## Časopis pro pěstování matematiky a fysiky

## Vojtěch Jarník

On the main theorem of the Minkowski geometry of numbers

Časopis pro pěstování matematiky a fysiky, Vol. 73 (1948), No. 1, 1--8
Persistent URL: http://dml.cz/dmlcz/123149

## Terms of use:

© Union of Czech Mathematicians and Physicists, 1948
Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
This paper has been digitized, optimized for electronic delivery and
stamped with digital signature within the project DML-CZ: The Czech
Digital Mathematics Library http://project.dml.cz

## On the main theorem of the Minkowski geometry of numbers.

Vojtěch Jarnik, Praha.
(Received April 7th 1947.)
In this note all numbers are real. We consider the $n$-dimensional space $\mathrm{R}_{n} ; X=\left[x_{1}, \ldots, x_{n}\right], Y=\left[y_{1}, \ldots, y_{n}\right]$ being two points and $\lambda$ a number, we put $X+Y=\left[x_{1}+y_{1}, \ldots, x_{n}+y_{n}\right], \lambda \bar{X}=\left[\lambda x_{1}, \ldots\right.$, $\left.\ldots, \lambda x_{n}\right]$. We always put $O=\left[0, \ldots, 0 \mid\right.$. A set $K \subset R_{n}$ will be called an S. C. B. (symmetrical convex body) if it is convex, closed, bounded, symmetrical about $O$ and contains at least one interior point. $V(\mathrm{~K})$ always denotes the volume of K; $\lambda \mathrm{K}$ denotes the set of all points $\lambda X$, where $X \in K$. Let $\Lambda$ be an $n$-dimensional lattice of determinant 1 and let K be an S. C. B. We denote by $\tau_{k}(\mathrm{~K})(k=1, \ldots, n)$ the least positive number $\tau$ such that $\tau \mathrm{K}$ contains at least $k$ lattice points $X^{1}, \ldots, X^{k}$ which are linearly independent, i. e. there is no relation $\lambda_{1} X^{1}+\ldots+\lambda_{k} X^{k}=0$ with $\left|\lambda_{1}\right|+\ldots+\left|\lambda_{k}\right|>0$. Thus we have $0<\tau_{1}(K) \leqq \tau_{\mathbf{g}}(K) \leqq \ldots \leqq \tau_{n}(K)$, and $K \subset K^{\prime}$ implies $\tau_{k}(K) \geq \tau_{k}\left(K^{\prime}\right)$. A famous theorem of Minkowski says that, if $K$ is an S. C. B., then

$$
\begin{equation*}
V(K) \prod_{i=1}^{n} \tau_{i}(K) \leqq 2^{n} \tag{1}
\end{equation*}
$$

and so, in particular,

$$
\begin{equation*}
V(K) \tau_{1}{ }^{n}(K) \leqq 2^{n} . \tag{2}
\end{equation*}
$$

Thus, if $K$ is an S. C. B. and $V(K) \geqq 2^{n}$, we have $\tau_{1}(K) \leqq 1$, and so $K$ contains at least one lattice point $-X \neq O$. Many results are known concerning (2), but (1) seems to have been much less studied. In this nóte we give some results concerning the inequality (1), chiefly in the simplest case $n=2$.

In the following, we put

$$
\begin{equation*}
\delta=\frac{1}{2} \sqrt{3}-\frac{1}{4} \pi, \varepsilon=1-\frac{1}{4} \pi . \tag{3}
\end{equation*}
$$

We shall say that a set $K \subset R_{2}$ belongs to the class $\rho_{1}\left(\rho_{1}>0\right)$, if $K$ is an 8. C. B. the boundary of which consists of a curve having a con-
tinuous radius of curvature which in none of its points is less than $\varrho_{1}$. J. G. van der Corput and H. Davenport ${ }^{1}$ ) have proved the following theorem: Let $\Lambda$ be a lattice of determinant l. If $K$ belongs to the class $\varrho_{0}\left(\varrho_{0}>0\right)$ and if $V(K) \geqq 4-4 \delta \varrho_{0}{ }^{2}$, then $K$ contains a lattice point other than $O$ (i. e. $\tau_{1}(K) \leqq 1$, i. e. $\tau_{1}{ }^{2}(K) \leqq 1$. In order to extend this theorem to the product $\tau_{1}(\mathrm{~K}) \tau_{2}(\mathrm{~K})$ it is convenient to give it in another form. Suppose $K$ belongs to the class $\varrho_{1}\left(\varrho_{1}>0\right)$ and $V(K)=4$. Then, if $0<\eta<\tau_{1}(K)$, the set $\left.{ }^{2}\right)$ $\left(\tau_{1}-\eta\right) K$ belongs to the class $\left(\tau_{1}-\eta\right) \varrho_{1}$ and the theorem just quoted gives

