \leq m-a+z_1 < n-n_0-a+\frac{a}{2} < n-n_0,$$

so that (11) is true. It remains the case where all m in $[n_1, n-n_0]$ are of the form $2a+b$, $b \in B$. *We suppose that this is true* in the next and final step.

STEP 10. We recall that $r_1 = \min(\mathbb{N} \backslash A) \notin B$. We remark that $2a+r_1$ belongs to $3A$, it is not equal to $3a$ nor is of the form $2a+b$ with $b \in B$. By relation (2), $2a+r_1$ is either of the form $b_1+b_2+b_3$ or of the form $a+b_1+b_2$ with $b_i \in B$, $i = 1,2,3$. None of the two relations $2a+r_1 = b_1+b_2+b_3$, $2a+r_1 = a+b_1+b_2$ is possible with all the b_i's, $i = 1,2,3$ or $i = 1,2$, strictly less than $\frac{a}{2}$. In view of the above facts and the relation (10), there is an integer $b_i \in B \cap [2a-n_0+1, 2a+r_1]$. Let $m_1 = n-n_0-r_1$. We verify that $n_1 \leq m_1 \leq n-n_0$: the non trivial inequality is equivalent to (1). As a consequence of the hypothesis formulated at the end of the ninth step, we get that $m_1 - 2a \in B$. Thus the number $m_1 - 2a + b_i$ belongs to $2B \subset 3B$ and verify

$$m_1 - 2a + b_i \leq n - n_0 - r_1 - 2a + 2a + r_1 = n - n_0$$

and

$$m_1 - 2a + b_i \geq n - n_0 - r_1 - 2a + 2a - n_0 + 1 = n - 2n_0 - r_1 + 1 \geq n - a + 1,$$

the latter inequality being a consequence of (1). Thus relation (11) is also valid in the this last case. The theorem is proved.

A more general context.

Concerning the question raised in section 1, it is known ([3],[4],[6]) that if A verifies $hA \sim \mathbb{N}$, then, except for at most $h-1$ elements a belonging to A, the set $B = A \setminus \{a\}$ is a basis and

$$G(B) \leq \frac{h^2 + 3h}{2}.$$

When $h = 3$, Nash [6] proved also that, if $B = A \setminus \{a\}$ is a basis, then $G(B) \leq 7$.

Let

$$\mathrm{x}(A) = \max_{a \in A^*} G(A \setminus \{a\}), \tag{12}$$

where A^* is the set of $a \in A$ such that $A \setminus \{a\}$ is a basis. Let also

$$\mathbf{X}(h) = \max_{A:\, hA \sim \mathbb{N}} \mathrm{x}(A).$$

It was proved in [1] and [6] that $\mathbf{X}(2) = 4$ and $\mathbf{X}(3) = 7$, respectively. I also proved ([3],[4]) that

$$\frac{h^2}{3} - h \leq \mathbf{X}(h), \tag{13}$$

for any $h \geq 2$. As stated above, in [6] Nash proved that for any $h \geq 2$,

$$\mathbf{X}(h) \leq \frac{h^2 + 3h}{2}.$$

The problem of determining the order of magnitude of $\mathbf{X}(h)$ remains open.

Sequences used to prove inequality (13) are unions of arithmetic sequences of the same difference, to which some isolated integers are added (in the sense of union). For example, the sequence $A = \{3\} \cup \{8n : n \in \mathbb{N}\} \cup \{8n+1 : n \in \mathbb{N}\}$ verifies $G(A) = 3$ and, if $B = A \setminus \{3\}$, then $G(B) = 7$.

Thus it seems that the difference $G(B) - G(A)$, which is positive, may be large only for few elements a belonging to A. We ask the following question: What happens if we accept a finite number of exceptions? In more precise terms, this consists in taking in relation (12) the limit superior instead of the maximum. Let

$$\mathrm{s}(A) = \limsup_{a \in A^*} G(A \setminus \{a\}), \tag{14}$$

and

$$\mathrm{S}(h) = \max_{A:\, hA \sim \mathbb{N}} \mathrm{s}(A).$$

Problem:. Evaluate $\mathrm{S}(h)$. Is it true that $\mathrm{S}(h) < \mathbf{X}(h)$ for all h?

In this paper I proved that $\mathbf{S}(3) \leq 6$ while $\mathbf{X}(3) = 7$, and in [2] I proved that $\mathrm{S}(2) = 3$ while it was known [1] that $\mathbf{X}(2) = 4$. In general, it is only known that

$$h + 1 \leq \mathrm{S}(h) \leq \mathbf{X}(h).$$

The first inequality is due to E.Härtter [5] (see also [7]).

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