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CRITICAL VALUES OF FISHER'S AND SIEGEL'S TEST[1]

JAROMÍR ANTOCH

Fisher's and Siegel's tests of periodicity belong among classical tools in the time series analysis. Unfortunately, since now the corresponding critical values and their approximations were published in rather limited scope. Main aim of present paper is to give extended tables of critical values for them based both on exact distribution of the test statistic and several related approximations.

1. FISHER'S TEST OF PERIODICITY

Testing periodicity in time series belongs among classical statistical problems. The solution based on periodogram has been proposed and thoroughly studied by Fisher [3.4, 5]. Since that time this test can be found in (almost) all books devoted to the time series analysis. Nevertheless, it is worth of noticing that critical values and their approximations available in the literature were calculated only for rather limited scope of sample sizes and levels (which almost did not change since the publication of Fisher's papers).

Let $X_1, \ldots, X_n$, $n = 2m+1$, be a series of random variables. Define periodogram by

$$I(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t e^{-it\lambda} \right|^2 \tag{1.1}$$

and normed periodogram by

$$Y_r = \frac{I(\lambda_r)}{\sum_{j=1}^{m} I(\lambda_j)}, \qquad r = 1, \ldots, m, \tag{1.2}$$

where $\lambda_r = 2\pi r/n$.

Assume that $X_1, \ldots, X_n$ are i.i.d. $N(0, \sigma^2)$ variables. For $Y = \max_{1 \le i \le m} Y_i$ Fisher proved that

$$P(Y > y) = 1 - \sum_{j=0}^{m} (-1)^j \binom{m}{j} (1 - jy)_+^{m-1}, \qquad 0 < y < 1, \tag{1.3}$$

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where $x_+ = x\, I[x > 0]$. Denote the exact critical value on the level α based on (1.3) by $a^*_m(\alpha)$.

Following two approximations of (1.3) are often used in practice, i.e.

$$P(Y > y) \cong m(1-y)^{m-1}, \qquad 0 < y < 1, \tag{1.4}$$

and

$$\lim_{m\to\infty} P\left(Y > \frac{z + \ln m}{m}\right) = 1 - \exp\{-e^{-z}\}, \qquad 0 < z < \infty. \tag{1.5}$$

While (1.4), suggested already by Fisher [3], is based on the use of first two terms of the sum in (1.3), the proof of (1.5) can be found in Anděl [1], e. g.

It is evident that the critical values $a^*_{m,1}(\alpha)$ and $a^*_{m,2}(\alpha)$ on the level α based on (1.4) and (1.5) are given by

$$a^*_{m,1}(\alpha) = 1 - \left(\frac{\alpha}{m}\right)^{1/(m-1)} \quad \text{and} \quad a^*_{m,2}(\alpha) = \frac{1}{m}\Big\{-\ln\Big(-\ln(1-\alpha)\Big) + \ln m\Big\}. \tag{1.6}$$

Both exact critical values $a^*_m(\alpha)$ and corresponding approximations $a^*_{m,1}(\alpha)$ and $a^*_{m,2}(\alpha)$ on the levels $\alpha = 0.05$, $\alpha = 0.025$ and $\alpha = 0.01$ can be found in Tables 1a–1c.

Table 1a. Critical values $a^*_m(\alpha)$ and their approximations $a^*_{m,1}(\alpha)$ and $a^*_{m,2}(\alpha)$ of Fisher's test on the level $\alpha = 0.05$.