$$
\begin{gather*}
4\left(\tau_{1}-\eta\right)^{2}<4-4 \delta\left(\tau_{1}-\eta\right)^{2} \varrho_{1}{ }^{2}, \\
\tau_{1}^{2}(K) \leqq\left(1+\delta \varrho_{1}^{2}\right)^{-1} . \tag{4}
\end{gather*}
$$

The constant $\delta$ is the best possible one (example: the circle $x^{2}+y^{2} \leqq 4 \pi^{-1}$ and the regular hexagonal lattice). It is perhaps not without interest to observe that the same inequality (4) holds also if $\tau_{1}{ }^{2}$ is replaced by $\tau_{1} \tau_{2}$ and that it can even be improved if $\tau_{1}<\tau_{2}$. This is shown by the following theorem.

Theorem 1. Let $\Lambda$ be a plane lattice of determinant $1_{2}$ let $\varrho_{1}>0$. Let K be a set of the class $\varrho_{1}$, let $V(\mathrm{~K})=4$. Then (putting $\tau_{i}(\mathrm{~K})=\tau_{i}$ ) we have

$$
\begin{equation*}
\tau_{1} \tau_{2} \leqq \frac{1+\left(1-\frac{\tau_{1}}{\tau_{2}}\right) \delta \varrho_{1}^{2}+\frac{\tau_{1}}{\tau_{2}} \varepsilon \varrho_{1}^{2}}{\left(1+\delta \varrho_{1}^{2}\right)\left(1+\varepsilon \varrho_{1}^{2}\right)} \tag{5}
\end{equation*}
$$

For $\tau_{1}=\tau_{2}$, this is exactly the formula (4). But, for $\tau_{1}<\tau_{2}$, (5) is sharper than (4). The right side of (5) is, for $\tau_{1}<\tau_{2}$, always less than $\left(1+\delta \rho_{1}{ }^{2}\right)^{-1}$ and greater than $\left(1+\varepsilon \varrho_{1}{ }^{2}\right)^{-1}$ and converges towards this value for $\tau_{1} \tau_{2}{ }^{-1} \rightarrow 0$ (when $\varrho_{1}$ remains fixed).

If we take in particular for $K$ the circle with the radius $\varrho_{1}=$ $=2 \pi^{-\frac{1}{2}}$ and for $\Lambda$ the lattice formed with the points $[0,0],\left[\tau_{1} \varrho_{1}, 0\right]$, [ $\left.\frac{1}{2} \tau_{1} \varrho_{1}, \tau_{1}{ }^{-1} \varrho_{1}-1\right]\left(0<\tau_{1} \leqq \pi^{\frac{1}{2}} 2^{-\frac{1}{3}} 3^{-\frac{1}{2}}\right)$ we obtain for $\tau_{1} \tau_{2}{ }^{-1}$ all values of the interval $0<\tau_{1} \tau_{2}{ }^{-1} \leqq 1$ and it is $\tau_{1} \tau_{2}=\left(\frac{1}{4} \tau_{1}{ }^{4}+\frac{1}{16} \tau^{2}\right)^{\frac{1}{2}}$, and so

$$
\tau_{1} \tau_{2}>\frac{1}{1+\varepsilon \varrho_{1}^{2}}, \lim _{\frac{\tau_{1}}{\tau_{2}} \rightarrow 1-} \tau_{1} \tau_{2}=\frac{1}{1+\delta \varrho_{1}^{2}}, \lim _{\substack{\tau_{1} \\ \tau_{2}}} \tau_{1} \tau_{2}=\frac{1}{1+\varepsilon \varrho_{1}^{2}} .
$$

Thus, though the formula (5) is surely not the best one of its kind, its limiting cases $\tau_{1} \tau_{2}^{-1} \rightarrow 1-, \tau_{1} \tau_{2}^{-1} \rightarrow 0+$ are so to say the best ones, at least for the particular value $\varrho_{1}=2 \pi^{-\frac{1}{2}}$.

