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# REPRESENTABILITY OF RECURSIVE P. MARTIN-LÖF TESTS

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The present paper, closely connected with [2], investigates the possibility of expressing P. Martin-Löf's complexity theory of strings in terms of Kolmogorov's complexity of strings which uses algorithms $\varphi$. We find for every recursive P. Martin-Löf test $V$ an algorithm $\varphi$ which in turn gives a P. Martin-Löf test $V(\varphi)$ such that $V \subset V(\varphi)$. The equality $V = V(\varphi)$ holds for some particular P. Martin-Löf tests called representable.

In this paper we continue our efforts to approach Kolmogorov's complexity theory of strings which uses algorithms $\varphi$ (see [3]) and P. Martin-Löf's complexity theory of strings which uses $M - L$ tests $V$ (see [5]). A very good up-to-date survey paper is [7]. The present authors have already done some attempts in this direction in [2]. We work within the general framework of a not necessarily binary alphabet (see [1]).

It has already been noticed that these theories are not equivalent (see [2]).

In this paper we find for every recursive $M - L$ test $V$ an algorithm $\varphi$ which in turn gives a $M - L$ test $V(\varphi)$ such that $V \subset V(\varphi)$ (see Theorem 2). The equality $V = V(\varphi)$ holds for some particular $M - L$ tests $V$ which we call *representable* (see Theorem 3). Such an equality $V = V(\varphi)$ would be a good interpretation of the somewhat unprecise term "equivalence" between Kolmogorov's and P. Martin-Löf's theories. In this respect see also Theorem 4.

The last section of our paper contains remarks and open problems.

## 1. BASIC NOTIONS

Throughout the paper $N$ will be the set of all natural numbers, i.e. $N = \{0, 1, 2, \ldots\}$. If $A$ is a finite set, card $A$ will be the number of elements in $A$.

For every non-empty sets $A$ and $B$ and for every function $f : A' \to B$ (where $A' \subset A$) we shall write $f : A \overset{\circ}{\to} B$. We shall say that $f$ is a *partial function* from $A$ to $B$. We consider that $f(x) = \infty$ in case $f$ is not defined in the point $x$.

If $f : A \overset{\circ}{\to} B$ is a partial function, then the domain of $f$ is the set $\mathrm{dom}\,(f) = \{x \in A \mid f(x) \neq \infty\}$; range $(f) = \{f(x) \mid x \in \mathrm{dom}\,(f)\}$; graph $(f) = \{(x, f(x)) : x \in \mathrm{dom}\,(f)\}$.

Let $X = \{a_1, a_2, \ldots, a_p\}$, $p \geqq 2$ be a finite alphabet. Denote by $X^*$ the free monoid generated by $X$ under concatenation, i.e. $X^*$ consists of all strings $x = x_1 x_2 \ldots x_n$, where the $x_i$ belong to $X$; also the null string $\lambda$ belongs to $X^*$. For every $a$ in $X$ and every natural $n > 0$, $a^n = aa \ldots a$ ($n$ copies of $a$). We shall consider that $a^0 = \lambda$. For every $x$ in $X^*$, $l(x)$ is the length of $x$, i.e. $l(x) = m$ in case $x = x_1 x_2 \ldots x_m$ and $l(\lambda) = 0$. For Recursive Function Theory see [4] and [6]. We shall consider *partial recursive functions* (*p.r. functions* in the sequel) $\varphi : X^* \times N \overset{\circ}{\to} X^*$ or $g : N - \{0\} \overset{\circ}{\to} X^* \times N$.

For every p.r. function $\varphi : X^* \times N \overset{\circ}{\to} X^*$, the *Kolmogorov complexity* induced by $\varphi$ is a function $K_\varphi : X^* \times N \to N \cup \{\infty\}$, defined by $K_\varphi(x \mid m) = \min\{l(y) \mid y \in X^*,\ \varphi(y, m) = x\}$ in case $x = \varphi(y, m)$ for some $y$ in $X^*$ and $K_\varphi(x \mid m) = \infty$, otherwise.

For every $W \subset X^* \times (N - \{0\})$ and for every natural $m \geqq 1$ we shall write $W_m = \{x \in X^* \mid (x, m) \in W\}$. We define the *critical level induced by* $W$ to be the function $m_W : X^* \to N \cup \{\infty\}$ given by $m_W(x) = \sup\{m \in N \mid m \geqq 1,\ x \in W_m\}$ in case such $m$ exists, and $m_W(x) = 0$, in the opposite case.

A non-empty recursively enumerable set $V \subset X^* \times (N - \{0\})$ will be called *Martin-Löf test* ($M - L$ test) if it possesses the following two properties:

1) For every natural $m \geqq 1$, $V_{m+1} \subset V_m$.

2) For every natural numbers $m$, $n$, $m \geqq 1$,

$$\mathrm{card}\,\{x \in X^* \mid l(x) = n,\ x \in V_m\} < p^{n-m}/(p - 1)\,.$$

We agree upon the fact that the empty set is a $M - L$ test.