m	$a^*_m(0.05)$	$a^*_{m,1}(0.05)$	$a^*_{m,2}(0.05)$
10	0.444 952 560 6	0.444 952 692 2	0.527 278 034 2
15	0.334 611 955 2	0.334 630 686 5	0.378 549 696 6
20	0.270 404 220 2	0.270 459 386 3	0.298 296 376 1
25	0.228 045 608 5	0.228 132 213 0	0.247 562 842 9
30	0.197 840 624 6	0.197 949 560 7	0.212 379 754 3
35	0.175 129 486 0	0.175 253 187 2	0.186 444 094 5
40	0.157 382 744 9	0.157 515 741 0	0.166 476 867 5
45	0.143 103 599 9	0.143 242 085 9	0.150 596 838 6
50	0.131 347 272 1	0.131 488 626 2	0.137 644 365 0
60	0.113 088 789 6	0.113 231 030 8	0.117 742 330 1
70	0.099 525 798 4	0.099 665 397 8	0.103 124 149 8
80	0.089 025 872 2	0.089 161 206 8	0.091 902 773 5
90	0.080 640 206 3	0.080 770 603 4	0.083 000 054 6
100	0.073 778 185 1	0.073 903 457 7	0.075 753 654 3
150	0.052 213 633 6	0.052 315 853 0	0.053 205 536 9
200	0.040 736 737 0	0.040 822 028 8	0.041 342 563 0
250	0.033 554 244 4	0.033 627 197 3	0.033 966 624 6
300	0.028 612 499 3	0.028 676 172 2	0.028 913 259 0
350	0.024 993 117 8	0.025 049 588 0	0.025 223 224 0
400	0.022 221 796 9	0.022 272 524 6	0.022 404 149 4
450	0.020 028 082 7	0.020 074 128 5	0.020 176 539 6
500	0.018 246 140 7	0.018 288 297 7	0.018 369 606 6
600	0.015 522 199 7	0.015 558 270 7	0.015 611 874 8
700	0.013 533 443 3	0.013 564 970 1	0.013 601 822 2
800	0.012 014 472 4	0.012 042 476 4	0.012 068 508 7
900	0.010 814 547 0	0.010 839 739 7	0.010 858 433 3
1000	0.009 841 528 3	0.009 864 425 1	0.009 877 950 5

Table 1b. Critical values $a^*_m(\alpha)$
and their approximations $a^*_{m,1}(\alpha)$ and $a^*_{m,2}(\alpha)$
of Fisher's test on the level $\alpha = 0.025$.

m	$a^*_m(0.025)$	$a^*_{m,1}(0.025)$	$a^*_{n,2}(0.025)$
10	0.486 095 733 5	0.4 860 957 335	0.59 788 3350
15	0.366 769 521 5	0.3 667 712 494	0.42 561 9306
20	0.296 583 781 5	0.2 965 944 508	0.33 359 8765
25	0.250 083 627 4	0.2 501 057 906	0.27 580 4233
30	0.216 860 254 8	0.2 168 925 847	0.23 591 4213
35	0.191 856 553 2	0.1 918 968 075	0.20 661 7091
40	0.172 310 849 0	0.1 723 569 406	0.18 412 8678
45	0.156 582 887 6	0.1 566 331 351	0.16 628 6832
50	0.143 634 865 2	0.1 436 879 786	0.15 176 5052
60	0.123 531 880 4	0.1 235 880 632	0.12 950 9636
70	0.108 607 486 1	0.1 086 645 327	0.11 321 0071
80	0.097 061 274 2	0.0 971 179 626	0.10 072 8236
90	0.087 846 254 8	0.0 879 019 128	0.09 084 5769
100	0.080 310 594 1	0.0 803 648 642	0.08 281 4744
150	0.056 667 981 8	0.0 567 142 358	0.05 791 2503
200	0.044 117 747 8	0.0 441 571 794	0.04 487 2231
250	0.036 279 407 5	0.0 363 135 713	0.03 679 0327
300	0.030 895 228 8	0.0 309 253 046	0.03 126 6657
350	0.026 957 171 4	0.0 269 840 106	0.02 724 0154
400	0.023 945 346 3	0.0 239 695 690	0.02 416 9795
450	0.021 563 661 9	0.0 215 857 295	0.02 174 5440
500	0.019 630 756 2	0.0 196 510 197	0.01 978 1107
600	0.016 679 366 7	0.0 166 967 822	0.01 678 8281
700	0.014 527 390 3	0.0 145 426 607	0.01 461 0679
800	0.012 885 578 8	0.0 128 991 760	0.01 295 1737
900	0.011 589 852 7	0.0 116 021 083	0.01 164 2355
1000	0.010 540 027 5	0.0 105 511 835	0.01 058 4025