[^0]Let us observe that $V(K), \tau_{i}(K)$ and the determinant of $\Lambda$ are invariant with regard to homogeneous linear transformations of determinant 1 , while $\varrho$ (the radius of curvature) is not. We can utilize this observation in order to improve our results: K being given, we may apply a transformation of determinant 1 in order to make $\varrho_{1}$ as great as possible; e. g. if $K$ is an ellipse, $V(K)=4$, we can treat $K$ as a circle of area 4 , i. e. we can regard $K$ as a curve of the class $2 \pi^{-\frac{1}{2}}$.

## § 1. Some auxiliary theorems.

Let $K \subset R_{n}$ be an $n$-dimensional S. C. B. and let $\Lambda$ be a lattice of determinant 1 . Write $\tau_{i}=\tau_{i}(K)$. Then there are $n$ linearly independent lattice points $P_{1}, P_{2}, \ldots, P_{n}$ such that $P_{i}$ lies on the boundary of $\tau_{i} K .{ }^{3}$ ) Denote by $M_{i}$ the $i$-dimensional subspace of $\mathrm{R}_{n}$ containing the points $O, P_{1}, \ldots, P_{i}(1 \leqq i \leqq n)$; let $M_{0}$ be the set consisting of the single point $O$. To every lattice point $P \neq O$ there is an $i(1 \leqq i \leqq n)$ so that $P \in M_{i}, P$ non $\in M_{i-1}$; we define then $\tau(P) \doteq$ $=\tau_{i}$. Thus we have $\frac{1}{\tau} P$ non $\epsilon \mathrm{K}$, if $0<\tau<\tau(P)$. All points $\frac{1}{\tau(P)} P$, where $P$ runs over all lattice points other than $O$, will be called critical points. We see: no critical point lies in the interior of K. We now prove the following theorem (theorem and proof similar to those in v. d. Corput-Davenport, Theorem 1).

Theorem 2. Let K, $\Lambda, P_{i}, \tau_{i}, \tau(P)$ have the meaning just explained. Then there is a polyhedron $\mathrm{K}^{\prime}$ with the following properties: (i) $\mathrm{K}^{\prime}$ is an $S . C$. B. (ii) $\mathrm{K}^{\prime} \subset \mathrm{K}^{\prime} .\left(\right.$ (iii) $\tau_{i}\left(\mathrm{~K}^{\prime}\right)=\tau_{i}(\mathrm{~K})$ for $i=1,2, \ldots, n$. $(i v)$ Every face of $\mathrm{K}^{\prime}$ contains at least one critical point in its interior.

Proof of Theorem 2. K is bounded and contains a neigh. bourhood of $O$; so there are evidently two numbers $a>0, b>0$ with the following properties:
(a) If $P$ is a point on the boundary of $K, A$ a tac-plane ( $(n-1)$ dimensional, ,Stützebene") to K at $P, d$ the distance from $O$ to $P$ and $d^{\prime}$ the distance from $O$ to A , then $d^{\prime}>a d$.
(b) If $L$ is an S. C. B., $L \supset K, \tau_{i}(L)=\tau_{2}$ for $i=1, \ldots, n$, then $L$ is contained in the sphere with the centre $O$ and radius $b$. This follows easily from the fact that $V(\mathrm{~L}) \leqq 2^{n}\left(\tau_{1} \ldots \tau_{n}\right)^{-1}$ and that $L$ contains a fixed neighbourhood of $\cdot O$.