The second condition enables us to say that $m_V$ takes only finite values for every $M - L$ test $V$, because in case $(x, m) \in V$, then $m \leqq l(x) - 1$ (directly from the definition).

For every p.r. function $\varphi : X^* \times N \overset{\circ}{\to} X^*$ we can obtain the particular $M - L$ test $V(\varphi) = \{(x, m) \in X^* \times (N - \{0\}) \mid K_\varphi(x \mid l(x)) < l(x) - m\}$, see [1]. We shall call a $M - L$ test $V$ *representable* in case there exists a p.r. function $\varphi : X^* \times N \overset{\circ}{\to} X^*$ such that $V = V(\varphi)$, see [2].

The lexicographical order on $X^*$ induced by $a_1 < a_2 < \ldots < a_p$ is given by $\lambda < a_1 < a_2 < \ldots < a_p < a_1 a_1 < a_1 a_2 < \ldots < a_1 a_p < a_2 a_1 < \ldots$. The enumeration of $X^*$ in this order will be $y(1) = \lambda$, $y(2) = a_1$, $y(3) = a_2, \ldots, y(p + 1) = a_p$, $y(p + 2) = a_1 a_1, \ldots$. It follows that $a_p^m = y(s(m))$, where $s(m) = 1 + p + p^2 + \ldots + p^m = (p^{m+1} - 1)/(p - 1)$. This enumeration of $X^*$ is recursive. In the sequel, the *lexicographical order* will mean this lexicographical order on $X^*$.

## 2. RESULTS

It is easily seen that there exist $M - L$ tests which are not recursive. For instance, take $A \subset \{a_1\}^* = \{\lambda, a_1, a_1^2, a_1^3, \ldots\}$ which is recursively enumerable and not recursive. Then $V = (A - \{\lambda, a_1\}) \times \{1\}$ is a non-recursive $M - L$ test.

The following theorem gives necessary and sufficient conditions under which a $M - L$ test is recursive.

**Theorem 1.** A $M - L$ test $V$ is recursive iff the function $m_V$ is recursive.

Proof. If $m_V$ is recursive we can compute $m_V(x)$ for every $x$ in $X^*$. Let $(x, m)$ be in $X^* \times (N - \{0\})$. If $m_V(x) \geqq m$, then $(x, m) \in V$; if $m_V(x) < m$, then $(x, m) \notin V$. Thus $V$ is recursive.

Now suppose $V$ is recursive which means that $\chi_V$ is a recursive function ($\chi_V$ is the characteristic function of $V$). It is easy to see that for every $x$ in $X^*$ we have $m_V(x) = \max\{m \in N \mid \chi_V(x, m) = 1\}$, in case $(x, 1) \in V$, and $m_V(x) = 0$, in case $(x, 1) \notin V$. This shows that $m_V$ is recursive. □

Actually, the object of our paper will be the study of some properties of *recursive* $M - L$ tests.

**Theorem 2.** Let $V \subset X^* \times N$ be a recursive $M - L$ test. Then there exists a p.r. function $\varphi : X^* \times N \xrightarrow{\circ} X^*$ such that $V \subset V(\varphi)$.

The p.r. function $\varphi$ can be taken to possess the following properties:

(a) The function $\varphi$ is injective.

(b) The graph of $\varphi$ is recursive.

(c) For every $x$ in $X^*$, we have the equivalence: $(x, 1) \in V$ iff $(x, 1) \in V(\varphi)$.

Proof. The set $A = \{(x, m_V(x)) \mid x \in V_1\}$ is obviously recursive. We distinguish two cases: i) *$V$ is infinite* and in this case there exists an injective recursive function $g : N - \{0\} \to X^* \times N$ such that $g(N - \{0\}) = A$; ii) *$V$ is finite* and $A$ has $q$ elements, and in this case there exists an injective (p.r.) function $g : \{1, 2, \ldots, q\} \to X^* \times N$ such that $g(\{1, 2, \ldots, q\}) = A$. In all cases, if $i$ is in the domain of $g$, we put $g(i) = (x_i, m_V(x_i))$. Moreover, due to the recursiveness of $V$, we can suppose that $g$ has the following "lexicographical" property: for all natural $1 \leqq i < j$:

u) $l(x_i) \leqq l(x_j)$,

v) if $l(x_i) = l(x_j)$, then $m_V(x_i) \geqq m_V(x_j)$.

We can define the procedure for $\varphi$.

For $i = 1$, $g(1) = (x_1, m_V(x_1))$ and we put $\varphi(z_1, l(x_1)) = x_1$, where $z_1 = y(s(l(x_1) - m_V(x_1) - 1))$. See the definition of $s$ in Section 1.

Next, let $i = 2$, so $g(2) = (x_2, m_V(x_2))$. In case $l(x_1) \neq l(x_2)$, we put $\varphi(z_2, l(x_2)) = x_2$, where $z_2 = y(s(l(x_2) - m_V(x_2) - 1))$. In case $l(x_1) = l(x_2)$, we consider the greatest element (according to the lexicographical order) of the set $\{y(1), y(2), \ldots$

$\ldots, y(s(l(x_2) - m_V(x_2) - 1))\} - \{z_1\}$, and we shall call this element $z_2$. Put $\varphi(z_2, l(x_2)) = x_2$.