2. SIEGEL'S TEST OF PERIODICITY

Siegel [11] suggested a new test statistic

$$T = \sum_{j=1}^{m} (Y_j - \zeta a^*_m)_+ \tag{2.1}$$

where $\zeta \in (0,1)$ is a constant. In our calculations we used $\zeta = 0.6$ following the suggestions from the literature. In the same paper Siegel derived exact distribution of T under H_0 that no periodic components are presented, i. e.

$$P(T > t) = \sum_{l=1}^{m} \binom{m}{l} \sum_{j=0}^{l-1} \binom{l-1}{j} \binom{m-1}{j} (-1)^{j+l+1} t^j (1 - \zeta l a^*_m - t)_+^{m-j-1} \tag{2.2}$$

and its first two moments

$$ET = (1 - \zeta a^*_m)^m \tag{2.3}$$

and

$$ET^2 = \frac{2(1-\zeta a^*_m)^{m+1} + (m-1)(1-2\zeta a^*_m)_+^{m+1}}{m+1}. \tag{2.4}$$

Denote the exact critical value on the level α based on (2.2) by $s^*_m(\alpha)$.

Table 1c. Critical values $a_m^*(\alpha)$ and their approximations $a_{m,1}^*(\alpha)$ and $a_{m,2}^*(\alpha)$ of Fisher's test on the level $\alpha = 0.01$.

m	$a_m^*(0.01)$	$a_{m,1}^*(0.01)$	$a_{m,2}^*(0.01)$
10	0.535 841 116 6	0.535 841 116 6	0.690 273 431 9
15	0.406 888 477 3	0.406 888 508 3	0.487 213 295 1
20	0.329 710 924 1	0.329 711 791 9	0.379 794 075 0
25	0.278 193 150 6	0.278 196 196 3	0.312 761 002 0
30	0.241 243 264 5	0.241 249 016 3	0.266 711 553 6
35	0.213 375 765 2	0.213 384 129 0	0.233 014 208 2
40	0.191 564 842 9	0.191 575 461 4	0.207 225 717 0
45	0.174 001 996 6	0.174 014 463 0	0.186 818 038 1
50	0.159 538 126 6	0.159 552 061 5	0.170 243 444 6
60	0.137 077 959 1	0.137 093 901 9	0.144 908 229 8
70	0.120 405 778 3	0.120 422 844 1	0.126 409 206 6
80	0.107 512 034 4	0.107 529 648 1	0.112 277 198 2
90	0.097 226 356 7	0.097 244 155 2	0.101 110 654 4
100	0.088 819 494 6	0.088 837 243 8	0.092 053 194 1
150	0.062 481 119 6	0.062 497 268 2	0.064 071 896 8
200	0.048 534 006 7	0.048 548 217 8	0.049 492 332 9
250	0.039 840 759 5	0.039 853 307 2	0.040 486 440 5
300	0.033 879 319 0	0.033 890 505 2	0.034 346 439 0
350	0.029 525 216 5	0.029 535 289 0	0.029 880 235 3
400	0.026 199 267 0	0.026 208 419 9	0.026 479 034 4
450	0.023 571 996 7	0.023 580 379 3	0.023 798 659 5
500	0.021 441 807 4	0.021 449 537 7	0.021 629 514 6
600	0.018 193 105 4	0.018 199 791 9	0.018 328 464 8
700	0.015 827 720 2	0.015 833 610 4	0.015 930 327 9
800	0.014 025 268 0	0.014 030 531 3	0.014 105 951 1
900	0.012 604 246 1	0.012 609 003 1	0.012 669 493 3
1000	0.011 453 958 0	0.011 458 297 8	0.011 507 904 5