We associate with every pair of critical points $Q,-Q$ a closed ${ }^{4}$ ) stripe $S(Q)=S(-Q)$ as follows: The segment $O Q$ has exactly one

[^1]point $Q_{1}$ in common with the boundary of $K$; we construct a tacplane $T$ to $K$ at $Q_{1}$ and construct then two planes $T_{1}, T_{2}$ through $Q,-Q$, parallel to $T$. Then $S(Q)$ is the stripe between $T_{1}, T_{2}$. Let $K^{\prime \prime}$, be the common part of all $\mathrm{S}(Q)$. Evidently $\mathrm{K} \subset \mathrm{K}^{\prime \prime}$ and $\tau_{i}\left(\mathrm{~K}^{\prime \prime}\right)=$ $\left.=\tau_{i}(\mathrm{~K}) \cdot{ }^{5}\right)$ [Proof: No oritical point lies in the interior of $\mathrm{K}^{\prime \prime}$, and so we have: if $P$ is a lattice point, $P \in M_{i}, P$ non $\in M_{i-1}$, then $\tau(P)=$ $=\tau_{i}$, and thus $P$ does not lie in the interior of $\tau_{i} K^{\prime \prime}$.] Following (b), $K^{\prime \prime}$ is bounded and so it is an S. C. B. Since, following (a), the broadth of $\mathbf{S}(Q)$ increases indefinitely together with the distance $O Q$, we see that all stripes $S(Q)$ but a finite number of them contain the bounded set $K^{\prime \prime}$ in their interior, so that only a finite numbre of these stripes can have points of $K^{\prime \prime}$ lying on their boundaries. Thus $K^{\prime \prime}$ is a polyhedron. If a face of $K^{\prime \prime}$ contains no critical point in its interior, we move this face, parallell to itself, away from $O$, carrying out the corresponding operation on the opposite face. We obtain so a variable S.C.B. $L \supset \mathrm{~K}^{\prime \prime}$ and, as long as no critical point enters into the interior of $L$, we have $\tau_{i}(\mathrm{~L})=\boldsymbol{\tau}_{i}$. Following (b) this movement must come to its end, either because the $(n-1)$-dimensional volume of the moving face becomes zero or because there appears some critical point in the interior of this face. After a finite number of such operations we come to a polyhedron $\mathrm{K}^{\prime}$ which satisfies not only (i), (ii), (iii), but also (iv).

Lemma 1. If $\tau_{1}=\ldots=\tau_{n}$, then $\mathrm{K}^{\prime}$ has at most $2\left(2^{n}-1\right)$ faces.
Proof (well known). We may suppose $\tau_{1}=1$, so that the critical points are exactly all lative points $P \neq O$. On every pair of opposite faces of $K^{\prime}$ we choose a pair of opposite lattice points $P,-P$, each in the interior of the corresponding face. If $2 k$ is the number of faces, we obtain so $k$ lattice points $P_{1}, \ldots, P_{k}$ suoh that $P_{i} \neq \pm P_{j}$ for $i \neq j$ and such that every interior point of any segment $P_{i} P_{j}^{\prime}\left(P_{i}^{\prime}=-P_{i}, i \neq j\right)$ lies in the interior of $\mathrm{K}^{\prime}$. It is obvious that no point $\frac{1}{2} P_{i}$ is a lattice point (since it lies in the interior of $\mathrm{K}^{\prime}$ ). So the points $P_{1}, \ldots, P_{k}$ belong modulo 2 to $2^{n}-1$ classes at most. Thus, if $k>2^{n}-1$, then there must be two indices $i, j$ $(i \neq j)$ such that $\frac{1}{2}\left(P_{i}-P_{j}\right) \neq O$ is a lattice point. But this point lies in the interior of $\mathrm{K}^{\prime}$ - contradiction.

Lemma 2. If $n=2, \tau_{1}<\tau_{2}$ in Theorem 2, then $K^{\prime}$ is a parallelogram or a hexagon. In the latter cäse, the point $\tau_{1}^{-1} P_{1}$ lies in the interior of a side of $\mathrm{K}^{\prime}$.

