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NONSMOOTH OPTIMAL DESIGN PROBLEMS FOR THE KIRCHHOFF PLATE WITH UNILATERAL CONDITIONS

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The form of directional derivative of the metric projection in the Sobolev space $H_0^2(\Omega)$ onto the convex set $K = \{f \in H_0^2(\Omega) \mid f \geq \psi\}$ is derived in [14].

In the present paper the result is used to obtain the first order optimality conditions for a class of nonsmooth optimal design problems for the Kirchhoff plate with an obstacle.

1. INTRODUCTION

The paper is concerned with the optimal design problems for the fourth order variational inequalities. Namely, the first order necessary optimality conditions are derived for the class of optimization problems under consideration.

The differential stability of metric projection in the Sobolev space $H_0^1(\Omega)$ onto the cone of nonnegative elements is considered by Mignot [9]. Mignot derived the form of the so-called conical differential of the metric projection. However, the technique of proof used by Mignot is based on potential theory in Dirichlet spaces, therefore, his argument cannot be directly applied in the Sobolev space $H_0^2(\Omega)$.

The differential stability of metric projection in the Sobolev space $H_0^2(\Omega)$ onto the cone of nonnegative elements is investigated by Rao and Sokolowski [14]. In particular, in [14] the sufficient conditions are obtained under which the set K is polyhedric at a given point $f \in K$. The question of polyhedricity is adressed in [14] since it implies directional differentiability of the metric projection onto K with an explicit form of the differential [5, 9], i.e., the so-called conical differential of the metric projection onto the cone of nonnegative elements. It follows, we refer the reader to [19] for the details, that the conical differential is given as a metric projection onto the intersection of a tangent cone with a supporting hyperplane.

The paper is organized as follows. In Section 2 the necessary optimality conditions for an optimal design problem for the Kirchhoff plate with an obstacle are derived. In Section 3 the optimal design of an obstacle is considered.

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The variational inequality for the Kirchhoff model of an elastic plate with an obstacle is used in the present paper, however, it seems that similar results can be derived as well as for the von Kármán plate model.

We refer the reader to [5,9] for the related results on the differential stability of metric projections in Hilbert space. Some applications of the differential stability of metric projection onto convex sets in Sobolev spaces are presented in [1]–[3], [6], [12]–[19]. In particular the sensitivity analysis of solutions of constrained optimization problems is studied in [16, 17]. The applications to optimal design problems are given in [1]–[3], [6], [7], [19]. We refer the reader to [4] for general results on variational inequalities.

2. OPTIMAL DESIGN PROBLEM

We derive the necessary optimality conditions for an optimal design problem for the Kirchhoff plate with an obstacle. We refer the reader to [1], [8] where such a problem is defined, and to [10], [19] for the related results on nonsmooth optimization problems for the linear elliptic systems. Let

$$a(h;\cdot,\cdot):\ H_0^2(\Omega)\times H_0^2(\Omega)\rightarrow \mathbb{R}$$

be the following bilinear form associated to the Kirchhoff plate [1], [10], $\Omega\subset\mathbb{R}^2$ is a smooth domain with the boundary Γ,

$$a(h;y,\varphi)=\int_\Omega h^3(x)\,b_{ijkl}\frac{\partial^2 y}{\partial x_i\partial x_j}(x)\,\frac{\partial^2\varphi}{\partial x_k\partial x_l}(x)\,\mathrm{d}x,\quad \forall\, y,\varphi\in H_0^2(\Omega) \tag{2.1}$$

here we use the summation convention over the repeated indices $i,\ j,\ k,\ l=1,2$.

The bilinear form (2.1) is defined in standard way for the following fourth order elliptic operator:

$$\begin{aligned}(Aw)(x_1,x_2) &= \frac{\partial^2}{\partial x_1^2}\left(h^3\frac{\partial^2 w(x_1,x_2)}{\partial x_1^2}\right)+\frac{\partial^2}{\partial x_2^2}\left(h^3\frac{\partial^2 w(x_1,x_2)}{\partial x_2^2}\right)+\\ &+ \nu\frac{\partial^2}{\partial x_1^2}\left(h^3\frac{\partial^2 w(x_1,x_2)}{\partial x_2^2}\right)+\nu\frac{\partial^2}{\partial x_2^2}\left(h^3\frac{\partial^2 w(x_1,x_2)}{\partial x_1^2}\right)+\\ &+ 2(1-\nu)\frac{\partial^2}{\partial x_1\partial x_2}\left(h^3\frac{\partial^2 w(x_1,x_2)}{\partial x_1\partial x_2}\right)\end{aligned}$$

where: $h:\Omega\rightarrow\mathbb{R}$ is thickness of the plate, $h\in L^\infty(\Omega)$, ν is Poisson ratio which characterizes plate material, $\nu\in(0,0.5)$.