Continuing the procedure we reach the step $i > 1$. There are two cases. In the first case $l(x_i) \neq l(x_j)$, for all $j < i$. In this case we put $\varphi(z_i, l(x_i)) = x_i$, where $z_i = y(s(l(x_i) - m_V(x_i) - 1))$. In the second (opposite) case let $j(1) < j(2) < \ldots < j(k) < i$ be all indices $j < i$ such that $l(x_i) = l(x_j)$. In fact, $j(2) = j(1) + 1$, $j(3) = j(2) + 1, \ldots$, due to the properties of the enumeration function $g$. We define $z_i$ to be the greatest element (in lexicographical order) of the set $\{y(1), y(2), \ldots, y(s(l(x_i) - m_V(x_i) - 1))\} - \{z_{j(1)}, z_{j(2)}, \ldots, z_{j(k)}\}$ and put $\varphi(z_i, l(x_i)) = x_i$. Notice that $\varphi$ acts as a function, because if $l(x_i) = l(x_j)$ we have $z_i \neq z_j$.

The construction is possible and the motivation follows. Put $l(x_i) = l(x_{j(1)}) = l(x_{j(2)}) = \ldots = l(x_{j(k)}) = l$. We have $m_i = m_V(x_i) \leqq m_k = m_V(x_{j(k)}) \leqq m_{k-1} = m_V(x_{j(k-1)}) \leqq \ldots \leqq m_1 = m_V(x_{j(1)})$. For every natural $t \in \{1, 2, \ldots, k\} \cup \{i\}$ let $\boldsymbol{B}_t = \{y(1), y(2), \ldots, y(s(l - m_t - 1))\}$. Notice that $\boldsymbol{B}_1 \subset \boldsymbol{B}_2 \subset \ldots \subset \boldsymbol{B}_k \subset \boldsymbol{B}_i$ and $\boldsymbol{B}_u = \boldsymbol{B}_v$(for $u < v$) iff $m_u = m_v$. We shall try to describe in a detailed manner the action of $\varphi$ and this will complete the motivation.

Clearly, $z_{j(1)} = y(s(l - m_1 - 1))$. In order to obtain $z_{j(2)}$, we distinguish two possible cases: a) $m_1 > m_2$ (and in this case $z_{j(2)} = y(s(l - m_2 - 1))$; *b)* $m_1 = m_2$ (and in this case $\boldsymbol{B}_1 = \boldsymbol{B}_2$, so $z_{j(2)}$ must be $y(s(l - m_2 - 1) - 1))$. It is to be seen that in case b) one has $s(l - m_2 - 1) - 1 \geqq 1$ (in other words the construction is possible) because $2 \leqq \text{card}\,\{x \in \boldsymbol{X}^* \mid l(x) = l \text{ and } (x, m_2) \in \boldsymbol{V}\} \leqq (p^{l-m_2} - 1)/(p - 1) = s(l - m_2 - 1)$. The case when strict inclusion occurs between the $\boldsymbol{B}_t'$s being clearly favorable, we focus our attention to the "bad" situation $m_h = m_{h+1} = m_{h+2} = \ldots = m_r = m$ $(1 \leqq h \leqq r \leqq i)$. Here, in case $h > 1$, we consider $m_{h-1} < m_h$. We have $\boldsymbol{B}_h = \boldsymbol{B}_{h+1} = \ldots = \boldsymbol{B}_r$. The construction gives: $z_{j(h)} = y(s(l - m - 1))$, $z_{j(h+1)} = y(s(l - m - 1) - 1, \ldots$, $z_{j(r)} = y(s(l - m - 1) - (r - h))$. It remains to show that $s(l - m - 1) - (r - h) \geqq 1$, i.e. $r - h + 1 \leqq (p^{l-m} - 1)/(p - 1)$. This inequality follows from $r - h + 1 \leqq \text{card}\,\{x \in \boldsymbol{X}^* : l(x) = l,\ (x, m) \in \boldsymbol{V}\} \leqq (p^{l-m} - 1)/(p - 1)$.

It is worth to add the fact that in case $\boldsymbol{V}$ is finite and the set $\boldsymbol{A}$ (see the beginning of the proof) has $q$ elements, the procedure stops at step $q$.

The injectivity of $\varphi$ is derived from the injectivity of $g : (x_i, m_V(x_i)) \neq (x_j, m_V(x_j))$ iff $x_i \neq x_j$ or $m_V(x_i) \neq m_V(x_j)$. This implies that for different $i$ and $j$ one must obtain different values $\varphi(z_i, l(x_i)) = x_i$ and $\varphi(z_j, l(x_j)) = x_j$.