χ_0^2-approximation

Best up to now proposed approximation to the exact distribution is that based on the non-central chi-square distribution with zero degrees of freedom $\chi_0^2(\lambda)$ suggested by Siegel [9, 10]. More precisely, he suggested to approximate the exact distribution of T by that of $c\chi_0^2(\lambda)$ where

$$P\left(\chi_0^2(\lambda) \le t\right) = 1 - e^{-(\lambda+t)/2} \sum_{l=1}^{\infty} \frac{(\lambda/2)^l}{l!} \sum_{j=0}^{l-1} \frac{(t/2)^j}{j!}, \qquad t \ge 0, \tag{2.5}$$

$$c = \frac{2(1-\zeta a_m^*)^{m+1} + (m-1)(1-2\zeta a_m^*)_+^{m+1} - (m+1)(1-\zeta a_m^*)^{2m}}{4(m+1)(1-\zeta a_m^*)^m} \tag{2.6}$$

and

$$\lambda = \frac{(1-\zeta a_m^*)^m}{c}. \tag{2.7}$$

The choice of constants c and λ ensure that both T and $c\chi_0^2(\lambda)$ have the same first two moments given by (2.3) and (2.4). Denote the critical value on the level α based on (2.5) – (2.7) by $s_{m,1}^*(\alpha)$. We can see from the results presented in Section 3 that this approximation gives excellent fit to the exact distribution even for small values of m. Its basic disadvantage is that it is computationally rather complex and "costly".

Numerical problems connected with the calculation of the cumulative distribution function and quantiles of $\chi_0^2(\lambda)$, which usually can be neither found in general statistical packages nor calculated with desired precision lead Anděl [2] to a proposal of several simpler approximations based only on the quantiles of standard normal and central chi-square distributions which are easily available. We shall present here only resulting formulas while for their derivation we refer to Anděl [2].

χ^2-approximation

The distribution function F of the statistic $c\chi_0^2$ is composed from two parts. More precisely, $F = p_1F_1 + p_2F_2$, where F_1 is the distribution function of the random variable degenerated in zero and F_2 is a distribution function of a continuous distribution. It was shown by Anděl [2] that F_2 can be reasonably approximated by $h\chi^2$, where χ^2 has classical χ^2-distribution with q degrees of freedom. Therefore, approximate critical value $s_{m,2}^*(\alpha)$ for $s_m^*(\alpha)$ on the level α based on this, so called χ^2-approximation, is given by

$$s_{m,2}^*(\alpha) = h\chi_q^2\left(1 - \frac{\alpha}{p_2}\right), \tag{2.8}$$

where $\chi_q^2(\beta)$ is the β-quantile of the chi-square distribution with q degrees of freedom,

$$p_2 = 1 - e^{-\lambda/2}, \quad \nu = \frac{c\lambda}{p_2}, \quad \sigma^2 = \frac{c^2\lambda(4+\lambda)}{p_2} - \nu^2, \quad h = \frac{\sigma^2}{2\nu}, \quad q = \frac{2\nu^2}{\sigma^2}, \tag{2.9}$$

c and λ are given by (2.6) and (2.7).

Wilson – Hilferty approximation

Applying well known Wilson – Hilferty transformation between the quantiles of normal and χ^2-distributions in (2.8) (see Johnson and Kotz [6] for details), we obtain immediately another simple approximation for $s_m^*(\alpha)$ of the form

$$s_{m,3}^*(\alpha) = hq\left(\frac{u_{1-\alpha/p_2}}{3}\sqrt{\frac{2}{q}} + 1 - \frac{2}{9q}\right)^3, \tag{2.10}$$

where u_β is the β-quantile of the standard normal distribution and h, p_2 and q are given by (2.9).

N-approximation

If we approximate F_2 by the normal distribution $N(\nu, \sigma^2)$ instead by $h\chi^2$, we can arrive to the following approximation of $s_m^*(\alpha)$ on the level α, so called N-approximation $s_{m,4}^*(\alpha)$, which is given by

$$s_{m,4}^*(\alpha) = \nu + \sigma\Phi^{-1}\left(1 - \frac{\alpha}{p_2}\right), \tag{2.11}$$

where ν, p_2 and σ^2 are given by (2.9).