Proof. We can make of the segment $O P_{1}$ the side of a fundamental parallelogram of the lattice 1 ; we choose the straight line

[^2]$O P_{1}$ for the $x$-axis of a rectangular system $O x, O y$. If $\lambda$ is the length of $O P_{1}$; then evidently the $y$-coordinate of every lattice point is a multiplum of $\lambda^{-1}$. Let $y_{0}$ be the greatest $y$-coordinate occuring in $K^{\prime}$. As the boundary of $K^{\prime}$ contains the points $\pm \tau_{1}{ }^{-1} P_{1}$, it is obvious that $K^{\prime}$ contains a parallelogram $P$ the area of which is $2 \lambda \tau_{1}{ }^{-1} y_{0}$. Since $\tau_{1} \tau_{2} V\left(K^{\prime}\right) \leqq 4$, we have $y_{0} \leqq 2\left(\tau_{2} \lambda\right)^{-1}$; here we have either $P=K^{\prime}$ and $K^{\prime}$ is a parallelogram, or $y_{0}<2\left(\tau_{2} \lambda\right)^{-1}$. The $y$-coordinate of every critical point, not situated on the $x$-axis, is a multiplum of $\left(\tau_{2} \lambda\right)^{-1}$. Thus, if $y_{0}<2\left(\tau_{2} \lambda\right)^{-1}$, all critical points situated on the boundary of $\mathrm{K}^{\prime}$ above the $x$-axis, have the same $y$-coordinate, namely $\left(\tau_{2} \lambda\right)^{-1}$, and so there are at most two pairs of sides of $\mathrm{K}^{\prime}$ the interior of which contains critical points which are not situated on the $x$-axis. Thus, $\mathrm{K}^{\prime}$ has either 4 or 6 sides; and, in the latter case, two of its sides must contain the critical points $\pm \tau_{1}^{-1} P_{1}$ in their interior.

Lemma 3. Let $n=2, \varrho_{1}>0$; let K belong to the class $\varrho_{1}$; let $\mathrm{K} \subset \mathrm{H}$ or $\mathrm{K} \subset \mathrm{P}$ where H is a hexagon, P a parallelogram. Then
in the former case,

$$
V(\mathrm{~K}) \leqq V(\mathrm{H})-4 \delta \varrho_{1}^{2}
$$

$$
V(\mathrm{~K}) \leqq V(\mathrm{P})-4 \epsilon \varrho_{1}{ }^{2}<V(\mathrm{P})-4 \delta \varrho_{1}{ }^{2}
$$

in the latter case.
Proof. ${ }^{6}$ ) There is a hexagon $\mathrm{L} \subset \mathrm{H}$ (in the latter case a paralle$\operatorname{logram~LCP}$ ) such that K is inscribed into L . Let $L$ and $M$ be two consecutive points of contact of $L$ and $K$ and let the tangents at $L$ and $M$ meet at an angle $\alpha$. We form a rectangular system of coordinate axes $O x, O y$, with the tangent at $L$ as the $x$-axis. If $P=$ $=[x, y]$. is any point on the boundary of $K$ between $L$ and $M$, we denote by $l=l(x, y)$ the length intercepted on the tangent at $P$ between $P$ and the $x$-axis. Then the area between the are $L M$ and the tangents at $L$ and $M$ is

$$
a=\frac{1}{2} \int_{0}^{\alpha} l^{2} \mathrm{~d} \psi=\frac{1}{2} \int_{0}^{\alpha} y^{2} \sin ^{-2} \psi \mathrm{~d} \psi ;
$$

here, if the radius of curvature is denoted by $\varrho$,

$$
\frac{\mathrm{d} y}{\mathrm{~d} \psi}=\varrho \sin \psi, y \geqq \int_{\varphi}^{\varphi} \varrho_{1} \sin \varphi \mathrm{~d} \varphi=2 \varrho_{1} \sin ^{2} \frac{1}{2} \psi
$$

Thus

$$
a \geqq 2 \varrho_{1}^{2} \cdot \int_{0}^{\alpha} \sin ^{4} \frac{1}{2} \psi \cdot \sin ^{-2} \psi \mathrm{~d} \psi=\varrho_{1}{ }^{2}\left(\operatorname{tg} \frac{1}{2} \alpha-\frac{1}{2} \alpha\right) .
$$

Let the angles between the consecutive sides of the hexagon $L$ be

[^3]$\alpha_{1}, \alpha_{2}, \alpha_{3}$, so that $\alpha_{1}+\alpha_{2}+\alpha_{3}=\pi$ (if $L$ is a parallelogram, we take $\alpha_{3}=0$ ). Thus
$$
V(\mathrm{~K}) \leqq V(\mathrm{~L})-2 \varrho_{1}^{2} s, s=\sum_{i=1}^{3}\left(\operatorname{tg} \frac{1}{2} \alpha_{i}-\frac{1}{2} \alpha_{i}\right) .
$$