Our next task is to prove the inclusion $\boldsymbol{V} \subset \boldsymbol{V}(\varphi)$. Indeed, in case $(x, m)$ is in $\boldsymbol{V}$ let $(x, m_V(x)) = (x_i, m_V(x_i))$ in the enumeration given by $g$. So $m \leqq m_V(x_i)$ and $x_i = \varphi(z_i, l(x_i))$ where the length of $z_i$ is less than $l(x_i) - m_V(x_i) - 1$, which shows that $\boldsymbol{K}_\varphi(x \mid l(x)) \leqq l(x_i) - m_V(x_i) - 1 < l(x) - m_V(x)$, i.e. $(x, m_V(x)) \in \boldsymbol{V}(\varphi)$. Consequently, $(x, m)$ is in the $\boldsymbol{M} - \boldsymbol{L}$ test $\boldsymbol{V}(\varphi)$.

Moreover, we can prove here also point (c), because it is seen that for every $x$ in $\boldsymbol{X}^*$ such that $(x, 1) \in \boldsymbol{V}(\varphi)$ there exists a natural $i$ such that $x = x_i$ and $(x_i, m_V(x_i)) \in \boldsymbol{V}$, which implies $(x, 1) \in \boldsymbol{V}$.

All it remains to prove is point (b), i.e. the recursiveness of the graph of $\varphi$. This is seen taking arbitrarily $((z, l), x) \approx (z, l, x)$ in $X^* \times N \times X^*$ and checking if $(z, l, x)$ belongs to the graph of $\varphi$, according to the following decision algorithm:

1. If $m_V(x) = 0$, NO. Stop.
2. If $l(x) \neq l$, NO. Stop.
3. Choose $i$ such that $g(i) = (x_i, m_V(x_i))$ and $x = x_i$.
4. Run the first $i$ steps in the procedure defining $\varphi$ in order to find $z_i$.
5. If $z = z_i$, YES. Stop.
6. If $z \neq z_i$, NO. Stop. □

*Remark.* It is obvious that for a given recursive $M - L$ test $V$ there are many p.r. functions $\varphi : X^* \times N \overset{\circ}{\to} X^*$ such that $V \subset V(\varphi)$, e.g. our construction depends on the enumeration function $g$.

**Theorem 3.** Let $V$ be a $M - L$ test. Consider the following assertions:

(1) $V$ is representable.

(2) For every natural $m \geqq 1$, one has

$$(*) \qquad \text{for all} \quad n \geqq m + 1\,, \quad \operatorname{card}\{x \in X^* \mid l(x) = n, m_V(x) = m\} \leqq p^{n-m-1}\,.$$

Then (1) ⇒ (2) and in case $V$ is recursive the implication (2) ⇒ (1) holds too.

Proof. (1) ⇒ (2). The hypothesis is that $V = V(\varphi)$ for some p.r. function $\varphi : X^* \times \times N \overset{\circ}{\to} X^*$.

Fix the natural numbers $n > m > 0$. For every $x$ in $X^*$ with $l(x) = n$ and such that $m_V(x) = m$ there exists $y$ in $X^*$ with $l(y) < l(x) - m$ and $\varphi(y, l(x)) = x$. We have $l(y) \leqq n - m - 1$.

We shall show that $l(y) = n - m - 1$. Supposing by contradiction $l(y) \leqq n - - m - 2$, let $l(y) = n - m - 1 - h$ with $h > 0$. This will lead us to the false relation $(x, m + h) \in V$. Indeed, $l(y) = n - m - h - 1 < n - m - h$ and $\varphi(y, l(x)) = = x$ show that $(x, m + h) \in V(\varphi) = V$.

The just proved equality $l(y) = n - m - 1$ shows that

$$\operatorname{card}\{x \in X^* \mid l(x) = n, m_V(x) = m\} \leqq \operatorname{card}\{y \in X^* \mid l(y) = n - m - 1\} = = p^{n-m-1}\,,$$

and the assertion (2) is proved.

Assuming that $V$ is recursive we shall prove (2) ⇒ (1). The hypothesis is that $(*)$ holds for every $m \geqq 1$. We shall show that $V = V(\varphi)$, where $\varphi$ is the p.r. function constructed in Theorem 2, namely we shall show that $V(\varphi) \subset V$.

Take $(x, m)$ in $V(\varphi)$. In any case $(x, 1) \in V$ (see Theorem 2). We shall prove that $(x, m) \in V$ by proving that $m_V(x) = m_{V(\varphi)}(x)$. Since $V \subset V(\varphi)$ (see Theorem 2) we have $m_{V(\varphi)}(x) \geqq m_V(x)$ and all it remains to prove is that $m_V(x) \geqq m_{V(\varphi)}(x)$.

Supposing the contrary, it follows that $(x, m_V(x) + 1) \in V(\varphi)$, hence there exists $z$ in $X^*$ with $l(z) < l(x) - m_V(x) - 1$ and $\varphi(z, l(x)) = x$.

Let $g(i) = (x_i, m_V(x_i))$ where $x = x_i$ in the enumeration given by $g$ (see the construction of $\varphi$ in the proof of Theorem 2). We let the procedure giving $\varphi$ run $i$ steps and we obtain the string $z_i$ such that $\varphi(z_i, l(x_i)) = x_i$. We shall show that $l(z_i) = l(x_i) - m_V(x_i) - 1 = l(x) - m_V(x) - 1$, thus deriving a contradiction (in view of the injectivity of $\varphi$).