Normal approximation

It can be shown that statistic T is asymptotically normal, see Siegel [9] for details. Unfortunately, this approximation is rather bad. Nevertheless, for completeness of our results we present in Tables 2a – 2c also approximate critical values $s^*_{m,5}(\alpha)$ on the level α based on it.

Table 2a. Critical values $s^*_m(\alpha)$ and their approximations $s^*_{m,1}(\alpha) - s^*_{m,5}(\alpha)$ of Siegel's test on the level $\alpha = 0.05$.

m	$s^*_m(0.05)$	$s^*_{m,1}(0.05)$	$s^*_{m,2}(0.05)$	$s^*_{m,3}(0.05)$	$s^*_{m,4}(0.05)$	$s^*_{m,5}(0.05)$
10	0.181 291 621 8	0.177 537 960 1	0.176 636 029 2	0.175 077 251 6	0.168 347 534 8	0.150 727 793 2
15	0.140 175 143 6	0.137 746 057 3	0.137 045 780 5	0.135 834 796 6	0.130 677 196 7	0.116 907 231 1
20	0.115 943 423 2	0.114 206 143 2	0.113 627 432 5	0.112 628 819 6	0.108 138 470 4	0.097 056 245 7
25	0.099 732 543 2	0.098 411 301 8	0.097 915 454 9	0.097 061 273 9	0.092 958 630 4	0.083 776 451 5
30	0.088 019 804 3	0.086 971 925 0	0.086 537 068 6	0.085 788 247 1	0.081 954 750 3	0.074 169 979 0
35	0.079 106 085 1	0.078 249 135 5	0.077 861 522 9	0.077 193 372 4	0.073 567 111 0	0.066 847 273 8
40	0.072 063 480 1	0.071 346 056 2	0.070 996 386 0	0.070 392 226 8	0.066 935 769 4	0.061 051 524 1
45	0.066 339 197 6	0.065 727 365 8	0.065 408 997 8	0.064 856 985 1	0.061 545 193 7	0.056 332 367 6
50	0.061 581 925 2	0.061 052 271 2	0.060 760 268 4	0.060 251 661 1	0.057 066 327 0	0.052 403 667 8
60	0.054 101 036 4	0.053 689 933 5	0.053 440 098 9	0.052 999 812 2	0.050 028 435 8	0.046 210 973 7
70	0.048 458 130 9	0.048 127 497 7	0.047 910 086 6	0.047 521 319 9	0.044 726 967 5	0.041 525 614 6
80	0.044 029 715 2	0.043 756 678 5	0.043 565 096 4	0.043 216 697 0	0.040 573 041 8	0.037 838 571 0
90	0.040 449 355 2	0.040 219 253 0	0.040 048 795 4	0.039 732 953 1	0.037 220 089 6	0.034 850 179 1
100	0.037 486 626 7	0.037 289 575 4	0.037 136 750 6	0.036 847 767 5	0.034 450 050 7	0.032 371 672 8
150	0.027 934 646 9	0.027 828 858 4	0.027 733 526 3	0.027 530 266 9	0.025 555 964 2	0.024 339 114 1
200	0.022 651 731 7	0.022 586 541 2	0.022 522 913 0	0.022 366 073 1	0.020 668 838 6	0.019 862 264 5
250	0.019 246 021 0	0.019 203 392 9	0.019 159 730 6	0.019 032 175 5	0.017 534 065 9	0.016 959 432 5
300	0.016 845 026 8	0.016 816 640 1	0.016 786 570 0	0.016 679 230 7	0.015 332 766 3	0.014 903 431 8
350	0.015 049 626 4	0.015 030 990 1	0.015 010 662 0	0.014 918 137 4	0.013 691 860 0	0.013 360 034 4
400	0.013 649 735 3	0.013 638 156 3	0.013 625 070 6	0.013 543 876 3	0.012 415 685 9	0.012 152 623 1
450	0.012 523 571 5	0.012 517 309 0	0.012 509 755 9	0.012 437 509 7	0.011 391 191 3	0.011 178 471 9
500	0.011 595 364 5	0.011 593 228 4	0.011 589 991 1	0.011 524 990 7	0.010 548 240 8	0.010 373 478 4
600	0.010 149 384 6	0.010 153 174 5	0.010 156 109 4	0.010 102 124 0	0.009 237 691 3	0.009 115 301 9
700	0.009 069 238 0	0.009 077 003 6	0.009 083 999 2	0.009 037 985 9	0.008 260 721 5	0.008 171 771 4
800	0.008 227 618 9	0.008 238 168 9	0.008 247 920 5	0.008 207 938 5	0.007 500 594 2	0.007 434 152 3
900	0.007 550 883 6	0.007 563 440 2	0.007 575 097 8	0.007 539 833 3	0.006 890 001 2	0.006 839 324 3
1000	0.006 993 283 6	0.007 007 313 4	0.007 020 301 1	0.006 988 823 7	0.006 387 244 2	0.006 347 957 0