We assume that

$$h\in U_{ad}=\{h\in L^\infty(\Omega)\mid 0<h_{\min}\le h(x)\le h_{\max},\ \text{ for a.e. } x\in\Omega\} \tag{2.2}$$

and we recall that the constants b_{ijkl}, $i,\ j,\ k,\ l=1,2$ satisfy the following conditions

$$\begin{aligned}&b_{ijkl}=b_{jikl}=b_{klij},\quad i,\ j,\ k,\ l=1,2\\ &b_{ijkl}\,\xi_{ij}\,\xi_{kl}\ge c\,\xi_{ij}\,\xi_{ij},\quad c>0,\end{aligned}$$

for all symmetric matrices $[\xi_{ij}]_{2\times 2}$.

We consider a boundary value problem with the homogenous boundary conditions. However, there is no additional difficulty to derive the same results for the problem with non-homogenous boudary conditions. It follows by our asumptions (2.2)–(2.4) that the bilinear form (2.1) is continuous, symmetric, and $H_0^2(\Omega)$–elliptic, i.e.,

$$a(h;y,y) \geq \alpha \|y\|^2_{H^2(\Omega)}, \quad \alpha > 0, \quad \forall\, y \in H_0^2(\Omega)\ .$$

Now let us denote

$$K = \{\varphi \in H_0^2(\Omega) \mid \varphi(x) \geq \psi(x) \text{ in } \Omega\} \tag{2.5}$$

where $\psi(\cdot) \in H^2(\Omega) \subset C(\bar{\Omega})$ is a given element such that the set (2.5) is non-empty, in particular $\psi(x) < 0$ on $\Gamma = \partial\Omega$. For a given element $h \in U_{ad}$ we denote by $w = w(h;x)$, $x \in \Omega$, the unique solution of the following variational inequality

$$w \in K : a(h;w,\varphi - w) \geq \int_\Omega f(\varphi - w)\,\mathrm{d}x, \quad \forall\, \varphi \in K$$

where $f \in H^{-2}(\Omega)$, $H^{-2}(\Omega)$ being the dual of $H_0^2(\Omega)$.

The solution to the above variational inequality is nothing else but the metric projection of an element in the space $H_0^2(\Omega)$ onto the convex set K with respect to the scalar product defined by the bilinear form (2.1), i.e. $w \doteq P_K(g)$ for some $g \in H_0^2(\Omega)$ which means that

$$\begin{aligned} w \in K : &\quad a(h; w - g, \varphi - w) \geq 0 \quad \forall\, \varphi \in K, \\ \text{where } g \in H_0^2(\Omega) : &\quad a(h; g, \varphi) = \int_\Omega f\varphi \mathrm{d}x \quad \forall\, \varphi \in H_0^2(\Omega) \end{aligned}$$

It is shown in [14], that under some assumptions on the support of a Radon measure defined for solutions to the variational inequality under consideration, the conical differential at a given $h \in L^\infty(\Omega)$ of the metric projection onto the set (2.5) exists for any direction. It implies that the mapping

$$L^\infty(\Omega) \ni h \to w(h;\cdot) \in H_0^2(\Omega)$$

at the given point $h \in U_{ad}$ is directionally differentiable, furthermore, the explicit form of differential is obtained. This is given in

Lemma 1. Let μ be the Radon measure defined by

$$\int \varphi\,\mathrm{d}\mu = a(h;w,\varphi) - \int_\Omega f\varphi\,\mathrm{d}x \quad \forall\, \varphi \in H_0^2(\Omega)$$

and assume that the support $F = \operatorname{spt}\mu$ of the measure μ satisfies the following condition:

For any $\varphi \in H_0^2(\Omega)$, $\varphi = 0$ on $F = \operatorname{spt}\mu$, it follows that $\varphi \in H_0^2(\Omega \setminus F)$.