Now the reader must remember the action of $\varphi$ (see the proof of Theorem 2). In case $l(x_i) \neq l(x_j)$ for all $j < i$, we have $l(z_j) = l(x_i) - m_V(x_i) - 1$, and the proof is finished in this case. In case $l(x_{j(1)}) = l(x_{j(2)}) = \ldots = l(x_{j(k)}) = l(x_i)$, $1 \leqq j(1) < j(2) < \ldots < j(k) < i$, we have analysed several possibilities, according to the existence of some equalities in the sequence of inequalities: $m_V(x_{j(1)}) \geqq m_V(x_{j(2)}) \geqq \ldots \geqq m_V(x_{j(k)}) \geqq m_V(x_i)$. In the case of the strict inequality $m_V(x_i) < m_V(x_{j(k)})$ we saw that $l(z_i) = l(x_i) - m_V(x_i) - 1$, and again the proof is finished. The most complicated case is when $m_V(x_i) = m_V(x_{j(k)}) = m_V(x_{j(k-1)}) = \ldots = m_V(x_{j(k-r)})$, where $0 \leqq r < k$. In this case we must put $z_i = y(s(l(x_i) - m_V(x_i) - 1) - (r + 1))$. In any case we have $r + 2$ elements $x$ such that $l(x) = n$ and $m_V(x) = m$ (we put $l(x_i) = n$ and $m_V(x_i) = m$) and the hypothesis gives $r + 2 \leqq p^{n-m-1} = \text{card}\,\{z \in X^* \mid l(z) = n - m - 1\}$. But $y(s(n - m - 1))$ is the last element (in lexicographical order) of the set $\{z \in X^* \mid l(z) = n - m - 1\} = H$. It follows that $z_i \in H$, which shows that the length of $z_i$ is $n - m - 1$ and the proof is finished in this case too. $\square$

The next result establishes a precise connection between the Kolmogorov complexity $K_\varphi$ and the critical level $m_V$ in case $V = V(\varphi)$.

**Theorem 4.** Let $V$ be a representable $M - L$ test and let $\varphi : X^* \times N \overset{\circ}{\to} X^*$ be a p.r. function such that $V = V(\varphi)$.

The following assertions hold for all $x$ in $X^*$:

(a) $m_V(x) = 0$ iff $K_\varphi(x \mid l(x)) \geqq l(x) - 1$.

(b) If $m_V(x) > 0$, then $K_\varphi(x \mid l(x)) = l(x) - m_V(x) - 1$.

In the particular case when $\varphi$ has the additional property that range $(\varphi) = \{x \in X^* \mid (x, 1) \in V\} = V_1$, point (a) can be stated more precisely, namely:

(a′) $m_V(x) = 0$ iff $K_\varphi(x \mid l(x)) = \infty$.

Proof. (a) Assume $m_V(x) = 0$, therefore $(x, 1) \notin V = V(\varphi)$. This shows that for every $y$ in $X^*$ with $l(y) < l(x) - 1$ we have $\varphi(y, l(x)) \neq x$. Then, either $\varphi(y, l(x)) \neq x$ for all $y$ in $X^*$ (which shows that $K_\varphi(x \mid l(x)) = \infty$), or there exists $y$ in $X^*$ with $\varphi(y, l(x)) = x$, but this $y$ must have $l(y) \geqq l(x) - 1$. So, $K_\varphi(x \mid l(x)) \geqq l(x) - 1$.

Assume now that $K_\varphi(x \mid l(x)) \geqq l(x) - 1$. There are two cases:

i) if $K_\varphi(x \mid l(x)) = \infty$, then $\varphi(y, l(x)) \neq x$ for all $y$ in $X^*$ and then $(x, 1) \notin V(\varphi)$ a.s.o.

ii) if $K_\varphi(x \mid l(x)) < \infty$, then there exists at least one $y$ in $X^*$ with $\varphi(y, l(x)) = x$ and one must have $l(y) \geqq l(x) - 1$. This shows that $(x, 1) \notin V(\varphi)$.

(b) According to the hypothesis, there exists $y$ in $X^*$ such that $\varphi(y, l(x)) = x$.

We have: $m_V(x) = m_{V(\varphi)}(x) = \max \{m \geqq 1 |$ there exists $y$ in $X^*$ with $l(y) < l(x) - m$ and $\varphi(y, l(x)) = x\} = \max \{m \geqq 1 |$ there exists $y$ in $X^*$ with $m < l(x) - l(y)$ and $\varphi(y, l(x)) = x\}$. The last maximum is attained for those $y$ in $X^*$ which have minimum length, i.e. for those $y$ in $X^*$ with $l(y) = K_\varphi(x \mid l(x))$. So, $m_V(x) = l(x) - K_\varphi(x \mid l(x)) - 1$.

