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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 9 (2001), No. 1, 33--38

Persistent URL: http://dml.cz/dmlcz/120567

A note on linear preserves of ternary cubic forms

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Abstract: We describe the structure of the group of linear automorphisms of any ternary cubic form.

Key Words: ternary cubic form; automorphism of a form; linear preserver

Mathematics Subject Classification: Primary 11E76; Secondary 14H52.

1. Introduction

Let $F = F(X_1, \ldots, X_n)$ be a form of degree r in n variables over a field K. We say that $\sigma \in \mathrm{Aut}(K^n)$ is *an automorphism of* F (or linear preserver of F) if

$$F(\sigma(X_1, \ldots, X_n)) = F(X_1, \ldots, X_n).$$

The set $\mathrm{Aut}_K F$ of all automorphisms of F forms a group, which we call the automorphism group of F. It is known that $\mathrm{Aut}_K F$ is finite when F is nonsingular and $r \geq 3$ (see [6]). So it is natural to ask after Suzuki [11] about the structure or at least the order of $\mathrm{Aut}_K F$. Many authors have examined this problem. Unfortunately there are no universal methods of solving it, so the authors usually confine their considerations to certain families of forms. O'Ryan and Shapiro [5] explored trace forms of degree d of central simple algebras over fields of characteristic not dividing $d!$, Guralnick [3] and Shapiro [8] described linear preservers of trace forms of matrix algebras over fields of positive characteristic, the authors of this paper (together with Sładek) observed forms over ordered fields, Wesołowski [12] found linear preservers of trace forms of etalé algebras, whereas Berchenko and Olver [1] were mainly interested in binaries over $\mathbb{R}$ and $\mathbb{C}$.

In the literature one can find many results on isomorphisms of various kinds of algebraic sets. Any form determines in a natural way a projective algebraic set and any linear preserver of this form determines an automorphism of that projective algebraic set. It suggests an additional way of examination of linear preserves. In the paper we apply the results on automorphism groups of elliptic curves to obtain the structure of groups of linear preserves of ternary cubics.

In the sequel we assume that *all fields are of characteristic 0* (though some of our results are true for fields of characteristic $\neq 2,3$). Since we have the natural injection $\mathrm{Aut}_K F \hookrightarrow \mathrm{Aut}_{\overline{K}} F \otimes \overline{K}$, we also assume that K is algebraically closed.

In [4] Orlik and Solomon cite the following result which is due to Bott and Tate: *For a given nonsingular form F of degree $r \geq 3$ in n variables over K the order of the group $Aut_K F$ is bounded by a function of r and n.*

For fixed r and n let us define

$$g(r,n) := \max\{|\mathrm{Aut}_K F|;\ K -\ \text{field},\ F \in F_{n,r}(K),\ F - \text{nonsingular}\},$$

where $F_{n,r}(K)$ denotes the space of all forms of degree r in n variables over a field K. By the result of Bott and Tate we have $g(r,n) < \infty$ for $r \geq 3, n \in \mathbb{N}$. Berchenko and Olver [1] describe automorphisms of binary forms. From their paper it follows that $g(r,2) = (6r-12)r$. Our investigation gives $g(3,3) = 162$.

2. Some information about elliptic curves

Suppose $F = F(X,Y,Z)$ is a nonsingular ternary cubic form. Then

$$E_F(K) := \{(X:Y:Z) \in \mathbb{P}^2 : F(X,Y,Z) = 0\}$$

is a projective elliptic curve which with an appriopate operation + and distinguished element O is an abelian group. By the Nagell's algorithm the equation of the above curve can be reduced to the Weierstrass form

$$E(a,b) : Y^2 Z = X^3 + aXZ^2 + bZ^3.$$

Elliptic curves over an algebraically closed field are classified by so called j-invariant. For $E(a,b)$ we have

$$j = \frac{1728(4a^3)}{4a^3 + 27b^2}.$$

In the literature the group $\mathrm{Aut}_K E_F$ of automorphisms of an elliptic curve is well described. Since we want to find the connection between $\mathrm{Aut}_K E_F$ and $\mathrm{Aut}_K F$ it make sense to recall necessary information on $\mathrm{Aut}_K E_F$.

Theorem 1. (see [8] p. 103) *Let $\mathrm{Aut}_K(E_F, O)$ be the group of automorphisms of an elliptic curve E_F fixing the zero element (that is, the group of all isogenies). Then*

$$Aut_K(E_F, O) \cong \begin{cases} \mu_2(K), & \text{if } j(E_F) \neq 0, 1728 \\ \mu_6(K), & \text{if } j(E_F) = 0 \\ \mu_4(K), & \text{if } j(E_F) = 1728 \end{cases}$$

where $\mu_n(K)$ stands for the group of n-th roots of unity.