Putting $t_{i}=\operatorname{tg} \frac{1}{2} \alpha_{i}$ we have $t_{2} t_{3}+t_{3} t_{1}+t_{1} t_{2}=1$, whence $\left(t_{1}+\right.$ $\left.+t_{2}+t_{3}\right)^{2} \geqq 3\left(t_{2} t_{3}+t_{3} t_{1}+t_{1} t_{2}\right)=3$, and so $s \geqq \sqrt{3}-\frac{1}{2} \pi$. In the case of a parallelogram, we have $\frac{1}{2} \alpha_{2}=\frac{1}{2} \pi-\frac{1}{2} \alpha_{1}, \alpha_{3}=0$, and so $t_{2}=t_{1}^{-1}, t_{3}=0, s=t_{1}+t_{1}^{-1}-\frac{1}{2} \pi \geqq 2-\frac{1}{2} \pi$.

## § 2. Proof of Theorem 1.

Let $K \subset R_{2}$ be a set of the class $\varrho_{1}, V(K)=4$. Put $\tau_{i}=\tau_{i}(K)$ ( $i=1,2$ ). We construct the hexagon or parallelogram $K^{\prime}$.having the properties indicated in Theorem 2, Lemma 1 and 2. We have $\boldsymbol{\tau}_{1} \tau_{2} \cdot V\left(K^{\prime}\right) \leqq 4$.


If $\tau_{1}=\tau_{2}$, then Lemma 3 gives

$$
4=V(\mathrm{~K}) \leqq V\left(\mathrm{~K}^{\prime}\right)-4 \delta \rho_{1}{ }^{2} \leqq 4\left(\tau_{1} \tau_{2}\right)^{-1}-4 \delta \varrho_{1}{ }^{2},
$$

which is the same as (5) for $\tau_{1} \tau_{2}^{-1}=1$.
Thus we may suppose in the following that $\tau_{1}<\tau_{2}$. If $K^{\prime}$ is, a parallelogram, then (see Lemma 3)

$$
\begin{gather*}
4=V(K) \leqq V\left(K^{\prime}\right)-4 \epsilon \varrho_{1}^{2} \leqq 4\left(\tau_{1} \tau_{2}\right)^{-1}-4 \epsilon \varrho_{1}^{2}, \\
\tau_{1} \tau_{2} \leqq\left(1+\varepsilon \varrho_{1}^{2}\right)^{-1}, \tag{6}
\end{gather*}
$$

and thus (5) holds.
Thus we may suppose that $K^{\prime}$ is a hexagon; and we put $\tau_{2} K^{\prime \prime}=$ $=\mathrm{H}$. Let H be the hexagon $A B C A^{\prime} B^{\prime} C^{\prime}$ (see the figure; we write $\left.X^{\prime}=-X\right)$. Then (Lemma 2) the point $\tau_{2} \tau_{1}{ }^{-1} P_{1}$ is situated in the interior of a side, say $A B$. Let $D D^{\prime}$ be the straight line parallel to $A B$ through $O$. The notation is chosen so that $C$ lies on the same side
of $D D^{\prime}$ as $A$ and $B$ (in an extreme case, we may have $C=D$, $\left.C^{\prime}=D^{\prime}\right)$. H contains in its interior no lattice point which is not situated on the straight line $O P_{1}$. The straight lines $A B, A^{\prime} B^{\prime}, A^{\prime} C$, $A C^{\prime}$ determine a parallelogramm $A G A^{\prime} G^{\prime}$ which will be denoted by $\mathrm{Q}(\mathrm{Q} \supset \mathrm{H})$. Through the points $P_{1}, P_{1}^{\prime}$ we construct straight lines $E F, E^{\prime} F^{\prime}$, parallel to $D D^{\prime}$. Denote by $S$ the closed stripe bounded by the straight lines $E F, E^{\prime} F^{\prime}$; let $\mathrm{H}^{\prime}$ be the common part of H and S. Obviously $\tau_{1}\left(\mathrm{H}^{\prime}\right)=1$ and so $V\left(\mathrm{H}^{\prime}\right) \leqq 4$.