In the particular case, all it remains to prove is the implication: $m_V(x) = 0 \Rightarrow K_\varphi(x \mid l(x)) = \infty$. Indeed, $m_V(x) = 0$ implies $(x, 1) \notin V = V(\varphi)$, so $x \notin \text{range}(\varphi)$. □

**Corollary 5.** Let $V$ be a recursive representable $M - L$ test and let $\varphi : X^* \times N \overset{\circ}{\to} X^*$ be a p.r. function with the properties $V = V(\varphi)$ and range $(\varphi) = V_1$. Then the partial function $U_\varphi : X^* \overset{\circ}{\to} N$ given by $U_\varphi(x) = K_\varphi(x \mid l(x))$ is a p.r. function with recursive graph.

Proof. Relations (a′) and (b) in Theorem 4 applied to the present function $\varphi$ show that $U_\varphi$ is a p.r. function. The graph of $U_\varphi$ is recursive because the pair $(x, m) \in$ graph $(U_\varphi)$ iff $m_V(x) > 0$ and $m = l(x) - m_V(x) - 1$. Here we made use of the recursiveness of the function $m_V$ (see Theorem 1). □

*Remark.* The p.r. function $\varphi$ given by the proof of Theorem 3 is a function satisfying the property that range $(\varphi) = V_1$.

**Theorem 6.** Let $\varphi : X^* \times N \overset{\circ}{\to} X^*$ be a p.r. function such that range $(\varphi) = (V(\varphi))_1$. Then the following assertions are equivalent:

(i) The partial function $U_\varphi : X^* \overset{\circ}{\to} N$ given by $U_\varphi(x) = K_\varphi(x \mid l(x))$ is a p.r. function with recursive graph.

(ii) The $M - L$ test $V(\varphi)$ is recursive.

Proof. (i) $\Rightarrow$ (ii). The proof will be given by the following equivalences: $((x, m) \in V(\varphi)) \Leftrightarrow (U_\varphi(x) < l(x) - m) \Leftrightarrow (U_\varphi(x) \in \{0, 1, 2, \ldots, l(x) - m - 1\}) \Leftrightarrow (U_\varphi(x) = 0$ or $U_\varphi(x) = 1$ or ... or $U_\varphi(x) = l(x) - m - 1) \Leftrightarrow ((x, 0) \in \text{graph}(U_\varphi)$ or $(x, 1) \in \text{graph}(U_\varphi)$ or ... or $(x, l(x) - m - 1) \in \text{graph}(U_\varphi))$. We made the convention that in case $l(x) - m - 1 < 0$, the set $\{0, 1, 2, \ldots, l(x) - m - 1\}$ is empty.

(ii) $\Rightarrow$ (i). We put $V = V(\varphi)$ and apply Corollary 5 to this $V$ and this $\varphi$. □

The following theorem will furnish an interesting class of recursive representable $M - L$ tests.

**Theorem 7.** Let $V \subset X^* \times N$ have the following properties:

(a) The set $V$ is recursively enumerable.

(b) For every natural $m \geqq 1$ we have the inclusion $V_{m+1} \subset V_m$.

1. The following assertions are equivalent:

(i) For all natural $n > m \geqq 1$, we have:

$$\text{card}\,\{x \in X^* \mid l(x) = n\,,\ \ (x, m) \in V\} = (p^{n-m} - 1)/(p - 1)\,.$$

(ii) For all natural $n > m \geqq 1$, we have:

$$\text{card}\,\{x \in \boldsymbol{X}^* \mid l(x) = n\,,\ m_{\boldsymbol{V}}(x) = m\} = p^{n-m-1}\,.$$

2. If one of the above conditions (i) or (ii) is fulfilled for a set $\boldsymbol{V}$ having properties (a) and (b), then $\boldsymbol{V}$ is a recursive representable $\boldsymbol{M} - \boldsymbol{L}$ test. Such $\boldsymbol{M} - \boldsymbol{L}$ tests will be called *full*.

Proof. 1. (i) ⇒ (ii). The conditions (a), (b) and (i) insure that $\boldsymbol{V}$ is a $\boldsymbol{M} - \boldsymbol{L}$ test, hence $m_{\boldsymbol{V}}$ takes only finite values.

On the other hand, for every natural $j \geqq 0$ and $n \geqq j + 1$ one can see, using condition (b), that $\{x \in \boldsymbol{X}^* \mid l(x) = n,\ m_{\boldsymbol{V}}(x) = n - (j + 1)\} = \{x \in \boldsymbol{X}^* \mid l(x) = n\,, (x, n - j - 1) \in \boldsymbol{V}\} - \{x \in \boldsymbol{X}^* \mid l(x) = n,\ (x, n - j) \in \boldsymbol{V}\}$. Consequently, $\text{card}\,\{x \in \boldsymbol{X}^* \mid l(x) = n, m_{\boldsymbol{V}}(x) = n - (j + 1)\} = ((p^{n-(n-j-1)} - 1)/(p - 1)) - ((p^{n-(n-j)} - 1)/(p - 1)) = p^j$, using the hypothesis and condition (b). Taking $n - (j + 1) = m$, we obtain $\text{card}\,\{x \in \boldsymbol{X}^* \mid l(x) = n, m_{\boldsymbol{V}}(x) = m\} = p^{n-m-1}$.