Let $t_Q : E_F(K) \to E_F(K)$ denote the translation map: $t_Q(P) = P + Q$ for $P, Q \in E_F(K)$ and let $\bar{E}_F(K) := \{t_Q \mid Q \in E_F(K)\}$. Of course, the group $\bar{E}_F(K)$ is isomorphic with $E_F(K)$ and actually

$$\mathrm{Aut}_K(E_F) = \bar{E}_F(K) \rtimes \mathrm{Aut}_K(E_F, O)$$

with the natural action of $\mathrm{Aut}_K(E_F, O)$ on $\bar{E}_F(K)$ (see [8] p. 75). Let $\mathrm{Aut}^0_K(E_F)$ and $\bar{E}^0_F(K)$ denote the subgroups of $\mathrm{Aut}_K(E_F)$ and $\bar{E}_F(K)$, respectively, consisting of all automorphisms which map the set of points of inflections to itself. Then

$$\mathrm{Aut}^0_K(E_F) = \bar{E}^0_F(K) \rtimes \mathrm{Aut}_K(E_F, O).$$

Since an elliptic curve (over algebraically closed field) has nine points of inflection (together with O) and they form a subgroup of $E_F(K)$ isomorphic with $\mu_3(K)^2$ we get the following.

Proposition 2. *If F is a nonsingular ternary cubic form then*

$$Aut^0_K(E_F) = \mu_3(K)^2 \rtimes Aut_K(E_F, O).$$

Thus

$$Aut^0_K(E_F) \cong \begin{cases} \mu_3(K)^2 \rtimes \mu_2(K)\,, & \text{if } j(E_F) \neq 0, 1728 \\ \mu_3(K)^2 \rtimes \mu_6(K)\,, & \text{if } j(E_F) = 0 \\ \mu_3(K)^2 \rtimes \mu_4(K)\,, & \text{if } j(E_F) = 1728 \end{cases}$$

Corollary 3.

$$|Aut^0_K(E_F)| = \begin{cases} 18\,, & \text{if } j(E_F) \neq 0, 1728 \\ 54\,, & \text{if } j(E_F) = 0 \\ 36\,, & \text{if } j(E_F) = 1728 \end{cases}.$$

3. The structure of the automorphism group

Looking for the automorphism group of the fixed form F it is enough to consider instead of F any form isomorphic with F. We begin this section with the theorem due to H. Weber which will turn out to be very useful for us.

Theorem 4. (see [13] p. 401) *Let $F(X, Y, Z)$ be a nonsingular cubic form over an algebraically closed field K. Then F is equivalent to the form*

$$F_d\,(X, Y, Z) = X^3 + Y^3 + Z^3 - 3dXYZ, \quad d \in K$$

by a linear change of variables.

Remark 5. F_d is nonsingular if and only if $d^3 \neq 1$.

By the above we can confine our considerations to forms F_d with $d^3 \neq 1$.

Theorem 6. *If $d^3 \neq 1$ then the group $Aut_K F_d$ is a central extension of $\mu_3(K)$ by the group $Aut^0_K(E_{F_d})$.*

Proof. For any $\sigma \in \mathrm{Aut}_K F_d$ we can write

$$\sigma(X, Y, Z) = (\sigma_1(X, Y, Z), \sigma_2(X, Y, Z), \sigma_3(X, Y, Z))$$

where σ_i is a linear transformation for $i = 1, 2, 3$.
Let $\Phi : \mathrm{Aut}_K F_d \to \mathrm{Aut}_K E_{F_d}$ be the map sending σ to $\Phi(\sigma)$ defined as follows

$$\Phi(\sigma)(X : Y : Z) = (\sigma_1(X, Y, Z) : \sigma_2(X, Y, Z) : \sigma_3(X, Y, Z)).$$

Obviously Φ is a homomorphism with $\ker \Phi = \mu_3(K) \cdot \mathrm{Id}_{K^3}$ contained in the center of $\mathrm{Aut}_K F_d$. Since for $\sigma \in \mathrm{Aut}_K F_d$ the automorphism $\Phi(\sigma)$ maps the set of points of inflection to itself (preserves both zeros of the form and zeros of the Hessian) we have $\mathrm{Im}\Phi \subseteq Aut^0_K(E_{F_d})$. It left to show that Φ is an epimorphism or equivalently that $|Aut_K F_d| \geq 3|Aut^0_K(E_{F_d})|$. We can easily show the following 54 automorphisms of F_d:

$$\sigma(X_1, X_2, X_3) = (aX_{\tau(1)}, bX_{\tau(2)}, (ab)^{-1}X_{\tau(3)})$$

where $a, b \in \mu_3(K)$, $\tau \in S(3)$, which form a subgroup of $\mathrm{Aut}_K F_d$ isomorphic with $\mu_3(K)^2 \rtimes S(3)$. Thus $|Aut_K F| \geq 54$. From the Nagell's algorithm it follows that $E_{F_d} \cong E(a, b)$ where $a = -27(d^3 + 8)d$, $b = 54(d^6 - 20d^3 - 8)$. Thus j-invariant depends on d in the following way:

(1) $j(E_{F_d}) \neq 0, 1728 \quad \Leftrightarrow \quad d \neq 0, -2\epsilon^k, (1 \pm \sqrt{3})\epsilon^k$
(2) $j(E_{F_d}) = 1728 \quad \Leftrightarrow \quad d = (1 \pm \sqrt{3})\epsilon^k$
(3) $j(E_{F_d}) = 0 \quad \Leftrightarrow \quad d = 0$ or $d = -2\epsilon^k$

where ϵ denotes a primitive third root of unity and k = 0,1,2.
Because of isomorphism $F_d \cong F_{d\epsilon^k}$ it is sufficient to consider the following cases: $d = 0$, $d = -2$, $d = 1 \pm \sqrt{3}$.

Case (1) $d \neq 0, -2, 1 \pm \sqrt{3}$
Since in this case $3|Aut^0_K(E_{F_d})| = 54$ we are done.

Case (2) $d = 1 \pm \sqrt{3}$
We have to show that $|\mathrm{Aut}_K F_d| \geq 108$. To do this it is sufficient to indicate at least one automorphism of F_d different from those 54 listed above. It is easy to check that

$$\sigma(X, Y, Z) = (\frac{\sqrt{3}}{3}(X + Y + Z), \frac{\sqrt{3}}{3}(X + \epsilon Y + \epsilon^2 Z), \frac{\sqrt{3}}{3}(X + \epsilon^2 Y + \epsilon Z))$$

is an automorphism of the form $F_{1+\sqrt{3}}$.
In the case $d = 1 - \sqrt{3}$ we can take

$$\sigma(X, Y, Z) = -(\frac{\sqrt{3}}{3}(X + Y + Z), \frac{\sqrt{3}}{3}(X + \epsilon Y + \epsilon^2 Z), \frac{\sqrt{3}}{3}(X + \epsilon^2 Y + \epsilon Z)).$$

Case (3a) $d = 0$
For $d = 0$ we have $F_0(X, Y, Z) = X^3 + Y^3 + Z^3$ and we need 162 automorphisms. Those are

$$\sigma(X_1, X_2, X_3) = (aX_{\tau(1)}, bX_{\tau(2)}, cX_{\tau(3)})$$

where $a, b, c \in \mu_3(K)$, $\tau \in S(3)$.

Case (3b) $d = -2$

For $d = -2$ we have $F_{-2}(X, Y, Z) = X^3 + Y^3 + Z^3 + 6XYZ$. One can check that F_{-2} is the trace form Tr_A^3 of K-algebra $A = K[X]/(X^3 - 1)$, where

$$Tr_A^r : A \to K, \; Tr_A^r(a) := Tr_A(a^r).$$

Wesołowski [12] Theorem 3.3 describes the automorphism group of the trace form Tr_A^r. Since $A \cong K^3$ we obtain

$$\mathrm{Aut}_K F_{-2} \cong \mu_3(K)^3 \rtimes S(3).$$

Thus $|\mathrm{Aut}_K F_{-2}| = 162$ and this is what we need. ◇

Corollary 7. $g(3,3) = 162$

Remark 8. Analysing the proof of the previous theorem one can write down explicitly the structure of the group of automorphisms in case (1) and (3). Namely,

$$AutF_d \cong \begin{cases} \mu_3(K)^2 \rtimes S(3) & \text{if } d \neq 0, -2\epsilon^k, (1 \pm \sqrt{3})\epsilon^k \\ \mu_3(K)^3 \rtimes S(3) & \text{if } d = 0 \text{ or } d = -2\epsilon^k \end{cases}.$$

In case (2) we can only say that $Aut_K F_d$ is double cover of $\mu_3(K)^2 \rtimes S(3)$.

Remark 9. In the singular case ($d = \epsilon^k$) we have

$$F_{\epsilon^k}(X, Y, Z) \cong F_1(X, Y, Z) = X^3 + Y^3 + Z^3 - 3XYZ.$$

The form F_1 is known as the norm form N_A of K-algebra $A = K[X]/(X^3 - 1)$. A characterization of the automorphism group of the norm form can be found in [12] Theorem 3.4. Because $A \cong K^3$ we obtain

$$\mathrm{Aut}_K F_1 \cong \mathrm{Ker} N_A \rtimes S(3) \cong (K^*)^2 \rtimes S(3).$$

Thus, a group $\mathrm{Aut}_K F_1$ is infinite.

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Received: November 23, 2001