3. CRITICAL VALUES AND CONCLUSIONS

This section presents critical values calculated using all just described methods. If we take a look on (1.3) and especially (2.2), we can immediately see computational problems connected with the evaluation of exact critical values due especially to plenty of binomial coefficients involved. If one wishes to obtain "really exact" critical values, it is necessary to use either programming language enabling to work in arbitrary precision (like in C) or some of higher level programming languages offering this feature. In our case we have used for the calculation program *Mathematica*® v. 2.2. The main reason was that in such a case we might use aside the arbitrary precision several build in special functions. Moreover, resulting code which is available from the author on request is more "readable and understandable".

Tables 1a – 1c contain both exact critical values of Fisher's test of periodicity $a^*_m(\alpha)$ and two approximations described by (1.6). We can see immediately that while the values of $a^*_{m,1}(\alpha)$ give reasonable approximation even for small values of m, this is not the case of $a^*_{m,2}(\alpha)$, which is surprisingly imprecise even for very large sample sizes.

Tables 2a – 2c contain both exact critical values $s^*_m(\alpha)$ of Siegel's test of periodicity and five approximations described in Section 2. We can see immediately that while values of the approximations of $s^*_m(\alpha)$ based on both central and non-central chi-square distribution are quite precise even for small values of m, this is not the case of both N and normal approximations. *Globally said, we can conclude that both χ^2-approximation and Wilson – Hilferty approximation based on standard χ^2 distribution give excellent counterpart to both the exact critical values and their χ^2_0-approximation even for small sample sizes. Their main advantage is that they are much more easily available and more easily calculable.*

Table 2b. Critical values $s^*_m(\alpha)$ and their approximations $s^*_{m,1}(\alpha) - s^*_{m,5}(\alpha)$ of Siegel's test on the level $\alpha = 0.025$.