(ii) ⇒ (i). For every natural $n > m \geqq 1$ we have the equality

$$(**)\qquad \{x \in \boldsymbol{X}^* \mid l(x) = n, (x, m) \in \boldsymbol{V}\} = \{x \in \boldsymbol{X}^* \mid l(x) = n, m_{\boldsymbol{V}}(x) = m\} \cup \cup \{x \in \boldsymbol{X}^* \mid l(x) = n, m_{\boldsymbol{V}}(x) = m + 1\} \cup \ldots \cup \{x \in \boldsymbol{X}^* \mid l(x) = n, m_{\boldsymbol{V}}(x) = n - 1\}\,.$$

In fact, $\boldsymbol{A}_n = \{x \in \boldsymbol{X}^* \mid l(x) = n,\ m_{\boldsymbol{V}}(x) = n\} = \emptyset$, because $\boldsymbol{A}_n \subset \boldsymbol{A}_{n-1} = \{x \in \boldsymbol{X}^* \mid l(x) = n,\ m_{\boldsymbol{V}}(x) = n - 1\}$, according to condition (b) and $\text{card}\,\boldsymbol{A}_{n-1} = 1$. If $\boldsymbol{A}_n$ were be non empty, then $\text{card}\,\boldsymbol{A}_n = 1$, so $\boldsymbol{A}_n = \boldsymbol{A}_{n-1}$ and this is impossible. Again condition (b) guarantees also that $\boldsymbol{A}_u = \emptyset$, where $u > n$. Thus the proof of (**) is complete.

Consequently, (**) yields

$$\text{card}\,\{x \in \boldsymbol{X}^* \mid l(x) = n, (x, m) \in \boldsymbol{V}\} = = \sum_{j=m}^{n-1} \text{card}\,\{x \in \boldsymbol{X}^* \mid l(x) = n, m_{\boldsymbol{V}}(x) = j\} = \sum_{j=m}^{n-1} p^{n-j-1} = (p^{n-m} - 1)/(p - 1)\,.$$

2. All it remains to prove is that (i) implies the recursiveness of $\boldsymbol{V}$ (because in this case $\boldsymbol{V}$ will be a recursive $\boldsymbol{M} - \boldsymbol{L}$ test satisfying condition (2) in Theorem 3).

The case when $\boldsymbol{V}$ is finite is obvious.

Assume therefore that $\boldsymbol{V}$ is infinite and let $g : (\boldsymbol{N} - \{0\}) \to \boldsymbol{X}^* \times \boldsymbol{N}$ be an injective recursive function such that $g(\boldsymbol{N} - \{0\}) = \boldsymbol{V}$. Put $g(i) = (x_i, m_i)$ for all natural $i \geqq 1$.

We take arbitrarily $(x, m)$ in $\boldsymbol{X}^* \times \boldsymbol{N}$ and we describe an algorithm for testing if $(x, m)$ is in $\boldsymbol{V}$. Put $l(x) = n$. There exists a natural $q \geqq 1$ such that the set $\boldsymbol{G} = \{g(1), g(2), \ldots, g(q)\}$ contains all the elements $(y, m) \in \boldsymbol{V}$ with $l(y) = n$. Moreover, $q$ can be effectively found. For instance, $q$ can be taken to be the least natural number $h$ such that the set $\{g(1), g(2), \ldots, g(h)\}$ contains exactly $(p^{n-m} - 1)/(p - 1)$ pairs $(y, m)$ with $l(y) = n$. If $(x, m) \in \boldsymbol{G}$, then $(x, m) \in \boldsymbol{V}$ and if $(x, m) \notin \boldsymbol{G}$, then $(x, m) \notin \boldsymbol{V}$. □

**Example 8.** We shall exhibit an example of $M - L$ test $V$ which is full and we shall also construct the associate p.r. function $\varphi$ such that $V = V(\varphi)$ given by Theorem 3.

a) In order to give the $M - L$ test $V$ we shall denote, for every $n > m \geqq 1$, by $A(n, m)$ the set $\{(x, m) \in V \mid l(x) = n\}$. It is clear that the $M - L$ test $V$ will be completely determined if we shall give all the sets $A(n, m)$.

Put $A(n, m) = \{(y(s(n-1) + i), m) \mid i = 1, 2, \ldots, s(n - m - 1)\})$ (see Section 1). It is seen that for every $m \geqq 1$ one has

$$V_m = \bigcup_{n=m+1}^{\infty} \{y(s(n-1) + i) \mid i = 1, 2, \ldots, s(n - m - 1)\}\,.$$

The reader can see now that this $V$ is a full $M - L$ test. Moreover, an elementary computation gives the form of the function $m_V$. We have for all $n \geqq 2$:

$$m_V(y(s(n-1) + 1)) = n - 1\,,$$

and

$$m_V(y(s(n-1) + i)) = n - k - 1\,,$$

for every $1 \leqq k \leqq n - 2$, where $i \in \{s(k-1) + 1, s(k-1) + 2, \ldots, s(k)\}$ ; also

$$m_V(x) = 0\,,$$

for the other $x$ in $X^*$.