We shall distinguish two cases:
(i) $C$ does not lie in the interior of $S$. Then $\mathrm{H}^{\prime}$ is a parallelogram, and clearly $V(\mathrm{Q})=\tau_{2} \tau_{1}^{-1} V\left(\mathrm{H}^{\prime}\right) \leqq 4 \tau_{2} \tau_{1}^{-1}$. But $\mathrm{K} \subset \mathrm{K}^{\prime}=\tau_{2}^{-1} \mathrm{HC}$ $\subset \tau_{2}{ }^{-1} \mathrm{Q}$ and so (Lemma 3)

$$
V(\mathrm{~K}) \leqq \tau_{2}^{-2} V(\mathrm{Q})-4 \epsilon \varrho_{1}^{2}, 4 \leqq 4\left(\tau_{1} \tau_{2}\right)^{-1}-4 \epsilon \varrho_{1}^{2},
$$

which gives again (6); thus (5) holds even in this case.
(ii) $C$ lies in the interior of S . Then $\mathrm{H}^{\prime}$ is a hexagon $F E C F^{\prime} E^{\prime} C^{\prime}$. Let $K$ be the point of intersection of the straight lines $E F, C D$. Put $C K: D K=\alpha(0<\alpha \lesssim 1), B G: A G=\beta(0<\beta<1)$. Let $Q^{\prime}$ be the parallelogram $F K \overline{\bar{F}^{\prime}} K^{\prime}$. Construct a point $L$ on the segment $K F$ and a point $M$ on the segment $D K$ such that $L K=B G, D M=$ $=\tau_{1} \tau_{2}{ }^{-1} D C$. Denoting by $\mathrm{D}_{1}, \mathrm{D}_{2}$ the triangles $E K C, L E C$, by M the quadrilateral $L E C M$ (which degenerates into a triangle, if $C=D$ ) and by $\mathrm{H}^{\prime \prime}$ the hexagon $F L M F^{\prime} L^{\prime} M^{\prime}$ we have obviously (putting $\left.\tau_{1} \tau_{2}^{-1}=\lambda, 0<\lambda<1\right)$

$$
\begin{gathered}
V\left(\mathrm{Q}^{\prime}\right)=\lambda V(\mathrm{Q}), \quad V\left(\mathrm{H}^{\prime \prime}\right)=\lambda V(\mathrm{H}) \\
V\left(\mathrm{Q}^{\prime}\right)-V\left(\mathrm{H}^{\prime}\right)=2 V\left(\mathrm{D}_{1}\right), V\left(\mathrm{H}^{\prime}\right)-V\left(\mathrm{H}^{\prime \prime}\right)=2 V(\mathrm{M}) \geqq 2 V\left(\mathrm{D}_{2}\right) .
\end{gathered}
$$

But

$$
\begin{aligned}
\cdot 2 V\left(\mathrm{D}_{1}\right) & =V\left(\mathrm{Q}^{\prime}\right) \frac{C K}{2 D K} \cdot \frac{K E}{A G} \leqq V\left(\mathrm{Q}^{\prime}\right) \frac{1}{2} \alpha \beta \lambda \\
2 V\left(\mathrm{D}_{2}\right) & =V(\mathrm{Q}) \frac{C K}{2 D G} \cdot \frac{E L}{A G}>V(\mathrm{H}) \frac{1}{2} \alpha \lambda \beta(1-\lambda)
\end{aligned}
$$

Hence, putting $\frac{1}{2} \alpha \beta=\gamma\left(0<\gamma<\frac{1}{2}\right)$ and observing that $V\left(\mathrm{H}^{\prime}\right) \leqq 4$, we have

$$
\begin{gather*}
V(\mathrm{H}) \leqq \lambda^{-1}\left(V\left(\mathrm{H}^{\prime}\right)-2 V\left(\mathrm{D}_{2}\right)\right), \\
V(\mathrm{H})(1+\gamma(1-\lambda)) \leqq \lambda^{-1} V\left(\mathrm{H}^{\prime}\right) \leqq 4 \lambda^{-1},  \tag{7}\\
V\left(\mathrm{Q}^{\prime}\right)=V\left(\mathrm{H}^{\prime}\right)+2 V\left(\mathrm{D}_{1}\right) \leqq V\left(\mathrm{H}^{\prime}\right)+V\left(\mathrm{Q}^{\prime}\right) \gamma \lambda, \\
V\left(\mathrm{Q}^{\prime}\right)(1-\gamma \lambda) \leqq V\left(\mathrm{H}^{\prime}\right) \leqq 4, \\
V(\mathrm{Q})(\mathrm{l}-\gamma \bar{\lambda}) \leqq 4 \lambda^{-1 .} . \tag{8}
\end{gather*}
$$