m	$s^*_m(0.025)$	$s^*_{m,1}(0.025)$	$s^*_{m,2}(0.025)$	$s^*_{m,3}(0.025)$	$s^*_{m,4}(0.025)$	$s^*_{m,5}(0.025)$
10	0.195 354 459 2	0.193 978 768 6	0.193 587 746 2	0.191 918 215 4	0.172 586 990 1	0.141 595 046 9
15	0.149 780 318 3	0.148 866 541 7	0.148 559 824 7	0.147 264 093 5	0.132 849 459 0	0.108 277 306 2
20	0.123 003 606 3	0.122 369 312 9	0.122 118 890 2	0.121 057 699 9	0.109 093 091 0	0.089 110 333 1
25	0.105 152 946 2	0.104 697 224 6	0.104 486 345 5	0.103 585 583 2	0.093 142 127 6	0.076 432 952 8
30	0.092 303 881 4	0.091 967 151 5	0.091 785 801 6	0.091 001 989 1	0.081 622 394 1	0.067 334 484 5
35	0.082 561 265 1	0.082 307 142 5	0.082 148 871 2	0.081 454 329 1	0.072 875 126 8	0.060 440 859 6
40	0.074 890 535 9	0.074 695 787 2	0.074 556 171 4	0.073 932 155 7	0.065 985 144 3	0.055 011 335 0
45	0.068 676 067 3	0.068 525 240 9	0.068 401 094 1	0.067 834 297 7	0.060 404 122 9	0.050 608 441 2
50	0.063 527 207 6	0.063 409 662 5	0.063 298 599 6	0.062 779 217 6	0.055 782 511 4	0.046 955 893 3
60	0.055 463 766 7	0.055 392 196 6	0.055 302 166 0	0.054 856 935 8	0.048 553 123 0	0.041 223 400 2
70	0.049 412 383 1	0.049 370 038 3	0.049 296 271 3	0.048 906 491 5	0.043 138 104 6	0.036 907 674 4
80	0.044 684 941 5	0.044 662 109 4	0.044 601 354 9	0.044 254 687 0	0.038 916 652 1	0.033 525 378 7
90	0.040 878 431 6	0.040 869 131 8	0.040 819 056 0	0.040 506 912 0	0.035 524 771 2	0.030 793 503 9
100	0.037 740 257 1	0.037 740 630 6	0.037 699 493 1	0.037 415 645 0	0.032 734 256 6	0.028 534 572 1
150	0.027 707 933 6	0.027 730 293 4	0.027 718 404 2	0.027 523 514 2	0.023 859 755 0	0.021 260 453 3
200	0.022 228 237 1	0.022 257 134 9	0.022 261 352 5	0.022 113 563 8	0.019 052 537 2	0.017 241 678 8
250	0.018 728 767 9	0.018 760 057 8	0.018 774 313 9	0.018 655 743 6	0.016 002 494 7	0.014 651 800 6
300	0.016 280 265 3	0.016 312 514 7	0.016 333 484 8	0.016 234 815 8	0.013 879 636 6	0.012 826 075 5
350	0.014 460 881 1	0.014 493 512 7	0.014 519 178 0	0.014 434 932 4	0.012 309 047 0	0.011 460 763 3
400	0.013 049 974 5	0.013 082 729 7	0.013 111 776 5	0.013 038 458 5	0.011 095 499 1	0.010 396 087 6
450	0.011 920 339 1	0.011 953 092 2	0.011 984 622 2	0.011 919 864 2	0.010 126 882 1	0.009 539 465 3
500	0.010 993 205 9	0.011 025 892 1	0.011 059 267 4	0.011 001 391 5	0.009 334 019 8	0.008 833 306 2
600	0.009 556 689 8	0.009 589 150 9	0.009 624 935 7	0.009 577 427 7	0.008 109 532 5	0.007 732 971 6
700	0.008 490 479 1	0.008 522 661 8	0.008 559 769 5	0.008 519 686 1	0.007 203 982 2	0.006 910 757 2
800	0.007 664 263 8	0.007 696 146 1	0.007 733 919 7	0.007 699 403 5	0.006 504 261 3	0.006 269 919 4
900	0.007 003 088 3	0.007 034 659 5	0.007 072 683 3	0.007 042 488 2	0.005 945 585 9	0.005 754 487 8
1000	0.006 460 618 1	0.006 491 872 2	0.006 529 875 4	0.006 503 125 6	0.005 488 054 7	0.005 329 688 6

Table 2c. Critical values $s_m^*(\alpha)$ and their approximations $s_{m,1}^*(\alpha) - s_{m,5}^*(\alpha)$ of Siegel's test on the level $\alpha = 0.01$.