An inspection of $A(n, 1)$ shows that for $n \geqq 2$ one has:

$$\operatorname{card}\{x \in X^* \mid l(x) = n, \text{ there exists an } m \geqq 1 \text{ such that } (x, m) \in V\} = s(n-2)\,.$$

b) In order to do the construction indicated in the proof of Theorem 2, we shall choose an enumeration function $g$ for the set $A = \{(x, m_V(x)) \mid x \in V_1\}$. This $g$ will satisfy the conditions u), v) required in the proof of Theorem 2 and it possesses the supplementary property (which completely determines $g$):

w) if for $i < j$ one has $l(x_i) = l(x_j)$ and $m_V(x_i) = m_V(x_j)$, then $x_i > x_j$ in lexicographical order. This means that for every $n > m \geqq 1$, the set $\{x \in X^* \mid l(x) = n, m_V(x) = m\}$ is ordered by the inverse of the lexicographical order.

The p.r. function $\varphi : X^* \times N \xrightarrow{\circ} X^*$ produced by the proof of Theorem 2 is given by

$$\varphi(y(i), n) = y(s(n-1) + i)\,,$$

for every $n \geqq 2$ and $i = 1, 2, \ldots, s(n-2)$.

An alternative of Theorem 7 (which was based upon the equalities (i) and (ii) guaranteeing the recursiveness of $V$) will be the following theorem. Here we shall actually replace the equalities (i) and (ii) in Theorem 7 by inequalities and we shall assume the recursiveness of $V$.

**Theorem 9.** Let $V \subset X^* \times N$ be a set having the following properties:

(a) The set $V$ is recursive.

(b) For every natural $m \geqq 1$, we have the inclusion $V_{m+1} \subset V_m$.

(c) For all natural $n > m \geqq 1$, we have

$$\operatorname{card}\{x \in \boldsymbol{X}^* \mid l(x) = n, m_V(x) = m\} \leqq p^{n-m-1}.$$

Under these assumptions the set $\boldsymbol{V}$ is a representable $\boldsymbol{M} - \boldsymbol{L}$ test.

Proof. In view of Theorem 3, all it remains to prove is the fact that $\boldsymbol{V}$ is a $\boldsymbol{M} - \boldsymbol{L}$ test. This can be done using the equality (**) in the proof of Theorem 6, which yields

$$\operatorname{card}\{x \in \boldsymbol{X}^* \mid l(x) = n, (x, m) \in \boldsymbol{V}\} = \sum_{j=m}^{n-1} \operatorname{card}\{x \in \boldsymbol{X}^* \mid l(x) = n, m_V(x) = j\} \leqq$$
$$\leqq \sum_{j=m}^{n-1} p^{n-j-1} = (p^{n-m} - 1)/(p - 1). \qquad \square$$

## 3. REMARKS AND OPEN PROBLEMS

Our representability theory (see also [2]) is an attempt to compare Kolmogorov's complexity theory of strings which uses algorithms [3] with P. Martin-Löf's complexity theory of strings which uses $\boldsymbol{M} - \boldsymbol{L}$ tests [5]. We have already seen that there exist non-representable $\boldsymbol{M} - \boldsymbol{L}$ tests [2], i.e. these theories are not equivalent. For instance, take $p = 2$, $\boldsymbol{X} = \{0, 1\}$ and $\boldsymbol{V} = \{(000,1), (010,1), (111,1)\}$.

In this direction we could obtain the following result concerning recursive sets:

If we call ***K-test*** a set $\boldsymbol{V} \subset \boldsymbol{X}^* \times \boldsymbol{N}$ having properties (a), (b) and (c) in Theorem 9, then Kolmogorov's complexity theory and P. Martin-Löf's complexity theory done only with $\boldsymbol{K}$-tests are equivalent. This means that for every p.r. function $\varphi : \boldsymbol{X}^* \times \boldsymbol{N} \xrightarrow{\circ} \boldsymbol{X}^*$ we can obtain the $\boldsymbol{K}$-test $\boldsymbol{V}(\varphi)$ and for every $\boldsymbol{K}$-test $\boldsymbol{V} \subset \boldsymbol{X}^* \times \boldsymbol{N}$ we can obtain a p.r. function $\varphi : \boldsymbol{X}^* \times \boldsymbol{N} \xrightarrow{\circ} \boldsymbol{X}^*$ such that $\boldsymbol{V} = \boldsymbol{V}(\varphi)$ (see Example 10 in [1], Theorems 3, 4 and 9).

We set the following natural open problems:

A) Does the equivalence (1)$\Leftrightarrow$(2) in Theorem 3 hold also for non-recursive $\boldsymbol{M}-\boldsymbol{L}$ tests $\boldsymbol{V}$? Equivalently, does the result in Theorem 9 hold also for non-recursive $\boldsymbol{M} - \boldsymbol{L}$ tests $\boldsymbol{V}$?

B) Does the result in Theorem 2 hold also for non-recursive $\boldsymbol{M} - \boldsymbol{L}$ tests $\boldsymbol{V}$?

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