But $K \subset K^{\prime}=\tau_{2}{ }^{-1} \mathrm{H} \subset \bar{\tau}_{2}{ }^{-1} \mathrm{Q}$ and so, by Lemma 3,

$$
\begin{aligned}
& 4\left(1+\varepsilon \varrho_{1}{ }^{2}\right)=V(K)+4 \epsilon \varrho_{1}^{2} \leqq \tau_{2}{ }^{2} \cdot V(\mathrm{Q}), \\
& 4\left(1+\delta \varrho_{1}{ }^{2}\right)=V(K)+4 \delta \varrho_{1}^{2} \leqq \tau_{2}{ }^{-2} V(\mathrm{H}) .
\end{aligned}
$$

Using (7), (8) we obtain

$$
\begin{equation*}
\tau_{1} \tau_{2} \leqq \operatorname{Min} \frac{1}{(1+\gamma(1-\lambda))\left(1+\delta \varrho_{1}^{2}\right)}, \frac{1}{(1-\gamma \lambda)\left(1+\varepsilon \varrho_{1}^{2}\right)} \tag{9}
\end{equation*}
$$

$\div$ This minimum atteins its greatest value if $\gamma$ is such that

$$
(1+\gamma(1-\lambda))\left(1+\delta \varrho_{1}{ }^{2}\right)=(1-\gamma \lambda)\left(1+\varepsilon \varrho_{1}^{2}\right) .
$$

Calculating $\gamma$ and putting it in (9), we obtain (5).

## K hlavní věte Minkowského geometrie čisel. (Obsah předešlého článku.)

Budiž K konvexni, omezená, uzavřená množina v rovině, jež obsahuje aspoň jeden vnitřní bod a je souměrná vzhledem $\mathbf{k}$ počátku; budiž $V(K)$ plocha množiny $K$; bud’te $\tau_{1}, \tau_{2}$,,minima"' množiny K ve smyslu Minkowského, takže podle základní věty Minkowského jest $\tau_{1} \tau_{2} V(K) \leqq 4$. Hlavnim výsledkem je tato věta: Je-li hranice množiny $K$ křivka se spojitě se měnícim poloměrem křivosti, jenž $v$ žádném bodě křivky není menší než jisté kladné číslo $\varrho_{1}$, platí (5) (viz též (3), kde jsou definována $\delta, \varepsilon$ ), jestliže. $V(K)=4$.

Mimo to je odvozena jedna věta o symetrických konvexnich tělesech $\mathrm{v} n$-rozměrném prostoru (Theorem 2).


[^0]:    ${ }^{1}$ ) On Minkowski's fundamental theorem in the geometry of numbers, Nederl. Akad, Wetensch. Proc. 49, 701-.707 = Indagationes Math. 7, 409-415 (1946).
    ${ }^{2}$ ) Where $\tau_{1}$ means $\tau_{1}(K)$.

[^1]:    ${ }^{3}$ ) Observe that the following considerations are true even if some or all the $\tau_{i}$ 's are equal.
    ${ }^{4}$ ) This means the boundary of the stripe belongs to it.

[^2]:    ${ }^{5}$ ) 8trictly speaking (we defined namely the $\tau_{i}$ 's,only for bounded sets): If $K^{\prime \prime \prime}$ is the common part of $K^{\prime \prime}$ with any sphere (with centre at $O$ ) containig $K$, then $\boldsymbol{\tau}_{i}\left(\mathrm{~K}^{\prime \prime \prime}\right) \ddot{\tau_{i}}(\mathrm{~K})$.

[^3]:    ${ }^{6}$ ) Almost litterally the same as in v. d. Corput-Davenport.