Then for $\varepsilon > 0$, ε small enough

$$\forall\, v \in L^\infty(\Omega) : w(h + \varepsilon v) = w(h) + \varepsilon q(v) + o(\varepsilon),$$

where $\|o(\varepsilon)\|_{H_0^2(\Omega)}/\varepsilon \to 0$ with $\varepsilon \downarrow 0$ and $q = q(v) \in H_0^2(\Omega), v \in L^\infty(\Omega)$, is given as the unique solution to the following variational inequality

$$q \in S: \quad a(h; q, \varphi - q) + a'_v(h; w(h), \varphi - q) \geq 0 \quad \forall \varphi \in S,$$

where

$$a'_v(h; y, \varphi) = \int_\Omega 3h^2(x)v(x)b_{ijkl}\frac{\partial^2 y}{\partial x_i \partial x_j}(x)\frac{\partial^2 \varphi}{\partial x_k \partial x_l}(x)\,\mathrm{d}x, \quad \forall y, \varphi \in H_0^2(\Omega)$$
$$S = \{\varphi \in H_0^2(\Omega) \mid \varphi = 0 \text{ on } \operatorname{spt}\mu,\ \varphi \geq 0 \text{ on } \Xi \setminus \operatorname{spt}\mu\}$$
$$\Xi = \{x \in \Omega \mid w(h; x) = \psi(x)\} \quad \text{is compact.}$$

Remark 1. The regularity condition required in Lemma 1 on the support $F = \operatorname{spt}\mu$ of the measure μ implies the conical differentiability of the metric projection in $H_0^2(\Omega)$ onto the convex set K, i.e., if the condition is satisfied then the set K is polyhedric [14] at $w(h) = P_K(g(h))$, where $g(h) \in H_0^2(\Omega)$ is given as the unique solution to the equation

$$a(h; g(h), \varphi) = \int_\Omega f\varphi\,\mathrm{d}x \quad \forall \varphi \in H_0^2(\Omega).$$

The following notation is introduced

$$\mathcal{U}_{ad} = U_{ad} \cap H^s(\Omega)$$

for some $s > 0$. Let us consider the following optimal design problem for the Kirchhoff plate, $\beta > 0$ is a given constant.

Problem (P): Find an element $h \in \mathcal{U}_{ad}$ which minimizes the functional

$$J(h) = \max_{x \in \Omega}[w(h; x)] + \frac{\beta}{2}\|h\|^2_{H^s(\Omega)}$$

over the set $\mathcal{U}_{ad}$.

It is clear that for any $\beta > 0$ there exists an optimal solution $h^\star \in \mathcal{U}_{ad}$ to the above problem. In the same way as in Lemma 1 we assume that the following condition is satisfied: For any $\varphi \in H_0^2(\Omega)$, $\varphi = 0$ on $F^\star = \operatorname{spt}\mu$, it follows that $\varphi \in H_0^2(\Omega \setminus F^\star)$, where $F^\star$ denotes the support of the Radon measure μ defined by

$$\int \varphi\,\mathrm{d}\mu = a(h^\star; w^\star, \varphi) - \int_\Omega f\varphi\,\mathrm{d}x \quad \forall \varphi \in H_0^2(\Omega)$$

and $w^\star$ is a solution to (2.5) for $h^\star$.

We cannot assert in general the existence of an optimal solution $h \in U_{ad}$ for $\beta = 0$. In such a case the notion of a generalized solution of problem (P) can be introduced [10]. We derive the necessary optimality conditions for problem (P) assuming that $\beta > 0$ and therefore there exists an optimal solution. The necessary optimality conditions of the same type can be obtained for a generalized solution to the problem (P) for $\beta = 0$.

Theorem 1. An optimal solution $h^\star \in \mathcal{U}_{ad}$ of the problem (P) satisfies the following first order optimality conditions

$$\max_{x \in \Omega^\star(h^\star)} \operatorname{sign}\{w(h^\star;x)\} q(v-h^\star;x) + \beta(h^\star, v-h^\star)_{H^s(\Omega)} \geq 0 \quad \forall\, v \in \mathcal{U}_{ad}\,,$$

where

$$\Omega^\star(h) = \left\{ x^\star \in \Omega \,|\, \max_{x\in\Omega} |w(h;x)| = w(h;x^\star) \right\} \quad \forall\, h \in \mathcal{U}_{ad}.$$

The proof of Theorem 1, follows by an application of Lemma 1 and a standard technique.

Remark 2. An optimal design problem for the Kirchhoff plate with a finite number of pointwise obstacles is considered in [3]. The results derived in [3] are not comparable with our result presented here, since we assume that an obstacle is smooth i.e., $\psi(.) \in H^2(\Omega)$. We refer also to [7] for the related results on optimal design of elastic plates.