m	$s_m^*(0.01)$	$s_{m,1}^*(0.01)$	$s_{m,2}^*(0.01)$	$s_{m,3}^*(0.01)$	$s_{m,4}^*(0.01)$	$s_{m,5}^*(0.01)$
10	0.214 407 337 9	0.216 973 362 1	0.216 965 516 5	0.215 545 159 7	0.178 803 034 8	0.128 116 592 5
15	0.163 715 457 7	0.164 871 142 5	0.164 839 468 9	0.163 700 504 9	0.136 913 065 9	0.096 039 859 1
20	0.133 711 589 5	0.134 448 128 5	0.134 419 773 2	0.133 484 311 9	0.111 771 397 0	0.078 170 486 7
25	0.113 703 477 4	0.114 254 610 5	0.114 232 511 4	0.113 442 825 5	0.094 886 996 4	0.066 546 331 7
30	0.099 323 723 6	0.099 772 677 8	0.099 756 614 3	0.099 075 231 2	0.082 711 380 6	0.058 292 764 0
35	0.088 444 769 6	0.088 828 759 2	0.088 818 032 0	0.088 219 973 8	0.073 486 620 8	0.052 087 728 4
40	0.079 900 498 2	0.080 238 977 5	0.080 232 898 7	0.079 700 835 6	0.066 238 804 2	0.047 229 979 3
45	0.072 995 766 8	0.073 300 161 7	0.073 298 135 4	0.072 819 594 7	0.060 383 235 9	0.043 310 046 0
50	0.067 289 252 1	0.067 566 862 3	0.067 568 389 7	0.067 134 109 5	0.055 546 887 4	0.040 071 526 0
60	0.058 383 708 8	0.058 621 339 3	0.058 628 801 9	0.058 263 415 1	0.048 009 461 4	0.035 014 010 6
70	0.051 730 440 0	0.051 939 171 0	0.051 951 394 7	0.051 637 117 5	0.042 390 714 8	0.031 227 568 4
80	0.046 554 325 2	0.046 740 915 7	0.046 757 049 5	0.046 482 164 5	0.038 029 832 4	0.028 273 409 9
90	0.042 402 440 6	0.042 571 408 7	0.042 590 815 1	0.042 347 195 8	0.034 540 269 9	0.025 896 320 2
100	0.038 991 588 8	0.039 146 146 6	0.039 168 331 1	0.038 950 111 9	0.031 680 289 2	0.023 937 096 1
150	0.028 176 762 6	0.028 286 046 3	0.028 317 552 8	0.028 177 408 6	0.022 665 383 8	0.017 670 573 5
200	0.022 342 804 5	0.022 428 244 5	0.022 464 984 0	0.022 364 631 4	0.017 848 149 6	0.014 239 801 7
250	0.018 652 777 5	0.018 723 636 4	0.018 763 622 4	0.018 687 157 7	0.014 824 084 2	0.012 042 663 5
300	0.016 091 154 5	0.016 152 270 7	0.016 194 381 9	0.016 133 723 0	0.012 737 704 1	0.010 501 164 8
350	0.014 200 326 1	0.014 254 531 7	0.014 298 072 1	0.014 248 572 3	0.011 205 646 9	0.009 352 822 2
400	0.012 742 444 2	0.012 791 528 1	0.012 836 039 0	0.012 794 792 3	0.010 029 632 8	0.008 460 214 7
450	0.011 581 123 2	0.011 626 281 4	0.011 671 445 8	0.011 636 520 4	0.009 096 463 0	0.007 744 015 5
500	0.010 632 318 6	0.010 674 386 1	0.010 719 977 8	0.010 690 028 1	0.008 336 655 7	0.007 155 041 4
600	0.009 170 811 5	0.009 208 351 7	0.009 254 343 0	0.009 231 676 6	0.007 171 277 8	0.006 240 093 2
700	0.008 093 626 1	0.008 128 033 5	0.008 174 040 7	0.008 156 399 7	0.006 316 615 9	0.005 558 838 1
800	0.007 263 926 4	0.007 296 051 0	0.007 341 845 7	0.007 327 844 1	0.005 661 025 3	0.005 029 458 2
900	0.006 603 470 5	0.006 633 862 4	0.006 679 304 6	0.006 668 036 3	0.005 140 979 2	0.004 604 779 3
1000	0.006 064 143 5	0.006 093 176 8	0.006 138 180 0	0.006 129 023 1	0.004 717 578 1	0.004 255 576 7

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