3. SHAPE OPTIMIZATION OF OBSTACLES

In this section it is assumed that the thickness of the plate is fixed. Let there be given a closed and convex set $\Psi_{ad} \subset H^2(\Omega)$ such that there exist elements $a \in H^{\frac{3}{2}}(\Gamma)$, $a(x) \leq a_0 < 0$ for all $x \in \Gamma$, $b \in H^{\frac{1}{2}}(\Gamma)$,

$$\psi_{|\Gamma} = a, \quad \frac{\partial \psi}{\partial n|\Gamma} = b \quad \forall\, \psi \in \Psi_{ad}.$$

The following notation is used

$$\mathcal{K}_\psi = \{\varphi \in H_0^2(\Omega) | \varphi \geq \psi \quad \text{in } \Omega\},$$

$w = w_\psi$, $\psi \in \Psi_{ad}$, is a solution to the variational inequality

$$w \in \mathcal{K}_\psi : a(h; w, \varphi - w) \geq \int_\Omega f(\varphi - w)\,\mathrm{d}x \quad \forall\, \varphi \in \mathcal{K}_\psi. \tag{3.1}$$

Let us consider the following nonsmooth shape optimization problem [8]

Problem (P'): Find an element $\psi^\star \in \Psi_{ad}$ which minimizes the functional

$$\mathcal{J}(\psi) = \max_{x\in\Omega} |w_\psi(h;x)|$$

over the set Ψ_{ad}.

Theorem 2. There exists[1] an optimal solution $\psi^\star \in \Psi_{ad}$ to the above problem.
Assume that the Radon measure ν defined by

$$\int \varphi \, \mathrm{d}\nu = a(h; w_{\psi^\star}, \varphi) - \int_\Omega f\varphi \, \mathrm{d}x \quad \forall \varphi \in H_0^2(\Omega)$$

satisfies the following condition:
For any $\varphi \in H_0^2(\Omega)$, $\varphi = 0$ on $F = \operatorname{spt} \nu$, it follows that $\varphi \in H_0^2(\Omega \setminus F)$.
Then the optimal solution $\psi^\star \in \Psi_{ad}$ satisfies the first order optimality conditions

$$\max_{x \in \Omega^\star(\psi^\star)} \operatorname{sign} \{w_{\psi^\star}(h; x)\} p_{\psi - \psi^\star}(h; x) \geq 0, \quad \forall \psi \in \Psi_{ad} \tag{3.2}$$

where

$$\Omega^\star(\psi) = \{x \in \Omega | \mathcal{J}(\psi) = w_\psi(h; x)\} \quad \forall \psi \in \Psi_{ad}$$

and $p^\star = p_{\psi - \psi^\star}$, $\psi \in \Psi_{ad}$, is given as a unique solution to the following variational inequality

$$p^\star \in \mathcal{S}_{\psi - \psi^\star} : \quad a(h; p^\star, \varphi - p^\star) \geq 0 \quad \forall \varphi \in \mathcal{S}_{\psi - \psi^\star}$$

The convex cone $\mathcal{S}_{\psi - \psi^\star}$ takes the following form

$$\mathcal{S}_{\psi - \psi^\star} = \{\varphi \in H_0^2(\Omega) | \varphi = \psi - \psi^\star \quad \text{on } \operatorname{spt} \nu \quad \varphi \geq \psi - \psi^\star \quad \text{on } \Xi \setminus \operatorname{spt} \nu\}.$$

Here

$$\Xi = \{x \in \Omega] w_{\psi^\star}(h; x) = \psi^\star(x)\}$$

The above theorem can be proved in the following way. Let $\chi \in H^2(\Omega)$ be an element such that

$$\chi_{|\Gamma} = a, \quad \frac{\partial \chi}{\partial n}_{|\Gamma} = b.$$

Then $z = w_\psi + \chi - \psi \in H_0^2(\Omega)$ is given as a unique solution to the following variational inequality

$$\chi \leq z \in H_0^2(\Omega) : \quad a(h; z, \varphi - z) \geq \int_\Omega f(\varphi - z) \, \mathrm{d}x - a(h; \chi + \psi, \varphi - z) \quad \chi \leq \varphi \in H_0^2(\Omega).$$

Under our assumptions the affine mapping

$$H^2(\Omega) \ni \psi \to z(\psi) \in H_0^2(\Omega)$$

is conically differentiable [14], which leads to the first order optimality conditions for the optimization problem under consideration. The optimality conditions for the composite cost fuctional with max type function are derived in the same way as e. g. in [10] for a linear plate model or in [19] in the case of multiple eigenvalues.

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